RESOLUTIONS OF ASSOCIATIVE AND LIE ALGEBRAS

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Abstract. Certain canonical resolutions are described for free associative and free Lie algebras in the category of non-associative algebras. These resolutions derive in both cases from geometric objects, which in turn reflect the combinatorics of suitable collections of leaf-labeled trees.

1. Introduction

We here describe certain explicit canonical resolutions for free associative and free (graded) Lie algebras, in the category of non-associative algebras. Both resolutions are based on the combinatorics of suitable collections of leaf-labeled trees.

The Lie case was needed for the second author's description of higher homotopy operations in rational homotopy theory, in [B2]: it turns out that in order to describe all such higher operations, one must resolve the differential graded Lie algebra L_* over $\mathbb{Q}$ (representing the rational homotopy type of a given space X) simplicially, by suitable free (differential) graded Lie algebras. The higher homotopy operations correspond to relations and syzygies for these free graded Lie algebras, thought of as non-associative algebras over $\mathbb{Q}$. Since we must replace all the Lie algebras by the corresponding free differential algebras in a functorial manner (to preserve the simplicial structure of the original resolution of L_*), we need canonical resolutions of free Lie algebras in the category of non-associative algebras, as described in this paper. The construction is closely related to "strongly homotopy Lie algebras" (see §3.21 below).

Our main interest is indeed in the Lie case. The associative case, which is based on work of Stasheff in [Sts], is included mainly as a preliminary illustration of the ideas involved, and to fix notation.

As might be expected, the resolutions, being canonical, are far from minimal: this is reflected in the fact that the resolution for the free Lie algebra $L = \mathbb{L}\langle x_1, \dots, x_g\rangle$ has generators in all dimensions $\leq g$, while if L is considered as a non-associative (skew-commutative) algebra, its homology vanishes above dimension 1, by Theorem 4.6 below (so that the generators for a minimal resolution are restricted to dimensions ≤ 1). Nevertheless, such canonical resolutions are often needed for functorial constructions (as noted above), and we hope the combinatorics involved may be of independent interest.

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1.1. **Notation and conventions.** A graded object over any category $\mathcal{C}$ is a sequence of objects $X_* = (X_0, X_1, \dots)$ from $\mathcal{C}$; we write $|x| = n$ if $x \in X_n$.

All vector spaces and algebras will be over a field $\mathbf{k}$ of characteristic 0 (though the application we have in mind is to the case of $\mathbf{k} = \mathbb{Q}$, the rationals). The (graded) vector space with the (graded) set X as its basis is denoted by $\mathbb{V}\langle X\rangle$, and the vector space dual of V is $V^\star := Hom_{\mathbf{k}}(V, \mathbf{k})$.

We denote by $\mathcal{A}lg$ the category of not-necessarily-associative algebras over $\mathbf{k}$, by $\mathcal{A}lg_a \subset \mathcal{A}lg$ the full subcategory of associative algebras, and by $\mathcal{A}lg_c \subset \mathcal{A}lg$ the full subcategory of skew-commutative not-necessarily-associative algebras, satisfying $xy = -yx$ for all x, y. $\mathcal{L}ie$ denotes the category of Lie algebras.

Similarly, we denote by $\mathcal{A}$ the category of graded not-necessarily-associative algebras, which we shall call GNAs. An object $A_* \in \mathcal{A}$ is thus a graded vector space $A_* = \oplus_{n=0}^{\infty} A_n$, equipped with a bilinear graded product $\cdot : A_p \otimes A_q \to A_{p+q}$ for each $p, q \geq 0$.

We denote by $\mathcal{A}_a \subset \mathcal{A}$ the full subcategory of graded associative algebras (GAAs), and by $\mathcal{A}_c \subset \mathcal{A}$ the full subcategory of graded-skew-commutative not-necessarily-associative algebras, satisfying $y \cdot x = (-1)^{|x||y|+1} x \cdot y$, which we call GCAs. $\mathcal{L} \subset \mathcal{A}_c$ denotes the subcategory of graded Lie algebras (GLAs); the product $[\,,\,] : L_p \otimes L_q \to L_{p+q}$ in a GLA L_* satisfies the (graded) Jacobi identity

$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0. \tag{1.2}$$

Note that we can embed $\mathcal{A}lg$ in $\mathcal{A}$ by thinking of $A \in \mathcal{A}lg$ as a graded algebra A_* with $A_0 = A$ and $A_i = \{0\}$ for $i \geq 1$; similarly for $\mathcal{A}lg_a \subset \mathcal{A}_a$, and so on. Thus results stated for graded algebras of various sorts include the ungraded versions as a special case.

There are also *differential* versions of all the above categories of graded algebras. In particular, a differential graded (not-necessarily-associative) skew-commutative algebra, called a DGCA, is a GCA $(A_*, \cdot) \in \mathcal{A}_c$, equipped with a differential (i.e., a map $\partial = \partial_n^A : A_n \to A_{n-1}$ for each $n > 0$ such that $\partial^2 = 0$) which is a graded derivation in the sense that if $x \in A_p$, $y \in A_q$ then $\partial(x \cdot y) = \partial(x) \cdot y + (-1)^p x \cdot \partial(y)$. The category of DGCAs is denoted by $d\mathcal{A}_c$. Similarly for differential graded not-necessarily-associative algebras, or DGNAs.

1.3. Notation. For any GNA $(A_*, \cdot) \in \mathcal{A}$, let $[x, y]$ denote $\frac{1}{2}(x \cdot y + (-1)^{|x||y|+1} y \cdot x)$. We then have $[y, x] = (-1)^{|x||y|+1}[x, y]$, so $(A_*, [\,,\,])$ is now a (non-associative) graded algebra with a graded-skew-commutative multiplication.

1.4. Definition. A *differential bigraded* (not-necessarily-associative) *skew-commutative algebra*, or DBGCA, is a bigraded vector space $A_{*,*} = \oplus_{p=0}^{\infty} \oplus_{s=0}^{\infty} A_{p,s}$, equipped with a bilinear graded product $\cdot : A_{p,s} \otimes A_{q,t} \to A_{p+q,s+t}$ for each $p, q, s, t \geq 0$ and a differential $\partial = \partial_{p,s}^A : A_{p,s} \to A_{p-1,s}$ satisfying $x \cdot y = (-1)^{(p+s)(q+t)+1} y \cdot x$ and $\partial(x \cdot y) = \partial(x) \cdot y + (-1)^{p+s} x \cdot \partial(y)$ for $x \in A_{p,s}$ and $y \in A_{q,t}$. The category of such DBGCAs will be denoted by $db\mathcal{A}_c$.

Each DBGCA $(A_{*,*}, \partial^A)$ has an *associated DGCA* (A_*, ∂^A), defined $A_n = \bigoplus_{p+q=n} A_{p,q}$ (same ∂^A); some authors re-index $A_{*,*}$ so that $\hat{A}_{p,s} = A_{p,p+s}$, and then A_* is obtained from $\hat{A}_{*,*}$ by ignoring the first (homological) grading. $n = p+s$ is called the *total* degree in $A_{*,*}$.

1.5. **Organization.** In section 2 we describe the simpler case of resolutions of free associative algebras, and in section 3 we describe resolutions of free Lie algebras. In section 4 we explain the connection to the homology of non-associative algebras.

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2. Associative algebras

We begin with a description of our canonical resolution for a free associative algebra by free non-associative algebras. We do so mainly because the underlying combinatorics, as well as the corresponding geometric objects, are more transparent in this case than for Lie algebras. For simplicity we deal here only with the non-graded case. First, some definitions. Fix once and for all a finite set $X = \{x_1, \dots, x_g\}$ (which we think of as a set of generators for a free algebra).

2.1. **Trees.** Recall that a *rooted plane tree* T (see [Stn]) consists of a (non-empty) finite set of nodes, with one designated node called the *root* $r(T)$; each node v has a linearly-ordered set of k_v other nodes, called its *children*; v is called their *parent*. If $k_v = 0$, then v is called a *leaf*; otherwise it is called an *internal node* of T, and the set of all internal nodes is denoted by $\mathrm{int}(T)$. The set of all leaves of T has the obvious natural linear order "from left to right". In this paper we require that $k_v \neq 1$ for all nodes v, i.e., all internal nodes have at least two children.

Note that the smallest rooted plane tree has a single node which is both the root and a leaf; in all other trees the root is an internal node.

2.2. Definition. Let $\mathcal{I}_n = \{1, 2, \dots, g\}^n$. For $I = (i_1, \dots, i_n) \in \mathcal{I}_n$, let $\mathcal{T}[I]$ denote the collection of all rooted plane trees with n leaves labeled $x_{i_1}, x_{i_2}, \dots, x_{i_n}$, in that order. Write $\mathcal{T}^n$ for $\bigcup_{I \in \mathcal{I}_n} \mathcal{T}[I]$, and $\mathcal{T} := \bigcup_{n=1}^{\infty} \mathcal{T}^n$.

Defining the *excess* of an internal node v of T to be $e(v) := k_v - 2$, the total excess of each tree determines a lower grading on $\mathcal{T}$ by

$$T \in \mathcal{T}^n_k \;\Leftrightarrow\; k = \sum_{v \in \mathrm{int}(T)} e(v). \tag{2.3}$$

Thus $\mathcal{T}^n_0$ consists precisely of the binary trees, for which every internal node has exactly two children; such trees correspond to complete parenthesizations on (the labels of) the leaves, e.g.: $(x_1((x_3x_2)x_1))$. More generally, trees in $\mathcal{T}^n_k$ $(0 \leq k \leq n-2)$ correspond to partial parenthesizations with $n-k-1$ pairs of parentheses (including an external pair, when $n \geq 2$) – e.g., $(x_1(x_3x_2)x_1) \in \mathcal{T}^4_1$.

2.4. **Associahedra.** Consider the $(n-2)$-dimensional associahedron $\mathbf{K}_{n-2}$ of [Sts, §§2,6], whose vertices are indexed by the possible "associations" (i.e., full parenthesizations) on n letters: it has a realization as a convex polytope in $\mathbb{R}^{n-2}$, and its boundary $\partial\mathbf{K}_{n-2}$ is thus homeomorphic to the $(n-3)$-sphere $\mathbf{S}^{n-3}$ (cf. [Z, p. 18]).

The dual polytope is simplicial, so that its boundary complex $\mathbf{P}_n$ is an $(n-3)$-dimensional simplicial complex, in which the top-dimensional faces correspond to the vertices of $\mathbf{K}_{n-2}$, i.e., to binary trees. In general, the k-simplices of $\mathbf{P}_n$ are in one-to-one correspondence with the trees in $\mathcal{T}^n_{n-3-k}$. Note that the indexing is the reverse of the one we described above: the binary trees now appear in the *top* dimension.

By choosing various sequences of labels $I \in \mathcal{I}_n = X^n_*$ to serve as the "letters", we obtain isomorphic copies of $\mathbf{P}_n$, which we denote by $\mathbf{P}_n[I]$, with the corresponding rational simplicial chain complexes being $C[I]_* := C_*(\mathbf{P}_n[I];\mathbf{k})$; similarly $C[I]^* := Hom_{\mathbf{k}}(C[I]_*;\mathbf{k})$ are the simplicial cochain complexes.

2.5. Definition. We denote by $\mathbb{A}\mathrm{lg}\langle X\rangle$ the *free non-associative algebra* generated by the set X. This is just the non-associative tensor algebra on the vector space $\mathbb{V}\langle X\rangle$, so we may write $\mathbb{A}\mathrm{lg}\langle X\rangle = \bigoplus_{i=0}^{\infty} A_n(X)$, where $A_n(X) = \mathbb{V}\langle \mathcal{T}^n_0\rangle$ (cf. §1.1). The multiplication in $\mathbb{A}\mathrm{lg}\langle X\rangle$ is defined by concatenation: if $T \in \mathcal{T}^p_0$ and $T' \in \mathcal{T}^q_0$, then $T\cdot T' \in \mathcal{T}^{p+q}_0$ is obtained by adjoining a root $r(T\cdot T')$ as the common parent of $r(T)$ and $r(T')$, in that order. The free associative algebra on X, denoted by $\mathbb{A}\mathrm{lg}_a\langle X\rangle$, and the free *graded* non-associative algebra on a graded set X_*, denoted by $\mathbb{A}\mathrm{lg}\langle X_*\rangle$, are defined similarly.

2.6. Definition. Given an *associative* algebra $B \in \mathcal{A}lg_a$, we may think of it as an object in $\mathcal{A}lg$. As such, it cannot be free (even if $B = \mathbb{A}\mathrm{lg}_a\langle X\rangle$, say), so we can try to *resolve* it: that is, construct a DGNA $(E_*, \partial^E) \in d\mathcal{A}$ which is free as a GNA, together with an augmentation $\varepsilon : E_0 \to B$ such that the augmented chain complex $E_* \to B$ – called a $(d\mathcal{A}$-)*resolution* – is acyclic. Of course, the same can be done for any GNA B (e.g., if B is a Lie algebra).

A bigraded $db\mathcal{A}$-resolution $F_{*,*}$ of a *graded* algebra $A_* \in \mathcal{A}_a$ is defined analogously.

2.7. **Constructing the resolution.** Since $\partial\mathbf{K}_{n-2} \simeq \mathbf{S}^{n-3}$, we have $\tilde{H}^i C[I]^* = \mathbf{k}$ for $i = n-3$ and $\tilde{H}^i C[I]^* = 0$ for $i \neq n-3$. Let us re-index each $C^* = C[I]^*$ by setting $\hat{C}_i = C^{n-4-i}$ for $-1 \leq i \leq n-4$, and $\hat{C}_{n-3} = \mathbf{k}$, so $\hat{C}_* = \hat{C}[I]_*$ is an acyclic augmented chain complex. Note that $\hat{C}_{-1}$ is the free vector space on all full parenthesizations of $i_1, \ldots, i_n$. Thus if we set

$$E_* = \bigoplus_{n=1}^{\infty} \bigoplus_{I\in\mathcal{I}_n} \hat{C}[I]_*,$$

we have a $d\mathcal{A}$-resolution of $E_{-1} \cong B = \mathbb{A}\mathrm{lg}_a\langle X\rangle$. Moreover, E_* has the structure of a DGNA, with the product extended bilinearly from the concatenation of trees defined in §2.5.

3. Lie Algebras

We can now deal with the analogous resolution of a free graded Lie algebra, thought of as an object in $\mathcal{A}_c$. Let $\mathbb{L}\langle X_*\rangle \in \mathcal{L}$ denote the free graded Lie algebra generated by the *graded* set $X_* = \{x_1, \ldots, x_g\}$. Ideally, we would like a Lie analogue of the associahedron $\mathbf{K}_{n-2}$ (cf. §2.4): i.e., a (combinatorial) topological space which encodes the combinatorics of the resolution of $\mathbb{L}\langle X_*\rangle$. Apparently this does not exist, in general; however, there is a version of the dual simplicial complex $\mathbf{P}_n$ –

namely, Boardman's "space of fully-grown trees" (see [Bo, §6]). This can be thought of as an n-dimensional generalization of the "Lie-hedron" of [MS] – but only for Lie expressions *without* repetitions (see §3.6 below).

In this section we again use the notation of §2.2, but now we must pay greater attention to the grading on X_*, as well as to the resulting signs. This is because in the case of Lie algebras we must deal separately with expressions in which the same generator appears more than once.

3.1. Definition. Let $I = (i_1, \dots, i_n)$ be an n-tuple of *distinct* indices of elements in X_*, and $T \in \mathcal{T}[I]$ a rooted plane tree with leaves labeled $x_{i_1}, \dots, x_{i_n}$, in that order. For each node $v \in \mathrm{int}(T)$ the symmetric group Σ_{k_v} permutes the k_v children of v, changing T into a combinatorially isomorphic tree $T' \in \mathcal{T}[I']$ (where I' is the permutation of I), and the actions of the symmetric groups at different nodes commute; so we define the *branch automorphism group* of T to be $\mathcal{B}\text{-}\mathcal{A}ut(T) := \prod_{v \in \mathrm{int}(T)} \Sigma_{k_v}$. (The elements $\varphi \in \mathcal{B}\text{-}\mathcal{A}ut(T)$ are, strictly speaking, not automorphisms of T, but only of the collection $\langle T \rangle$ of all rooted plane trees combinatorially isomorphic to T.)

Equivalently, we may think of $\mathcal{B}\text{-}\mathcal{A}ut(T)$ as the subgroup of the symmetric group Σ_n consisting of all linear orderings of the leaf labels $x_{i_1}, \dots, x_{i_n}$ of T which are compatible with the tree structure of T.

If we identify a tree T with the corresponding partially parenthesized expression α in the letters $x_{i_1}, \dots, x_{i_n}$, then we may write $\mathcal{B}\text{-}\mathcal{A}ut(\alpha)$ for $\mathcal{B}\text{-}\mathcal{A}ut(T)$, and think of the group as permuting letters or parenthesized sub-blocks of α.

3.2. Definition. For T as above, define the *degree* $|v|$ of any node v of T inductively by setting the degree of a leaf labeled by $x \in X_*$ to be $|x|$ (as in §1.1), and if $v \in \mathrm{int}(T)$ has children $u_1, \dots, u_k$, let $|v| := |u_1| + \cdots + |u_k| + k - 2$. In particular, the *total degree* of T, denoted by $|T|$, is defined to be the degree of its root $r(T)$. Thus $\mathcal{T}[I]$ is bigraded (with the *homological degree* defined by (2.3)) and we write $T \in \mathcal{T}[I]_{k,s}$ if $|T| = k$ and T is in homological degree s. If all the generators in X_* have degree 0, the two degrees are the same.

Note that the action of $\mathcal{B}\text{-}\mathcal{A}ut(T)$ respects the degrees of the nodes, so we may define the *Koszul sign* $\varepsilon(\varphi)$ of a branch automorphism $\varphi \in \mathcal{B}\text{-}\mathcal{A}ut(T)$ to be the product of Koszul signs $\mathrm{sign}_{X_*}(\sigma)$, taken over all the constituent permutations $\sigma \in \Sigma_{k_v}$. (The Koszul sign of a permutation acting on a graded set X_* is defined by letting $\mathrm{sign}_{X_*}((k, k+1)) = (-1)^{pq+1}$, for an adjacent transposition $(k, k+1)$ which switches two elements (in our case: nodes) of degrees p, q respectively.)

3.3. *Remark.* The Koszul sign we use actually differs by -1 from that usually used by algebraists, so as to conform to the topological usage needed for our application in [B2].

3.4. Definition. For T as above, we define the *complexity* $\mathrm{cx}(v)$ of any node v inductively by setting $\mathrm{cx}(r(T)) = 0$, where $r(T)$ is the root of T, and if $v \in \mathrm{int}(T)$ has k children, then $\mathrm{cx}(u) = \mathrm{cx}(v) + k$ for each child u of v.

3.5. Definition. For each $T \in \mathcal{T}[I] \subset \mathcal{T}^n_k$ as above, let $+\langle T \rangle \subset \mathcal{T}^n_k$ denote the collection of all trees T' obtainable from T under some $\varphi \in \mathcal{B}\text{-}\mathcal{A}ut(T)$ with $\epsilon(\varphi) = +1$, and similarly define $-\langle T \rangle$ (with $\epsilon(\varphi) = -1$). We think of $\pm\langle T \rangle$ as the equivalence class of the tree T, with respect to the relation of abstract combinatorial isomorphism, partitioned by sign into two subclasses.

Write $\hat{\mathfrak{I}}_n$ for the collection of (unordered) n-multisets of elements of X_*, and set $\hat{\mathfrak{T}}[\hat{I}] := \bigcup_{I\in\hat{I}} \bigcup_{T\in\mathfrak{T}[I]} \pm\langle T\rangle$. We may think of $\hat{\mathfrak{T}}[\hat{I}]$ as the collection of all rooted trees $\hat{T}$ with n leaves labeled $x_{i_1}, x_{i_2}, \dots, x_{i_n}$, *without* a specified planar embedding, but with a sign determining which of the two classes of possible embeddings we have chosen. Set $\hat{\mathfrak{T}}^n := \bigcup_{\hat{I}\in\hat{\mathfrak{I}}_n} \hat{\mathfrak{T}}[\hat{I}]$.

3.6. Definition. The *space of trees* is the n-dimensional simplicial complex whose k-simplices consist, in our notation, of the unsigned equivalence classes $\langle T\rangle = +\langle T\rangle \cup -\langle T\rangle$ of rooted trees $T \in \mathfrak{T}^{n+3}_{n-k}[\hat{I}]$, for some fixed set $\hat{I} \in \hat{\mathfrak{I}}_{n+3}$ of $n+3$ *distinct* labels. It is denoted by T_n in [RW, §1], but to avoid over-use of the letter T we shall denote it here by $\mathbf{M}^n$. See also [HW].

Thus, the k-simplices $\mathbf{M}^n_k$ are in one-to-one correspondence with the isomorphism classes of leaf-labeled trees – without a specified planar embedding – having exactly $k+2$ internal vertices, and with leaves labeled $x_{i_1}, \dots, x_{i_{n+3}}$, say, not necessarily in that order. As Robinson and Whitehouse show, $\mathbf{M}^n$ is homotopy equivalent to a wedge of $(n+2)!$ n-spheres (cf. [RW, Thm. 1.5]). However, we cannot use their results as they stand, since we need to be careful with signs. So we make the following definitions:

3.7. Definition. For $I = (i_1, \dots, i_n)$ as above, let $J_* = J[I]_*$ denote the (bi)graded vector space with $J[I]_k$ spanned by $\hat{\mathfrak{T}}[I]_{*,n-3-k}$ for $-1 \le k \le n-2$. (For simplicity we suppress the "topological" grading due to the grading of X_*, since it is not relevant at this stage.) We define a differential $\partial = \partial^J_k : J_k \to J_{k-1}$ as follows:

Represent any $\langle T\rangle \in \hat{\mathfrak{T}}[I]_{*,n-3-k}$ by a partially parenthesized expression α in the letters $x_{i_1}, \dots, x_{i_n}$; then $\partial[\hat{T}]$ will be represented by the sum of all expressions obtained from α by omitting a pair of parentheses (equivalently: by contracting one *internal edge* of T, i.e., an edge connecting two internal vertices) – with appropriate signs. These signs are determined recursively by the following three rules:

(1) If $\alpha = ((a_1a_2\dots a_k)b_1\dots b_m)$, where each a_i or b_j is a partially parenthesized expression (possibly just a generator $x \in X_*$), then the summand $(a_1a_2\dots a_kb_1\dots b_m)$ appears in the expansion of $\partial[\alpha]$ with the sign $(-1)^{m+1}$.
(2) If $\alpha = (ab_1\dots b_m)$, where a (and each b_j) is a partially parenthesized expression, then the sum comprising $(\partial[a]b_1\dots b_m)$ appears in the expansion of $\partial[\alpha]$ with the sign $(-1)^{m+1}$.
(3) If $\alpha = (a_1\dots a_kbc_1\dots c_m)$, where each of a_i, b and c_j is a partially parenthesized expression, then

$$\partial[\alpha] = (-1)^{(\sum_{i=1}^k |a_i|)|b|+k}\,\partial[(ba_1\dots a_kc_1\dots c_m)].$$

We set $\partial[x] := 0$ for any generator $x \in X_*$.

3.8. Example. For any partially parenthesized expressions a, b, c, d we have

$$\partial[((ab)c)] = (abc) + ((\partial[a]b)c) + (-1)^{|a|}((a\partial[b])c) + (-1)^{|a|+|b|}((ab)\partial[c])$$

and

$$\begin{aligned}\partial[(((ab)c)d)] =&((ab)cd) + ((abc)d) + (((\partial[a]b)c)d) + (-1)^{|a|}(((a\partial[b])c)d)\\ &+(-1)^{|a|+|b|}(((ab)\partial[c])d) + (-1)^{|a|+|b|+|c|}(((ab)c)\partial[d])\end{aligned}$$

3.9. *Note.* Rules (1) and (2) say that if $\alpha = ((\cdots((\bar{a})b^1_1 \dots b^1_{m_1})\cdots)b^t_1 \dots b^t_{m_t})$, where the expression $\bar{a} := a_1a_2 \dots a_k$ is initial in α (with only left parentheses preceding it), then the sequence $\alpha_{\bar{a}} = ((\cdots(a_1a_2 \dots a_k b^1_1 \dots b^1_{m_1})\cdots)b^t_1 \dots b^t_{m_t})$, obtained from α by omitting the outer parentheses around $\bar{a}$, appears in the expansion of $\partial[\alpha]$ with the sign $(-1)^{\mathrm{cx}(\bar{a})}$ (Def. 3.4).

Rule (3) says that if one wishes to omit the outer parentheses around $\bar{a}$ when it is not initial in α, and T is the rooted plane tree corresponding to α, then one must move $\bar{a}$ to the left of all its siblings, and similarly for all its ancestors, by a suitable branch automorphism $\varphi \in \mathcal{B}\text{-}\mathcal{A}ut(T)$ – which introduces the sign $\varepsilon(\varphi)$ – and then apply the previous rule. Note that this φ is not unique (unless we specify a preferred choice for the automorphism); but it is not hard to see that the various choices of φ differ by elements of the isotropy subgroup of $\mathcal{B}\text{-}\mathcal{A}ut(T)$ which leave the node corresponding to $\bar{a}$ fixed, and that the correspondence $\alpha \mapsto \alpha_{\bar{a}}$ commutes with the action of this subgroup. So the sign of the resulting class $\pm\langle\alpha_{\bar{a}}\rangle$ in the expansion of $\partial^J[\alpha]$ – and thus ∂^J itself – is well defined.

3.10. Lemma. *∂^J is a differential on J_*.*

Proof. Given a partially parenthesized expression $\alpha = ((\cdots(\bar{a}^1)\bar{a}^2)\dots\bar{a}^t)$, with each $\bar{a}^i = a^i_1a^i_2\dots a^i_{k_i}$, we must verify that, for any two pairs of parentheses in α, omitting them in the two possible orders yields opposite signs in the expansion of $\partial[\partial[\alpha]]$. Using the recursive rules of §3.7, one must check the following cases:

(i) $\alpha = ((\bar{a})(\bar{b})\bar{c})$, where $\bar{a} := a_1 \dots a_k$, $\bar{b} := b_1 \dots b_\ell$ and $\bar{c} := c_1 \dots c_m$. We see that if $|\bar{a}| := |a_1| + \cdots + |a_k|$ and so on, then the two orders of omitting parentheses yield

$$\begin{aligned}((\bar{a})(\bar{b})\bar{c}) \mapsto&(-1)^{2+m}(\bar{a}(\bar{b})\bar{c}) = (-1)^{(2+m)+|\bar{a}|(|\bar{b}|+\ell-2)+k}((\bar{b})\bar{a}\bar{c}) \mapsto\\ &(-1)^{(m+|\bar{a}|(|\bar{b}|+\ell)+k)+k+m+1}(\bar{b}\bar{a}\bar{c}) = (-1)^{(|\bar{a}|(|\bar{b}|+\ell)+1)+|\bar{a}||\bar{b}|+k\ell}(\bar{a}\bar{b}\bar{c}) =\\ &(-1)^{(|\bar{a}|+k)\ell+1}(\bar{a}\bar{b}\bar{c})\end{aligned}$$

and

$$\begin{aligned}((\bar{a})(\bar{b})\bar{c}) =&(-1)^{(|\bar{a}|+k)(|\bar{b}|+\ell)+1}((\bar{b})(\bar{a})\bar{c}) \mapsto (-1)^{((|\bar{a}|+k)(|\bar{b}|+\ell)+1)+2+m}(\bar{b}(\bar{a})\bar{c}) =\\ &(-1)^{(|\bar{a}|+k)\ell+\ell+1+m}((\bar{a})\bar{b}\bar{c}) \mapsto (-1)^{(|\bar{a}|+k)\ell}(\bar{a}\bar{b}\bar{c})\end{aligned}$$

respectively, which indeed differ in sign.

The remaining cases, namely:

(ii) $(((\bar{a})\bar{b})\bar{c}) \mapsto \pm(\bar{a}\bar{b}\bar{c},$
(iii) $(a(\bar{b})\bar{c}) \mapsto \pm(\partial[a]\bar{b}\bar{c})$,
(iv) $((a\bar{b})\bar{c}) \mapsto \pm(\partial[a]\bar{b}\bar{c})$,
(v) $(ab\bar{c}) \mapsto \pm(\partial[a]\partial[b]\bar{c})$,

are dealt with in a similar fashion. □

To show that $J[I]_*$ is acyclic except in the top dimension for each I, we mimic the geometric proof of Robinson and Whitehouse. This requires another

3.11. Definition. Given any subset $A = \{i_1, \dots, i_k\}$ of I with $k \geq 2$, let $\tilde{J}[A]_*$ denote the subcomplex of $J[I]_*$ spanned by all trees T in which $x_{i_1}, \dots, x_{i_k}$ all have the same parent node (i.e., in the corresponding expression α, the letters x_{i_j} are not separated by unbalanced pairs of parentheses). Since ∂^J is defined by omitting parentheses, this is clearly a subcomplex. Compare [RW, Def. 1.3], and [RW, Lemma 1.4] for the following

3.12. Lemma. *For any* $A \subseteq I$ *with* $|A| \geq 2$, *the complex* $\tilde{J}[A]_*$ *is acyclic.*

Proof. Write $\bar{x} := x_{i_1} \dots x_{i_k}$. We define a contracting homotopy $\delta = \delta^A_m : \tilde{J}[A]_m \to \tilde{J}[A]_{m+1}$ on basis elements $T \in \hat{\mathfrak{T}}[I]_{*,n-3-m}$ (or equivalently, on the corresponding partially parenthesized expression α), and extend linearly:

If T has a node whose leaves are precisely $x_{i_1}, \dots, x_{i_k}$ (i.e., if α has $(\bar{x})$ as a sub-expression), then $\delta(\alpha) := 0$; while if $\alpha = ((\cdots(\bar{x}\bar{a}^1)\cdots)\bar{a}^t)$ (where each $\bar{a}^i = a^i_1 a^i_2 \dots a^i_{k_i}$), we set $\delta(\alpha) := (-1)^{\mathrm{cx}(\bar{x})}((\cdots((\bar{x})\bar{a}^1)\cdots)\bar{a}^t)$. By requiring that $\delta(\varphi(\alpha)) = (-1)^{\varepsilon(\varphi)}\varphi(\delta(\alpha))$ for any $\varphi \in \mathcal{B}\text{-}\mathcal{A}ut(\alpha)$ (as long as both sides of the equation make sense), we have defined δ on all of $\tilde{J}[A]_*$.

Using the rules of §3.7 above, one may verify that δ is indeed a contracting homotopy for $\tilde{J}[A]_*$ (i.e., $\partial \circ \delta + \delta \circ \partial = id$). □

This implies the following variant of [RW, Thm. 1.5]:

3.13. Proposition. *For any n distinct indices* $I = (i_1, \dots, i_n)$ *we have* $H_i(J[I]_*) = 0$ *for* $-1 \leq i < n-3$, *and* $H_{n-3}(J[I]_*) \cong \mathbf{k}^{(n-1)!}$.

Proof. Let $C_* := \bigcup_{1 \leq k < \ell < n} \tilde{J}[(i_k, i_\ell)]_*$. (This is a subcomplex of $J[I]_*$.) The intersection $\bigcap_{t=1}^r \tilde{J}[(i_{k_t}, i_{\ell_t})]_*$ is acyclic (and nonempty) for any subcollection $(k_t, \ell_t)_{t=1}^r$ of pairs, as can be seen by using the contracting homotopy δ^A of Lemma 3.12, where the subset $A = A_r$ of $\{k_1, \ell_1, \dots, k_r, \ell_r\}$ is defined by induction on $1 \leq s \leq r$ by letting $A_1 := \{k_1, \ell_1\}$, $A_{s+1} = A_s$ if $A_s \cap \{k_{s+1}, \ell_{s+1}\} = \emptyset$, and $A_{s+1} = A_s \cup \{k_{s+1}, \ell_{s+1}\}$ otherwise. Lemma 3.12 and the Mayer-Vietoris sequence imply that C_* itself is acyclic, and $(n-3)$-dimensional.

Now any partially parenthesized expression $\alpha \in J[I]_*$ is actually in C_*, unless it is in fact *fully* parenthesized (corresponding to a binary tree, so in dimension $n-3$), and of the form $\alpha = ((\cdots((x_n x_{\sigma(1)})x_{\sigma(2)})\cdots)x_{\sigma(n-1)})$ for some $\sigma \in \Sigma_{n-1}$. Since $\partial[\alpha]$ is a cycle in C_{n-4}, there is some $\beta \in C_{n-3}$ (unique, for dimensional reasons) such that $\alpha - \beta$ is a cycle in $J[I]_*$. Thus $H_{n-3}(J[I]_*) \cong \mathbf{k}^{(n-1)!}$. □

3.14. **The leaf action.** Note that a multiset $\hat{I} = \{i_1, \dots, i_n\} \in \hat{\mathfrak{I}}_n$ (§3.5) may be thought of as the collection of orbits of ordered n-tuples $I \in \mathfrak{I}_n$ under the action of the symmetric group Σ_n. For simplicity we may take our set of labels to be simply $I = \mathbf{n} := (1, \dots, n)$, and think of Σ_n as acting on the leaves (i.e., on the labels) of any $T \in \mathfrak{T}[\mathbf{n}]_k$. This action, extended linearly to $J[\mathbf{n}]_*$, commutes with the differential and with the action of the branch automorphism groups, which therefore makes sense of the following

3.15. Definition. Any multiset $\hat{I} = (i_1 = \cdots = i_1; i_2 = \cdots = i_2; \ldots ; i_k = \cdots = i_k)$, with a total of $n = n_1 + n_2 + \cdots + n_k$ entries counted with repetitions, may be thought of as the orbit set of $\mathbf{n}$ under the action a *leaf automorphism group* $\mathcal{L}$-$\mathcal{A}ut(\hat{I}) = \Sigma_{n_1} \times \cdots \Sigma_{n_k} \subseteq \Sigma_n$. We define the corresponding chain complex $J[\hat{I}]_*$ to be the quotient of $J[\mathbf{n}]_*$ under the action of $\mathcal{L}$-$\mathcal{A}ut(\hat{I})$ – though this is no longer associated to a geometric object in the way that $J[\mathbf{n}]_*$ was associated (up to signs) to $\mathbf{M}^n$.

3.16. **The resolution** $F_{*,*}$**.** To produce our candidate for the $db\mathcal{A}_c$-resolution of $L = \mathbb{L}\langle X_*\rangle$, we must again re-index, as in §2.7, by setting $G[\hat{I}]_i := (J[\hat{I}]^{n-3-i})^\star$ (vector space dual) for $0 \leq i \leq n-3$, so that $\langle G[\mathbf{n}]_*, \partial^\star\rangle$ is (up to sign) the re-indexed cochain complex for $\mathbf{M}^n$. We then define

$$F_{*,*} := \bigoplus_{n=0}^{\infty} \bigoplus_{\hat{I}\in\hat{\mathfrak{I}}_{n+3}} G[\hat{I}]_*. \tag{3.17}$$

(We have re-inserted the "topological" grading into our notation at this stage, to call attention to the fact that we have constructed a *bigraded* resolution.)

3.18. *Remark.* Note that $G[\hat{I}]_*$ once more reverses the indexing, so that for I consisting of distinct indices, at least, $G[\hat{I}]_k$ is spanned by all trees of lower (homological) degree k, as defined in (2.3). Similarly, $\partial^\star(T)$, which we defined by the vector space dual of ∂^J, could be described directly as the signed sum of all trees obtained from T by *adding* internal edges – or equivalently, adding parentheses to the corresponding partially parenthesized expression α, with the signs again given by §3.7. This is in fact more natural algebraically, as the following examples show:

3.19. Example. For any partially parenthesized expressions a, b, and c in $F_{*,*}$, one has

$$\begin{aligned}\partial^\star[(abc)] =&((ab)c) + (-1)^{|a||b|+|a||c|}((bc)a) + (-1)^{|a||c|+|b||c|}((ca)b)\\ &+ ((\partial^\star[a]b)c) + (-1)^{|a|}((a\partial^\star[b])c) + (-1)^{|a|+|b|}((ab)\partial^\star[c])\end{aligned}$$

(compare Example 3.8). In particular, for any three generators $x, y, z \in X_*$ one has

$$\partial^\star[(xyz)] = ((xy)z) + (-1)^{|x||y|+|x||z|}((yz)x) + (-1)^{|x||z|+|y||z|}((zx)y),$$

which up to the action of $\mathcal{B}$-$\mathcal{A}ut(T)$ is the usual graded Jacobi identity of (1.2).

Similarly, for $x, y, z, w \in X_*$

(3.20)

$$\begin{aligned}\partial^F[(xyzw)] =& - ((xy)zw) + (-1)^{|y||z|}((xz)yw) + (-1)^{|y||w|+|z||w|+1}((xw)yz) +\\ &(-1)^{|x||y|+|x||z|+1}((yz)xw) + (-1)^{|x||y|+|x||w|+|z||w|+1}((yw)xz) +\\ &(-1)^{(|x|+|y|)(|z|+|w|)}((zw)xy) + ((xyz)w) + (-1)^{|z||w|+1}((xyw)z) +\\ &(-1)^{|y|(|z|+|w|)}((xzw)y)(-1)^{|x|(|y|+|z|+|w|+1)}((yzw)x),\end{aligned}$$

which can be thought of as a "second order Jacobi identity".

$F_{*,*}$ has an augmentation $\varepsilon : F_{*,0} \to \mathbb{L}\langle X_*\rangle$, which takes any fully parenthesized expression in the elements x_i to the corresponding iterated Lie bracket. In fact, with the product structure extended linearly from concatentation of trees, as in §2.5, $F_{*,*}$

is a DBGCA (see §1.4; or a DGCA, when X_* is ungraded). The product is graded-skew-commutative, and Rule (2) of §3.7 implies that $\partial^\star[a\cdot b]=\partial^\star[a]\cdot b+(-1)^{|a|}a\cdot\partial^\star[b]$ for any $a,b\in F_{*,*}$.

3.21. *Remark.* In fact, $F_{*,*}$ is not merely a free bigraded skew-commutative not-necessarily-associative algebra, but also is the free *strongly homotopy Lie algebra* on the graded set X_*. The analogous singly-graded objects, first introduced by Stasheff and Schlessinger in [SS2] (see also [SS2]) play a role in deformation theory, in rational homotopy theory, and in mathematical physics. See also [GK, §1.3.9], and [LM, 2.1], where these are called $L(\infty)$-*structures.* Martin Markl has pointed out to us that the resolution for free Lie algebras we define can also be obtained by the methods of [GK] and [M].

3.22. Theorem. *$F_{*,*}$ is a resolution of $L=\mathbb{L}\langle X_*\rangle$.*

Proof. It is clear from the construction that $H_0(F_{*,*})\cong L$, and that $F_{*,*}$ is free as a DBGCA, so it suffices to show that $F_{*,*}$ is acyclic in positive degrees. Since $F_{*,*}$ is defined as a direct sum of chain complexes (3.17), it is enough to consider each summand separately. Thus, for each $\hat{I}\in\hat{\mathfrak{I}}_{n+3}$ (fixed for the remainder of the proof), it suffices to show that $J[\hat{I}]_*$ is acyclic in degrees $<n$.

To do so, first consider the corresponding multiset I' *without* repetitions. Because $J[I']_*$ is acyclic by Proposition 3.13 above, it has (many possible) contracting chain homotopies. We now proceed to make a specific choice of such a homotopy (dependent on the original I):

Assume that $I=I'/\mathfrak{L}\text{-}\mathcal{A}ut(\hat{I})$ for $G=\mathfrak{L}\text{-}\mathcal{A}ut(\hat{I})\subseteq\Sigma_n$ as above. Since ∂_{k+1} commutes with the action of G, the summand $Im(\partial_{k+1})$ of $J[I']_k$ is invariant under this action. Thus, by Maschke's Theorem (see [CR, 10.8]), for each $0<k\leq n-3$ we may choose a splitting

$$J[I']_k=Im(\partial_{k+1})\oplus S_k,$$

where S_k is also invariant under the action of G, and of course $\partial_k|_{S_k}$ is an isomorphism onto $Im(\partial_k)\subseteq J[I']_{k-1}$ (because $Im(\partial_{k+1})=Ker(\partial_k)$).

We may thus define a linear map $\delta'_k:J[I']_k\to J[I']_{k+1}$ by $\delta'_k(\partial_{k+1}T_i)=T_i$, and $\delta'_k|_{S_k}\equiv 0$; this is a contracting homotopy for $J[I']_*$. Moreover, it commutes with the action of G, so it induces a contracting homotopy δ on $J[\hat{I}]_*$, which is thus acyclic. $\square$

4. Homology of DGLs

We may use the resolutions constructed above to calculate the homology of a free Lie or associative algebra, considered as a non-associative algebra. We first recall Quillen's definition of homology in model categories:

4.1. Definition. An object X in a category $\mathcal{C}$ is said to be *abelian* if it is an abelian group object – that is, if $Hom_{\mathcal{C}}(Y,X)$ has a natural abelian group structure for any $Y\in\mathcal{C}$. When $\mathcal{C}$ is $\mathcal{L}ie$, $\mathcal{A}lg$, $\mathcal{A}lg_a$, $\mathcal{L}$, or $\mathcal{A}$, for example, this is equivalent to requiring that all products vanish in X.

The full subcategory of abelian objects in $\mathcal{C}$ is denoted by $\mathcal{C}_{ab}\subset\mathcal{C}$. It is equivalent to the category $\mathcal{V}ect$ of vector spaces if $\mathcal{C}=\mathcal{L}ie$, $\mathcal{A}lg$, $\mathcal{A}lg_a$, and so on, and to the category $\mathcal{V}$ of graded vactor spaces if $\mathcal{C} = \mathcal{L}$ or $\mathcal{A}$; so we see that $\mathcal{C}_{ab}$ is an abelian category, in the cases of interest to us. We then have an *abelianization* functor $Ab : \mathcal{C} \to \mathcal{C}_{ab}$, along with a natural transformation $\theta : Id \to Ab$ having the appropriate universal property. In all the examples above, $Ab(X) = X/I(X)$, where $I(X)$ is the ideal in $X \in \mathcal{C}$ generated by all non-trivial products.

4.2. **Homology of algebras.** Let $\mathcal{C}$ be a category as above, which also has a model category structure (see [Q2, II, §1]). In [Q1, II, §5] (or [Q3, §2]), Quillen defines the *homology* of an object $X \in \mathcal{C}$ to be the total left derived functor $\mathbf{L}(Ab)$ of Ab, applied to X (cf. [Q1, I, §4]).

In more familiar terms, this means that we construct a resolution $A \to X$ (i.e., replace X by a weakly equivalent cofibrant object $A \in \mathcal{C}$), and then define the i-th homology group of X by $H_i X := H_i(Ab(A))$, where $Ab(A)$ is (equivalent to) a chain complex in an abelian category, so its homology is defined as usual. One must verify, of course, that this definition is independent of the choice of the resolution $A \to X$.

If $\mathcal{C}$ itself does not have a closed model category structure, one often defines the homology of $X \in \mathcal{C}$ by embedding $\mathcal{C}$ in some category which does have such a structure, which in most cases may be taken to be $s\mathcal{C}$, the category of simplicial objects over $\mathcal{C}$ (see [Q1, II, §4]). Thus, if $\iota : \mathcal{C} \hookrightarrow s\mathcal{C}$ is the embedding of categories defined by taking $\iota(C)$ to be the constant simplicial object equal to C in all dimensions, then $H_i(C) := \pi_i(\mathbf{L}(Ab \circ \iota)C)$.

This is the approach usually taken for $\mathcal{C} = \mathcal{L}ie$, $\mathcal{A}lg$, $\mathcal{A}$, and so on: to define the homology of a graded Lie algebra $L_* \in \mathcal{L}$, say, one chooses a free simplicial resolution $A_{\bullet,*} \to L_*$ and then calculates the homotopy groups of the simplicial graded vector space $Ab(A_{\bullet,*}) \in s\mathcal{V}$.

As for graded Lie algebras and skew-commutative algebras, one can define closed model category structures on $s\mathcal{A}_c$ and $db\mathcal{A}_c$ (see [BS, §2], and [B1, §4]), and because we are working over a field of characteristic 0, we have the following analogue of [Q2, Props. 2.3 & 4.6, Thm. 4.4]

4.3. Proposition. *There are adjoint functors* $s\mathcal{A} \underset{N^*}{\overset{N}{\rightleftarrows}} db\mathcal{A}$, *which induce equivalences of the corresponding homotopy categories* $ho(s\mathcal{A}) \approx ho(db\mathcal{A}_c)$. N^* *takes free DBGCAs to free simplicial GCAs.*

Proof. See [B2, Props. 2.9, 7.2, 7.3]. □

Thus we may use DGCAs (resp. DBGCAs) instead of simplicial commutative algebras (resp. simplicial GCAs) as our free resolutions – as in §2.6 – and replace the homotopy groups by the homology groups of the corresponding (bigraded) chain complex.

4.4. *Remark.* We gave the definition of homology in its simplicial version, which applies to more general types of universal algebras, in order to emphasize that our methods do not apply to associative or Lie algebras over an arbitrary (commutative) ground ring $\mathbf{k}$, because in that case one cannot resort to differential graded algebras as resolutions. (The case of $\mathbf{k} = \mathbb{Z}$ would have been of special interest.)

4.5. **Calculating the homology.** In particular, we may use the resolutions $E_* \to \mathbb{A}\mathrm{lg}_a\langle X\rangle$ and $F_{*,*} \to \mathbb{L}\langle X_*\rangle$ defined above to calculate the homology of a free associative or (graded) Lie algebra, considered as an object in $\mathcal{A}lg$ or $\mathcal{A}$. Explicitly, if $E_\bullet$ is the simplicial algebra corresponding to the DGNA E_*, then $H_n(\mathbb{A}\mathrm{lg}_a\langle X\rangle)$ is defined to be the n-th homotopy group of the simplicial vector space $Ab(E_\bullet)$, where the abelianization functor is applied in each simplicial dimension separately; and similarly for $H_*(\mathbb{L}\langle X_*\rangle)$.

However, the definition of the correspondence between E_* and $E_\bullet$ (cf. [B2, Proof of Prop. 2.9]) implies that the indecomposables in the two cases are in bijective correspondence, so that in fact we may calculate $H_n(\mathbb{A}\mathrm{lg}_a\langle X\rangle)$ as the n-th homology group of the differential vector space (i.e., chain complex) $Ab(E_*) := E_*/I(E_*)$. This simply means that we must replace by 0 all trees in E_* whose roots have only two children, and compute the homology of the resulting chain complex. Similarly for $F_{*,*}$.

4.6. Theorem. *$H_i(\mathbb{L}\langle X_*\rangle) = 0$ for $i \geq 2$.*

Proof. As before, let $I = (i_1, \ldots, i_n)$ be some n-tuple of distinct indices of elements in the graded set X_*, and let $N_* = N[I]_*$ denote the subcomplex of $J_* = J[I]_*$ spanned by all trees T with $k_{r(T)} \geq 3$. We will say that a subcomplex $C_* \subset J_*$ is *ℓ-coconnected* if $H_i(C_*) = 0$ for $i \leq n - 3 - \ell$.

(I) Given any subset $A = \{i_1, \ldots, i_k\}$ of I, let $\tilde{N}[A]_*$ denote the subcomplex of $N[I]_*$ spanned by all trees T in which $x_{i_1}, \ldots, x_{i_k}$ all have the same parent node (compare §3.11 above). We claim that $\tilde{N}[A]_*$ is k-coconnected, for any A with $k \geq 2$.

This is shown essentially as in the proof of Lemma 3.12. We define a (partial) contracting homotopy $\delta : \tilde{N}[A]_i \to \tilde{N}[A]_{i+1}$ for $i < n - 3 - k$ as follows:

Write $\bar{x} := x_{i_1}, \ldots, x_{i_k}$. If α has $(\bar{x})$ as a sub-expression, then $\delta[\alpha] = 0$; if $\alpha = ((\cdots(\bar{x}\bar{a}^1)\cdots)\bar{a}^t)$, we set $\delta[\alpha] = (-1)^{\mathrm{cx}(\bar{x})}((\cdots((\bar{x})\bar{a}^1)\cdots)\bar{a}^t)$ (and require that $\delta[\varphi(\alpha)] = (-1)^{\varepsilon(\varphi)}\varphi(\delta[\alpha])$ for any $\varphi \in \mathcal{B}\text{-}\mathcal{A}ut(\alpha)$). Any other basis element of $\tilde{N}[A]_*$ is in the subcomplex $\bar{x}J[I \setminus A]_*$ – i.e., of the form $\alpha = (\bar{x}\bar{a})$ for some $\bar{a} = a_1 \cdots a_t$ for some $t \geq 1$, where $(\bar{a}) \in J[I \setminus A]_*$ (again, up to the $\mathcal{B}\text{-}\mathcal{A}ut(\alpha)$-action). But $\bar{x}J[I \setminus A]_*$ is isomorphic to the complex $J[I \setminus A]_*$ shifted up by k, and this has a contracting homotopy δ' in degrees $< n - k - 3$ by Proposition 3.13; set $\delta[(\bar{x}\bar{a})] := \delta'[(\bar{a})]$ if $t = 1$, and $\delta[(\bar{x}\bar{a})] := \delta'[(\bar{a})] + ((\bar{x})\bar{a})$ if $t \geq 2$.

(II) We now show that $N_* = N[I]_*$ is 2-coconnected. If we denote the sequence of chain complexes

$$(\tilde{N}[(i_1, i_2)]_*,\ \tilde{N}[(i_1, i_3)]_*, \ldots, \tilde{N}[(i_1, i_{n-1})]_*,\ \tilde{N}[(i_2, i_3)]_*, \ldots, \tilde{N}[(i_{n-2}, i_{n-1})]_*)$$

by $(D^t_*)_{t=1}^m$, and set $C^t_* = \bigcup_{i=1}^t D^i_*$ for $1 \leq t \leq m$, then the chain complex $C^m_* = \bigcup_{1 \leq k < \ell < n} \tilde{N}[(i_k, i_\ell)]_*$ is in fact all of N_*, since any $\alpha \in J[I]_*$ not in C^m_* is of the form $\alpha = ((\cdots((x_n x_{\sigma(1)})x_{\sigma(2)})\cdots)x_{\sigma(n-1)})$ for some $\sigma \in \Sigma_{n-1}$ – so not in N_*.

Note that each D^i_* is 2-coconnected by (I) above, and in fact for any subset $\{s_1, \ldots, s_\ell\} \subset \{1, 2, \ldots, m\}$ the complex $\bigcap_{i=1}^\ell D^{s_i}_*$ is $(\ell + 1)$-coconnected (by

Lemma 3.12 and the argument in the first paragraph of the proof of Proposition 3.13). Since $C_*^t = C_*^{t-1} \cup D_*^t$, where $C_*^{t-1} \cap D_*^t = (C_*^{t-2} \cap D_*^t) \cup (D_*^{t-1} \cap D_*^t)$, in the Mayer-Vietoris sequence

$$\begin{aligned} \dots H_i(C_*^{t-2} \cap D_*^{t-1} \cap D_*^t) &\to H_i(C_*^{t-2} \cap D_*^t) \oplus H_i(D_*^{t-1} \cap D_*^t) \to H_i(C_*^{t-1} \cap D_*^t) \\ &\xrightarrow{\partial} H_{i-1}(C_*^{t-2} \cap D_*^{t-1} \cap D_*^t) \to H_{i-1}(C_*^{t-2} \cap D_*^t) \oplus H_{i-1}(D_*^{t-1} \cap D_*^t) \dots \end{aligned}$$

we see that $C_*^{t-1} \cap D_*^t$ is 3-coconnected if $C_*^{t-2} \cap D_*^{t-1} \cap D_*^t$ is 4-coconnected, say. Thus we can show by descending induction on $0 \leq \ell < t$ that $C_*^{t-\ell} \cap D_*^{s_1} \cap \cdots D_*^{s_\ell}$ is $\ell + 2$-coconnected for any subset $\{s_1, \dots, s_\ell\} \subset \{1, 2, \dots, m\}$.

(III) We have shown that N_* is 2-coconnected, which means that after re-indexing as in §3.16, we obtain a bigraded chain complex $\tilde{F}_{*,*}$ (direct sums as in (3.17)), which now has no homology *above* dimension 1, since the subcomplexes $N[I]_*$ are invariant under the action of the leaf automorphism groups on $J[I]_*$ (as in the proof of Theorem 3.22). □

Evidently $H_0(\tilde{F}_{*,*}) \cong X_*$, since the indecomposables of the original complex $F_{*,*}$, in homological dimension 0, consist simply of a set of generators for the free non-associative algebra on X_*. The calculation of $H_1(\tilde{F}_{*,*})$ will be dealt with elsewhere.

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