Cyclic permutations, shuffles, and quasi-symmetric functions

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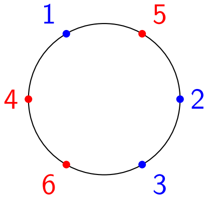
Bar-Ilan University

Algebraic Combinatorics Online Workshop
April 21, 2020



(This talk is being recorded)

Based on joint work with

Ira Gessel (Brandeis)
Vic Reiner (Minnesota)
Yuval Roichman (Bar-Ilan)

Special thanks to Darij Grinberg (Drexel)

Outline

Permutations, shuffles, and descents

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- $A =$ a finite set of size a (alphabet)

$$\begin{aligned} S_A &:= \text{the set of all permutations of } A \\ &= \text{bijections } u : [a] \to A \quad \text{(bijective words)} \end{aligned}$$

Example: $A = \{1, 3, 5, 7, 8\}$, $\qquad u = 51783 \in S_A$

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- $A, B =$ disjoint finite sets; $u \in S_A, v \in S_B$

$$u \sqcup\!\sqcup\, v := \text{ the set of all shuffles of } u \text{ and } v$$

Example:
$A = \{1, 2, 3, 5\}, B = \{4, 6, 7\}, u = 1235 \in S_A, v = 764 \in S_B$

$$1723654 \in u \sqcup\!\sqcup\, v$$

Permutations, shuffles, and descents

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  The descent set of $u \in S_A$ is

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Question:

What is the distribution of $\mathrm{des}(w)$ for $w \in u \sqcup\!\sqcup\, v$?

Permutations, shuffles, and descents

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In particular, what are the smallest and largest values of $\operatorname{des}(w)$?

Example: $u = 1432, v = 65; (a, b, i, j) = (4, 2, 2, 1)$

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$$\begin{aligned} u \sqcup\!\sqcup\, v = \{ & 1\underset{\wedge}{4}\underset{\wedge}{3}2\underset{\wedge}{6}5,\ 1\underset{\wedge}{4}3\underset{\wedge}{6}25,\ 14\underset{\wedge}{6}\underset{\wedge}{3}25,\ 1\underset{\wedge}{6}\underset{\wedge}{4}\underset{\wedge}{3}25,\ \underset{\wedge}{6}1\underset{\wedge}{4}\underset{\wedge}{3}25, \\ & 1\underset{\wedge}{4}3\underset{\wedge}{6}\underset{\wedge}{5}2,\ 14\underset{\wedge}{6}3\underset{\wedge}{5}2,\ 1\underset{\wedge}{6}\underset{\wedge}{4}3\underset{\wedge}{5}2,\ \underset{\wedge}{6}1\underset{\wedge}{4}3\underset{\wedge}{5}2,\ 14\underset{\wedge}{6}\underset{\wedge}{5}\underset{\wedge}{3}2, \\ & 1\underset{\wedge}{6}4\underset{\wedge}{5}\underset{\wedge}{3}2,\ \underset{\wedge}{6}14\underset{\wedge}{5}\underset{\wedge}{3}2,\ 1\underset{\wedge}{6}\underset{\wedge}{5}\underset{\wedge}{4}\underset{\wedge}{3}2,\ \underset{\wedge}{6}1\underset{\wedge}{5}\underset{\wedge}{4}\underset{\wedge}{3}2,\ \underset{\wedge}{6}\underset{\wedge}{5}1\underset{\wedge}{4}\underset{\wedge}{3}2 \} \end{aligned}$$

$$\sum_{w \in u \sqcup\!\sqcup\, v} q^{\operatorname{des}(w)} = 3q^2 + 9q^3 + 3q^4$$

Permutations, shuffles, and descents

Theorem (Stanley '72; Goulden '85, Stadler '99)

If $|A| = a$, $|B| = b$, $A \cap B = \varnothing$,
$u \in S_A$, $\mathrm{des}(u) = i$, $v \in S_B$, $\mathrm{des}(v) = j$ *then*

$$\#\{w \in u \sqcup\!\sqcup\, v \,:\, \mathrm{des}(w) = k\} = \binom{a+j-i}{k-i}\binom{b+i-j}{k-j}$$

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Remarks:

- Does not depend on u and v (only on $\mathrm{des}(u)$ and $\mathrm{des}(v)$).
- Does not depend on the relative order of A and B.

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- Does not depend on u and v (only on $\mathrm{des}(u)$ and $\mathrm{des}(v)$).
- Does not depend on the relative order of A and B.
- Actually holds on the level of descent sets.
- Follows from multiplication of quasi-symmetric functions.

Permutations, shuffles, and descents

Motivating Question:

What is the cyclic analogue?

Cyclic permutations, shuffles, and descents

Cyclic permutations, shuffles, and descents

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  Example: $u = 1234$, $v = 56789$

  $$w = 734819562 \in u \sqcup\!\sqcup_c v$$

Cyclic permutations, shuffles, and descents

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where $u(a+1) := u(1)$.

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  Example: $u = 2\underset{\wedge}{4}15\underset{\wedge}{6}\underset{\wedge}{3} \in S_{[6]}$

  $$\mathrm{Des}(u) = \{2, 5\}, \quad \mathrm{cDes}(u) = \{2, 5, 6\}$$

  Example: $v = 3\underset{\wedge}{4}15\underset{\wedge}{6}2 \in S_{[6]}$

  $$\mathrm{cDes}(v) = \mathrm{Des}(v) = \{2, 5\}$$

Cyclic permutations, shuffles, and descents

Remarks:

- $\mathrm{cdes}(u)$ is invariant under cyclic shifts of u. Thus $\mathrm{cdes}([u])$ is well defined.
- Similarly, the cyclic shuffle $[u] \sqcup\!\sqcup_c [v]$ is well defined, and is cyclically invariant.

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Theorem (AGRR)

If $|A| = a$, $|B| = b$, $A \cap B = \varnothing$,
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$$\#\{[w] \in [u] \sqcup\!\sqcup_c [v] \ : \ \mathrm{cdes}([w]) = k\} = ?$$

Cyclic quasi-symmetric functions

Symmetric and quasi-symmetric functions

- A symmetric function is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two sequences $(i_1, \ldots, i_t)$ and $(i'_1, \ldots, i'_t)$ of distinct positive integers (indices), and any sequence $(m_1, \ldots, m_t)$ of positive integers (exponents), the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in f are equal.
- A quasi-symmetric function is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \ldots < i_t$ and $i'_1 < \ldots < i'_t$ of positive integers, and any sequence $(m_1, \ldots, m_t)$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in f are equal.

Cyclic quasi-symmetric functions

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Example:

$$x_1^4 x_2^2 x_3^5 + \ldots \in \text{QSym}$$

$$x_1^4 x_2^2 x_3^5 + x_1^2 x_2^5 x_3^4 + x_1^5 x_2^4 x_3^2 + \ldots \in \text{cQSym}$$

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Similar features

- Sym, QSym, and cQSym are graded rings,

$$\text{Sym} \subseteq \text{cQSym} \subseteq \text{QSym}$$

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- The n-th graded piece has a basis indexed by simple combinatorial objects:

$$\begin{aligned} \mathrm{Sym}_n: &\quad \{s_\lambda \,:\, \lambda \vdash n\} \quad \text{Schur functions} \\ \mathrm{QSym}_n: &\quad \{F_{n,J} \,:\, J \subseteq [n-1]\} \quad \text{Fundamental QSF} \\ \mathrm{cQSym}_n: &\quad \{\widehat{F}^c_{n,[J]} \,:\, \varnothing \neq J \subseteq [n] \text{ up to cyclic shifts}\} \\ &\quad \text{Normalized fundamental CQSF} \end{aligned}$$

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- Dimension:

$$\begin{aligned} \dim \mathrm{Sym}_n &= p(n) \sim c^{\sqrt{n}} \quad \text{(partitions)} \\ \dim \mathrm{QSym}_n &= 2^{n-1} \quad \text{(compositions)} \\ \dim \mathrm{cQSym}_n &= \frac{1}{n}\sum_{d|n} \varphi(d) 2^{n/d} - 1 \sim \frac{1}{n} 2^n \end{aligned}$$

Similar features (cont.)

- The involution ω:

$$\begin{aligned} \mathrm{Sym}_n: &\quad s_\lambda \leftrightarrow s_{\lambda'} \\ \mathrm{QSym}_n: &\quad F_{n,J} \leftrightarrow F_{n,[n-1]\setminus J} \\ \mathrm{cQSym}_n: &\quad \widehat{F}^c_{n,[J]} \leftrightarrow \widehat{F}^c_{n,[[n]\setminus J]} \end{aligned}$$

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- Multiplication corresponds to (cyclic) shuffling: For $u \in S_A$, $v \in S_B$, $A \cap B = \varnothing$, $A \cup B = C$,

$$F_{|A|,\mathrm{cDes}(u)} \cdot F_{|B|,\mathrm{cDes}(v)} = \sum_{w \in u \sqcup\!\sqcup v} F_{|C|,\mathrm{cDes}(w)}$$

$$F^c_{|A|,[\mathrm{cDes}(u)]} \cdot F^c_{|B|,[\mathrm{cDes}(v)]} = \sum_{[w] \in [u] \sqcup\!\sqcup_c [v]} F^c_{|C|,[\mathrm{cDes}(w)]}$$

Similar features (cont.)

- $s_{\lambda/\mu}$ is a linear combination, with nonnegative integer coefficients, of the basis elements (for cQSym - only when λ/μ is not a connected ribbon!):

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SYT}(\lambda/\mu)} F_{n,\mathrm{Des}(T)} \quad \text{[Gessel '84]}$$
$$= \sum_{[J]} m^c([J]) \widehat{F}^c_{n,[J]}$$

This follows from the existence of cyclic descents for SYT (Rhoades ['10], A-Reiner-Roichman ['18], A-Elizalde-Roichman ['19], Huang ['20])

Differences

- The need for normalization: $\widehat{F}^{c}_{n,[J]} = \frac{1}{d_J} F^{c}_{n,J}$, where

$$d_J := |Stab_{\mathbb{Z}/n\mathbb{Z}}(J)| = \#\{i \in \mathbb{Z}/n\mathbb{Z} \,:\, J + i \equiv J \pmod{n}\}$$

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- Linear dependence:

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- "Non-Escher" property: clearly

$$\mathrm{cDes}(u) \neq \varnothing, [n] \qquad (\forall u \in S_n)$$

but we would like to include $\widehat{F}^c_{n,[\varnothing]} = h_n = s_{(n)}$ and $\widehat{F}^c_{n,[[n]]} = e_n = s_{(1^n)}$.

Other proof ingredients

An unusual ring homomorphism

- Define a new product on $\mathbb{Z}[[q]]$ by

$$q^i \odot q^j := q^{\max(i,j)},$$

with the usual addition, to get the ring $\mathbb{Z}[[q]]_\odot$.

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- Consider the ring of multivariate formal power series $\mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, x_2, \ldots]]$ (with the usual addition and multiplication), and its subring $\mathbb{Z}[[\mathbf{x}]]_{\mathrm{bd}}$ consisting of bounded-degree power series.

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- Define a ring homomorphism $\Psi : \mathbb{Z}[[\mathbf{x}]]_{\text{bd}} \to \mathbb{Z}[[q]]_\odot$ by

$$\Psi(x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}) := q^{i_k} \quad (k > 0,\ i_1 < \ldots < i_k,\ m_1, \ldots, m_k > 0)$$

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  and $\Psi(1) := 1$.
- $$\Psi(F_{n,J}) = \frac{q^{|J|+1}}{(1-q)^n} \qquad (J \subseteq [n-1])$$

A triple binomial identity

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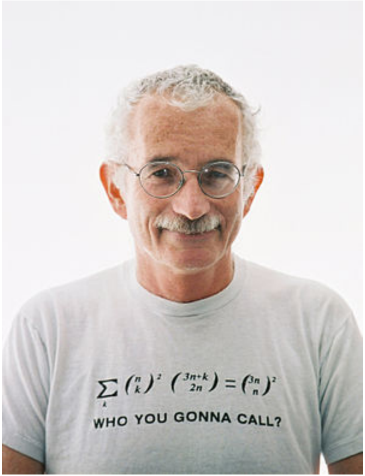
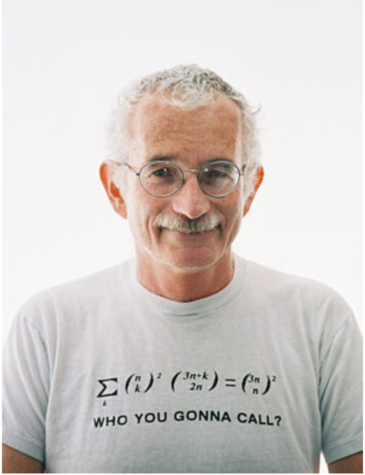
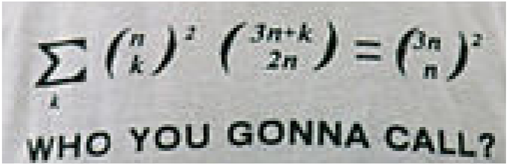


A triple binomial identity



Doron Zeilberger

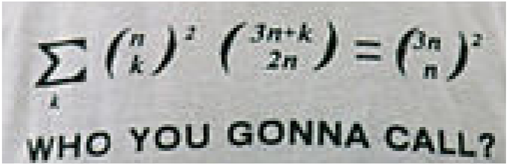
A triple binomial identity



$$\sum_k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$$

WHO YOU GONNA CALL?

A triple binomial identity



$$\sum_k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$$

WHO YOU GONNA CALL?

This is a special case of the triple-binomial identity

$$\sum_k \binom{m-x+y}{k}\binom{n-y+x}{n-k}\binom{x+k}{m+n} = \binom{x}{m}\binom{y}{n}$$

which is equivalent to the hypergeometric identity

$${}_3F_2\left(\left.\begin{matrix} a, b, -n \\ c, a+b-c-n+1 \end{matrix}\right| 1\right) = \frac{(c-a)^{\bar{n}}(c-b)^{\bar{n}}}{c^{\bar{n}}(c-a-b)^{\bar{n}}}$$

due to Pfaff (1797) and Saalschütz (1890). We use the general case.

… and the answer is:

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Theorem (AGRR)

If $|A| = a$, $|B| = b$ *with* $A \cap B = \varnothing$, *and* $u \in S_A$, $v \in S_B$ *with* $\mathrm{cdes}([u]) = i$, $\mathrm{cdes}([v]) = j$, *then the number of* $[w] \in [u] \sqcup_c [v]$ *with* $\mathrm{cdes}([w]) = k$ *is*

$$k\binom{a+j-i-1}{k-i}\binom{b+i-j-1}{k-j}+ (a+b-k)\binom{a+j-i-1}{k-i-1}\binom{b+i-j-1}{k-j-1}$$

$$= \frac{k(a-i)(b-j)+(a+b-k)ij}{(a+j-i)(b+i-j)}\binom{a+j-i}{k-i}\binom{b+i-j}{k-j}.$$

Summary

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- The ring cQSym of cyclic quasi-symmetric functions is intermediate between Sym and QSym.
- It has many properties in common with QSym, but also some interesting unique features.
- It has applications to combinatorial enumeration (and to other areas).

Thank You!