Standard Young Tableaux – Old and New

Ron Adin and Yuval Roichman

Department of Mathematics
Bar-Ilan University

Workshop on Group Theory in Memory of David Chillag
Technion, Haifa, Oct. '14

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3	5	7	
6	8		
9			

1	2	4	
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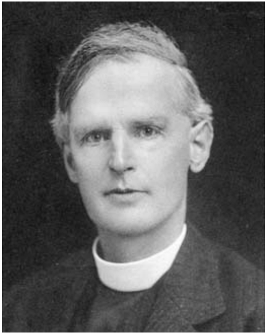
			9

David Chillag

Abstract

More than a hundred years ago, Frobenius and Young based the emerging representation theory of the symmetric group on the combinatorial objects now called Standard Young Tableaux (SYT).
Many important features of these classical objects have since been discovered, including some surprising interpretations and the celebrated hook length formula for their number.
In recent years, SYT of non-classical shapes have come up in research and were shown to have, in many cases, surprisingly nice enumeration formulas.
The talk will present some gems from the study of SYT over the years, based on a recent survey paper.
No prior acquaintance assumed.

Founders



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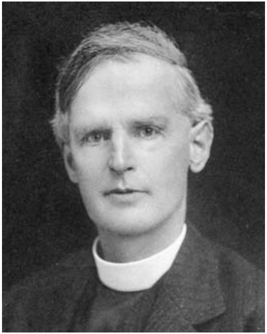

A. Young

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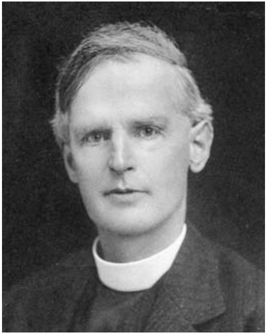

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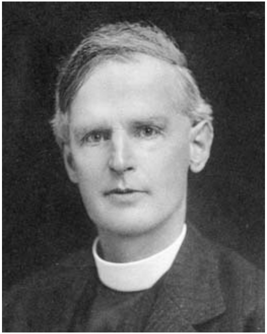

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F. G. Frobenius

P. A. MacMahon

Classical

Introduction

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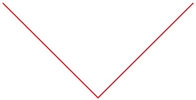

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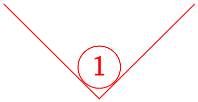

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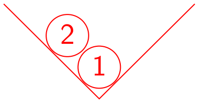

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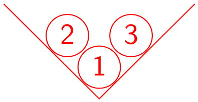

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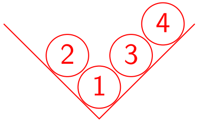

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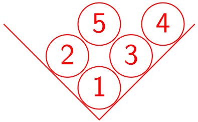

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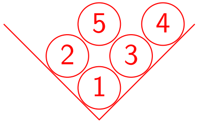

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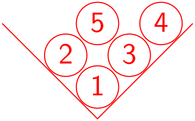

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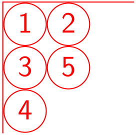
$\rightarrow$



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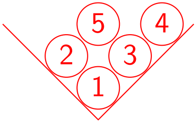

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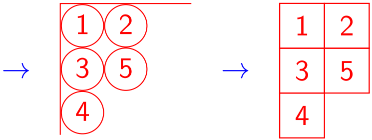

Diagrams and Tableaux

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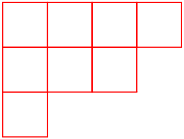
partition $\longleftrightarrow$ diagram/shape

$\lambda = (4, 3, 1) \vdash 8$ $[\lambda] =$ 

Diagrams and Tableaux

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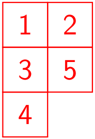
$$\lambda = (4,3,1) \vdash 8 \qquad [\lambda] = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Standard Young Tableau (SYT):

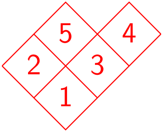
$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{SYT}(4,3,1).$$

Entries increase along rows and columns

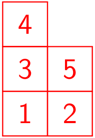
Conventions



English



Russian



French

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$$\lambda = (3,2), \qquad f^{\lambda} = 5$$

SYT and S_n Representations

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Corollary:

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

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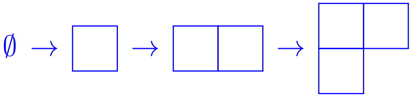

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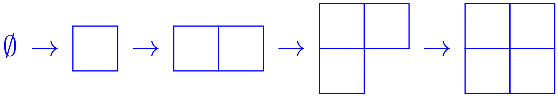

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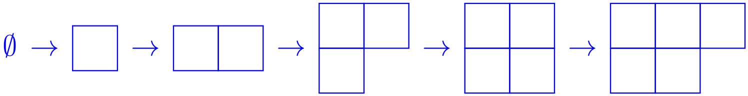

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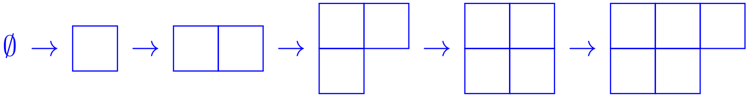

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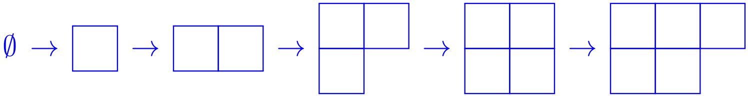

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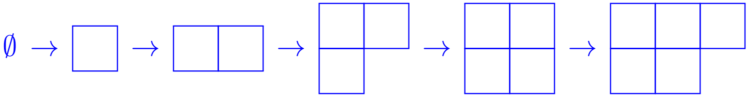

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$$\text{SYT}(\lambda) \quad \longleftrightarrow \quad \text{maximal chains in the Young lattice from } \emptyset \text{ to } \lambda$$

The number of such maximal chains is therefore f^λ.

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Each SYT of shape $\lambda = (\lambda_1, \dots, \lambda_t)$ corresponds to a lattice path in $\mathbb{R}^t$, from the origin 0 to the point λ, where in each step exactly one of the coordinates changes (by adding 1), while staying within the region

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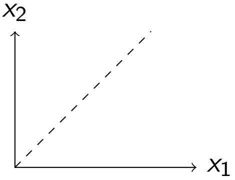

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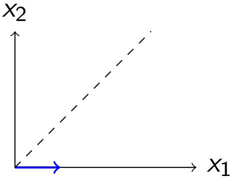

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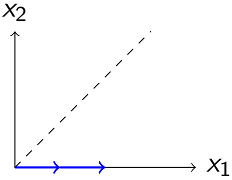

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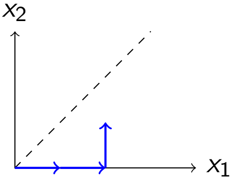

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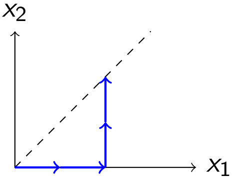

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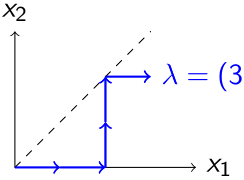

1	2	5
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Interpretation: Order Polytope

The order polytope corresponding to a diagram D is

$$P(D) := \{f : D \to [0,1] \,|\, c \leq_D c' \Longrightarrow f(c) \leq f(c') \,(\forall c,\, c' \in D)\},$$

where $\leq_D$ is the natural partial order between the cells of D. It is a closed convex subset of the unit cube $[0,1]^D$.

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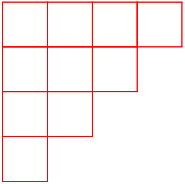
Observation:

$$\operatorname{vol} P(D) = \frac{f^D}{|D|!}.$$

Interpretation: Reduced Words (1)

The following theorem was conjectured and first proved by Stanley using symmetric functions. A bijective proof was given later by Edelman and Greene.

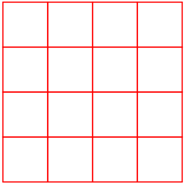
Theorem: [Stanley 1984, Edelman-Green 1987]
The number of reduced words (in adjacent transpositions) of the longest permutation $w_0 := [n, n-1, ..., 1]$ in S_n is equal to the number of SYT of staircase shape $\delta_{n-1} = (n-1, n-2, ..., 1)$.



Interpretation: Reduced Words (2)

An analogue for type B was conjectured by Stanley and proved by Haiman.

Theorem: [Haiman 1989]
The number of reduced words (in the alphabet of Coxeter generators) of the longest element $w_0 := [-1, -2, ..., -n]$ in B_n is equal to the number of SYT of square $n \times n$ shape.



Product and Determinantal Formulas

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Theorem: [Frobenius 1900, MacMahon 1909, Young 1927]

$$f^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{t} \ell_i!} \cdot \prod_{(i,j):\, i<j} (\ell_i - \ell_j).$$

Product and Determinantal Formulas

For a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$, let $\ell_i := \lambda_i + t - i$ $(1 \le i \le t)$.

Theorem: [Frobenius 1900, MacMahon 1909, Young 1927]

$$f^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{t} \ell_i!} \cdot \prod_{(i,j):\, i<j} (\ell_i - \ell_j).$$

Theorem (Determinantal Formula)

$$f^\lambda = |\lambda|! \cdot \det \left[\frac{1}{(\lambda_i - i + j)!} \right]_{i,j=1}^{t},$$

using the convention $1/k! := 0$ for negative integers k.

Hook Length Formula

The hook length of a cell $c = (i, j)$ in a diagram of shape λ is

$$h_c := \lambda_i + \lambda'_j - i - j + 1.$$

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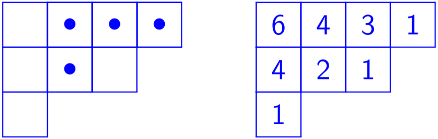

hook of $c = (1, 2)$ hook lengths

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	•	•	•
	•		

hook of $c = (1, 2)$

6	4	3	1
4	2	1	
1			

hook lengths

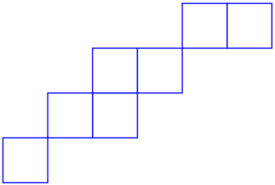
Theorem: [Frame-Robinson-Thrall, 1954]

$$f^\lambda = \frac{|\lambda|!}{\prod_{c \in [\lambda]} h_c}.$$

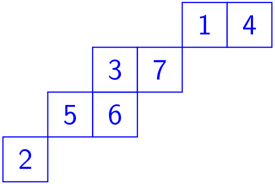
Still Classical

Skew Shapes

If λ and μ are partitions such that $[\mu] \subseteq [\lambda]$, namely $\mu_i \leq \lambda_i$ $(\forall i)$, then the skew diagram of shape λ/μ is the set difference $[\lambda/\mu] := [\lambda] \setminus [\mu]$ of the two ordinary shapes.



$= \quad [(6,4,3,1)/(4,2,1)]$



$\in \quad \mathrm{SYT}((6,4,3,1)/(4,2,1)).$

Skew Shapes and Representations

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$$\lambda/\mu \longrightarrow \chi^{\lambda/\mu}$$

skew shape of size n (reducible) character of S_n

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Skew Shapes and Representations

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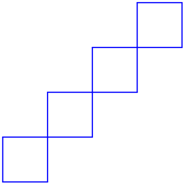
SYT(λ/μ)	$\longleftrightarrow$	basis of representation space
$f^{\lambda/\mu}$	$=$	$\chi^{\lambda/\mu}(id)$

For example,



$\longleftrightarrow$ the regular character $\chi^{\text{reg}}(g) = |G|\delta_{g,id}$ $(G = S_4)$

Skew Determinantal Formula

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions such that $\mu_i \le \lambda_i$ $(\forall i)$.

Theorem [Aitken 1943, Feit 1953]

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{t},$$

with the conventions $\mu_j := 0$ for $j > s$ and $1/k! := 0$ for negative integers k.

Unfortunately, no product or hook length formula is known for general skew shapes.

Shifted Shapes

A partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is strict if the part sizes λ_i are strictly decreasing: $\lambda_1 > \ldots > \lambda_t > 0$.

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The shifted diagram of shape λ is the set

$$D = [\lambda^*] := \{(i,j) \,|\, 1 \le i \le t,\ i \le j \le \lambda_i + i - 1\}.$$

Note that $(\lambda_i + i - 1)_{i=1}^{t}$ are weakly decreasing.

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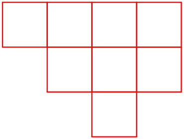
$$\lambda = (4,3,1) \implies [\lambda^*] =$$



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$$\lambda = (4,3,1) \implies [\lambda^*] = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \multicolumn{1}{c|}{} & & & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 8 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \in \mathrm{SYT}((4,3,1)^*).$$

Shifted Shapes and Representations

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Strict partitions λ of n essentially correspond to irreducible projective characters of S_n.

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$$g^{\lambda} := \#\,\mathrm{SYT}(\lambda^*)$$

Corollary:

$$\sum_{\lambda \models n} 2^{n-t}(g^{\lambda})^2 = n!$$

Shifted Formulas

Like ordinary shapes, the number g^λ of SYT of shifted shape λ has three types of formulas – product, hook length and determinantal.

Theorem [Schur 1911, Thrall 1952]

$$g^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{t} \lambda_i!} \cdot \prod_{(i,j):\, i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

Theorem

$$g^\lambda = \frac{|\lambda|!}{\prod_{c \in [\lambda^*]} h_c^*}$$

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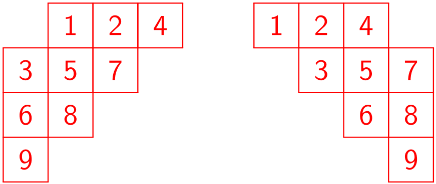
Non-Classical

Truncated Shapes

	1	2	4
3	5	7	
6	8		
9			

1	2	4	
	3	5	7
		6	8
			9

Truncated Shapes



classical non-classical

Truncated Shapes

	1	2	4
3	5	7	
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classical
skew

1	2	4	
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non-classical
shifted, truncated

Truncated Shapes

$$\begin{array}{cccc} & 1 & 2 & 4 \\ 3 & 5 & 7 & \\ 6 & 8 & & \\ 9 & & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{array}$$

classical skew	non-classical shifted, truncated
$\#\,\mathrm{SYT} = 768$	$\#\,\mathrm{SYT} = 4$

Truncated Shifted Staircase

The number of SYT whose shape is a shifted staircase with a truncated corner came up in a combinatorial setting, counting the number of shortest paths between antipodes in a certain graph of triangulations.

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$$\begin{aligned} \lambda &= (9,9,8,7,6,5,4,3,2,1) \\ N &= 54 \text{ (size)} \end{aligned}$$

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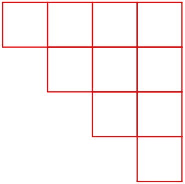
$$\begin{aligned} \lambda &= (9,9,8,7,6,5,4,3,2,1) \\ N &= 54 \text{ (size)} \\ \\ g^{\lambda} &= 116528733315142075200 \\ &= 2^6 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \end{aligned}$$

The largest prime factor is $< N$!!!

Shifted Staircase

Let $\delta_n := (n, n-1, \ldots, 1)$, a strict partition (shifted staircase shape).



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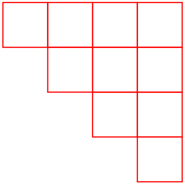


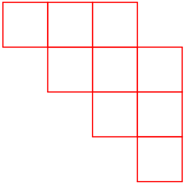
Corollary: (of Schur's product formula for shifted shapes)
The number of SYT of shifted staircase shape δ_n is

$$g^{\delta_n} = N! \cdot \prod_{i=0}^{n-1} \frac{i!}{(2i+1)!},$$

where $N := |\delta_n| = \binom{n+1}{2}$.

Truncated Shifted Staircase

The following enumeration problem was actually the original motivation for the study of truncated shapes, because of its combinatorial interpretation.



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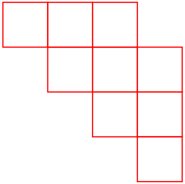


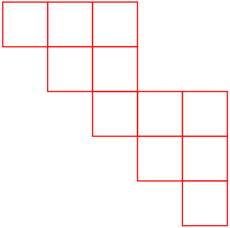
Theorem: [A-King-Roichman, Panova] The number of SYT of truncated shifted staircase shape $\delta_n \setminus (1)$ is equal to

$$g^{\delta_n} \frac{C_n C_{n-2}}{2\, C_{2n-3}},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Truncated Shifted Staircase

More generally, truncating a square from a shifted staircase shape:

$[\delta_5 \setminus (2^2)] =$ 

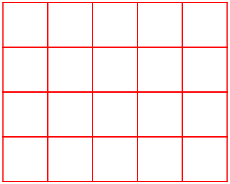
Theorem: [AKR] The number of SYT of truncated shifted staircase shape $\delta_{m+2k} \setminus ((k-1)^{k-1})$ is

$$g^{(m+k+1,\dots,m+3,m+1,\dots,1)} g^{(m+k+1,\dots,m+3,m+1)} \cdot \frac{N!M!}{(N-M-1)!(2M+1)!},$$

where $N = \binom{m+2k+1}{2} - (k-1)^2$ is the size of the shape and $M = k(2m+k+3)/2 - 1$.

Similarly for truncating "almost squares" $(k^{k-1}, k-1)$.

Rectangle

$[(5^4)] =$ 

Observation:

The number of SYT of rectangular shape (n^m) is

$$f^{(n^m)} = (mn)! \cdot \frac{F_m F_n}{F_{m+n}},$$

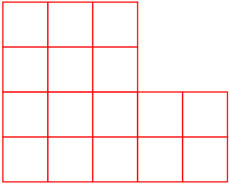
where

$$F_m := \prod_{i=0}^{m-1} i!.$$

Truncated Rectangle

Truncate a square from the NE corner of a rectangle:

$$[(5^4) \setminus (2^2)] =$$



Theorem: [AKR]
The number of SYT of truncated rectangular shape $((n+k-1)^{m+k-1}) \setminus ((k-1)^{k-1})$ (and size N) is

$$\frac{N!(mk-1)!(nk-1)!(m+n-1)!k}{(mk+nk-1)!} \cdot \frac{F_{m-1}F_{n-1}F_{k-1}}{F_{m+n+k-1}}.$$

Similar results were obtained for truncation by almost squares.

Truncated Rectangle

Not much is known for truncation of rectangles by rectangles. The following formula was conjectured by AKR and proved by Sun.

Theorem: [Sun]
For $n \geq 2$

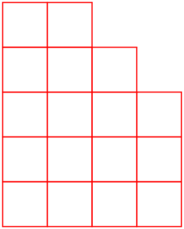
$$f^{(n^n)\setminus(2)} = \frac{(n^2-2)!(3n-4)!^2 \cdot 6}{(6n-8)!(2n-2)!(n-2)!^2} \cdot \frac{F_{n-2}^2}{F_{2n-4}}.$$

Theorem: [Snow]
For $n \geq 2$ and $k \geq 0$

$$f^{(n^{k+1})\setminus(n-2)} = \frac{(kn-k)!(kn+n)!}{(kn+n-k)!} \cdot \frac{F_k F_n}{F_{n+k}}.$$

Truncated Rectangle

Truncate a rectangle by a (shifted) staircase.

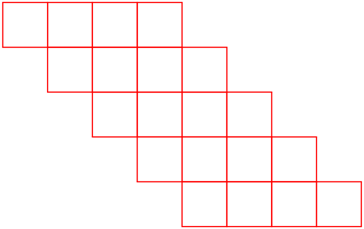
$[(4^5) \setminus \delta_2] =$ 

Theorem: [Panova]
Let $m \geq n \geq k$ be positive integers. The number of SYT of truncated shape $(n^m) \setminus \delta_k$ is

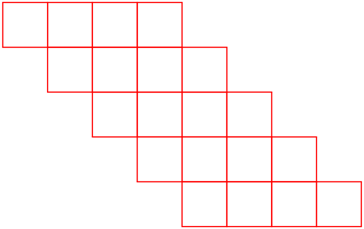
$$\binom{N}{m(n-k-1)} f^{(n-k-1)^m} g^{(m,m-1,\ldots,m-k)} \frac{E(k+1,m,n-k-1)}{E(k+1,m,0)},$$

where $N = mn - \binom{k+1}{2}$ is the size of the shape and $E(r,p,s) = \ldots$.

Shifted Strip



Shifted Strip



Theorem: [Sun]

The number of SYT of truncated shifted shape with n rows and 4 cells in each row is the $(2n-1)$-st Pell number

$$\frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1}\right).$$

Open Problems

- Which non-classical shapes have nice/product formulas?
- A modified hook length formula?
- A representation theoretical interpretation?

Grazie per l'attenzione !