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Automorphisms of the category of free Lie algebras

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Abstract

We prove that every automorphism of the category of free Lie algebras is a semi-inner automorphism. This solves Problem 3.9 from [G. Mashevitzky, B. Plotkin, E. Plotkin, Electron. Res. Announc. Amer. Math. Soc. 8 (2002) 1–10] for Lie algebras.

Introduction

We start from an arbitrary variety of algebras Θ. Let us denote the category of free in Θ algebras $F = F(X)$, where X is finite, by Θ^0. In order to avoid the set theoretic problems we view all X as subsets of a universal infinite set X^0.

Our main goal is to study automorphisms $\varphi : \Theta^0 \to \Theta^0$ and the corresponding group $\mathrm{Aut}\,\Theta^0$ for various Θ.

In this paper we consider the case when Θ is the variety of all Lie algebras over an infinite field P. Our aim is to prove the following principal theorem:

Theorem 1. *Every automorphism of the category of free Lie algebras is a semi-inner automorphism.*

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This theorem solves Problem 3.9 from [19] for the case of Lie algebras.

Our primary interest to automorphisms of categories raised from the universal algebraic geometry (see [2–4,16,17,22,23,26–31,37], etc.). The motivations we keep in mind are inspired by the following observations.

Some basic notions of classical algebraic geometry can be defined for arbitrary varieties of algebras Θ. For every algebra $H \in \Theta$ one can consider geometry in Θ over H. This geometry gives rise to the category $K_\Theta(H)$ of algebraic sets in affine spaces over H [31]. The key question in this setting is when the geometries in Θ defined by different algebras H_1 and H_2 coincide. The coincidence of geometries means for us that the corresponding categories of algebraic sets $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are either isomorphic or equivalent.

It is known that the conditions on H_1 and H_2 providing isomorphism or equivalence of the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ depend essentially on the description of the automorphisms of the category Θ^0 (see [19,31]). This explains the interest to automorphisms of categories of free algebras of varieties.

Let $F = F(X) \in \Theta$ be a free algebra, i.e., an object of the category Θ^0. The group $\mathrm{Aut}(\Theta^0)$ is tied naturally with the following sequence of groups:

$$\mathrm{Aut}(F), \qquad \mathrm{Aut}\big(\mathrm{Aut}(F)\big), \qquad \mathrm{Aut}\big(\mathrm{End}(F)\big).$$

The groups $\mathrm{Aut}(F)$ are known for the variety of all groups (Nielsen's theorem [15]), for the variety of Lie algebras (P. Cohn's theorem [7]), for the free associative algebras over a field when the number of generators of F is $\leqslant 2$ [8,9,18,25] and for some other varieties. For free associative algebras with bigger number of generators the question is still open (see Cohn's conjecture [8]). The groups $\mathrm{Aut}(\mathrm{Aut}(F))$, $\mathrm{Aut}(\mathrm{End}(F))$ are known for the variety of all groups [10,12,33], and due to E. Formanek every automorphism of $\mathrm{End}(F)$ is inner. The groups $\mathrm{Aut}(\mathrm{Aut}(F))$, $\mathrm{Aut}(\mathrm{End}(F))$ are also known for some other varieties of groups and semigroups [11,14,20,34–36].

Suppose that a free algebra $F = F(X)$ generates the whole variety Θ. In this case there exists a natural way from the group $\mathrm{Aut}(\mathrm{End}(F))$ to the group $\mathrm{Aut}(\Theta^0)$. Thus, there is a good chance to reduce the question on automorphisms of the category Θ^0 to the description of $\mathrm{Aut}(\mathrm{End}(F))$.

$\mathrm{Aut}(F)$ is the group of invertible elements of the semigroup $\mathrm{End}(F)$. Every automorphism φ of the semigroup $\mathrm{End}(F)$ induces an automorphism of the group $\mathrm{Aut}(F)$. This gives a homomorphism $\tau : \mathrm{Aut}(\mathrm{End}(F)) \to \mathrm{Aut}(\mathrm{Aut}(F))$. The kernel of this homomorphism consists of automorphisms acting trivially in $\mathrm{Aut}(F)$. These automorphisms are called *stable*. We will prove that

(1) The homomorphism τ is not a surjection.
(2) If X consists of more than 2 elements then τ is an injection.
(3) If X consists of 2 elements then $\mathrm{Ker}\,\tau$ consists of scalar automorphisms (see Sections 2, 3).

The paper is organized as follows. In Section 1 we give the definitions of inner and semi-inner automorphisms of a category. In Section 2 the notations are introduced. Section 3 is dedicated to linearly stable automorphisms and we prove that every linearly stable automorphism is inner. In Section 4 we define the notion of a quasi-stable automorphism and prove that every quasi-stable automorphism is inner. In Section 5 we prove that every automorphism of the semigroup of endomorphisms of the free two generator Lie algebra is semi-inner. In Section 6 we prove the general reduction theorem for a large class of varieties and reduce the problem about $\mathrm{Aut}(\Theta^0)$ to the description of $\mathrm{Aut}(\mathrm{End}(F(x, y))$. Section 7 is dedicated to the proof of the main theorem. Finally, in Appendix we prove some auxiliary statements used in the text.

1. Inner and semi-inner automorphisms of a category

Recall the notions of category isomorphism and equivalence [21]. A functor $\varphi : \mathcal{C}_1 \to \mathcal{C}_2$ is called an *isomorphism of categories* if there exists a functor $\psi : \mathcal{C}_2 \to \mathcal{C}_1$ such that $\psi\varphi = 1_{\mathcal{C}_1}$ and $\varphi\psi = 1_{\mathcal{C}_2}$, where $1_{\mathcal{C}_1}$ and $1_{\mathcal{C}_2}$ are identity functors.

Let φ_1, φ_2 be two functors $\mathcal{C}_1 \to \mathcal{C}_2$. An *isomorphism of functors* $s : \varphi_1 \to \varphi_2$ is defined by the following conditions:

(1) To every object A of the category C_1 an isomorphism $s_A : \varphi_1(A) \to \varphi_2(A)$ in $\mathcal{C}_2$ is assigned.
(2) If $\nu : A \to B$ is a morphism in $\mathcal{C}_1$, then there is a commutative diagram in $\mathcal{C}_2$:

$$\begin{array}{ccc} \varphi_1(A) & \xrightarrow{s_A} & \varphi_2(A) \\ {\scriptstyle \varphi_1(\nu)}\downarrow & & \downarrow{\scriptstyle \varphi_2(\nu)} \\ \varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B). \end{array}$$

The isomorphism of functors φ_1 and φ_2 is denoted by $\varphi_1 \simeq \varphi_2$.

The notion of category equivalence generalizes the notion of category isomorphism. A pair of functors $\varphi : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi : \mathcal{C}_2 \to \mathcal{C}_1$ define a *category equivalence* if $\psi\varphi \simeq 1_{\mathcal{C}_1}$ and $\varphi\psi \simeq 1_{\mathcal{C}_2}$. If $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ then we get the notions of *automorphism* and *autoequivalence* of the category $\mathcal{C}$.

Definition 1.1. An automorphism φ of the category $\mathcal{C}$ is called inner if there exists an isomorphism of functors $s : 1_{\mathcal{C}} \to \varphi$.

This means that for every object A of the category $\mathcal{C}$ there exists an isomorphism $s_A : A \to \varphi(A)$ such that

$$\varphi(\nu) = s_B \nu s_A^{-1} : \varphi(A) \to \varphi(B),$$

for any morphism $\nu : A \to B$ in $\mathcal{C}$.

For every small category $\mathcal{C}$ denote the group of all automorphisms of $\mathcal{C}$ by $\mathrm{Aut}(\mathcal{C})$ and denote its normal subgroup of all inner automorphisms by $\mathrm{Int}(\mathcal{C})$.

From now on and till Section 5, Θ will denote the variety of all Lie algebras over the field P. Correspondingly, Θ^0 is the category of free Lie algebras over P.

Define the notion of a semi-inner automorphism of the category Θ^0. Consider, first, semimorphisms in the variety Θ. A *semimorphism* in Θ is a pair $(\sigma, \nu): A \to B$, where A and B are algebras in Θ, $\nu: A \to B$ a homomorphism of Lie rings, σ an automorphism of the field P, subject to condition $\nu(\lambda a) = \sigma(\lambda)\nu(a)$, where $\lambda \in P$, $a \in A$. If $\sigma = 1$ then ν is a homomorphism of Lie algebras and we write it as $(1, \nu)$. Semimorphisms are multiplied componentwise.

Thus, if $\mu: A \to B$ is a homomorphism, and $(\sigma, \nu_1): A \to A_1$, $(\sigma, \nu_2): B \to B_1$ are semi-isomorphisms, then

$$(\sigma, \nu_2)(1, \mu)(\sigma, \nu_1)^{-1} = (\sigma, \nu_2)(1, \mu)\big(\sigma^{-1}, \nu_1^{-1}\big) = \big(1, \nu_2\mu\nu_1^{-1}\big).$$

This means that $\nu_2\mu\nu_1^{-1}: A_1 \to B_1$ is a homomorphism of Lie algebras.

Definition 1.2. An automorphism φ of the category Θ^0 is called semi-inner, if for some $\sigma \in \mathrm{Aut}(P)$ there is a semi-isomorphism of functors $(\sigma, s): 1_{\Theta^0} \to \varphi$.

This definition means that for every object $F \in \Theta^0$ there exists a semi-isomorphism $(\sigma, s_F): F \to \varphi(F)$ such that

$$\varphi(\nu) = s_{F_2}\nu s_{F_1}^{-1}: \varphi(F_1) \to \varphi(F_2),$$

for any morphism $\nu: F_1 \to F_2$ in Θ^0.

Let now σ be an arbitrary automorphism of the field P. We will construct a semi-inner automorphism $\hat{\sigma}$ of the category Θ^0. Consider an arbitrary free finitely generated Lie algebra $F = F(X)$ and fix a Hall basis in F [1,6]. It can be shown [6] that if u, v are two elements of a Hall basis then $[u, v]$ is presented via elements of this basis with the coefficients (structure constants) belonging to the minimal subfield of the field P. Define a map $\sigma_F: F \to F$. Every element w of F has the form:

$$w = \lambda_1 u_1 + \cdots + \lambda_n u_n,$$

where $\lambda_i \in P$ and u_i belong to the Hall basis of F. Define

$$\sigma_F(w) = \sigma(\lambda_1)u_1 + \cdots + \sigma(\lambda_n)u_n.$$

We show that (σ, σ_F) is a semi-automorphism of the algebra F. It is clear that σ_F preserves the addition, and that $\sigma_F(\lambda w) = \sigma(\lambda)\sigma_F(w)$. It remains to check that σ_F preserves the multiplication. Let $w_1 = \sum_i \alpha_i u_i$, and $w_2 = \sum_j \alpha_j u_j$. Take

$$[u_i, u_j] = \sum_k n_k^{i,j} u_k^{i,j},$$

where $n_k^{i,j}$ belong to a minimal subfield. Then

$$[w_1, w_2] = \sum_{i,j} \alpha_i \beta_j [u_i, u_j] = \sum_{i,j,k} \alpha_i \beta_j n_k^{i,j} u_k^{i,j}.$$

Apply σ_F, then

$$\sigma_F [w_1, w_2] = \sum_{i,j,k} \sigma(\alpha_i) \sigma(\beta_j) n_k^{i,j} u_k^{i,j},$$

since the automorphism σ does not change the elements of a prime subfield. On the other hand

$$\bigl[\sigma_F(w_1), \sigma_F(w_2)\bigr] = \Bigl[\sum_i \sigma(\alpha_i) u_i, \sum_j \sigma(\beta_j) u_j\Bigr] = \sum_{i,j} \sigma(\alpha_i) \sigma(\beta_j) [u_i, u_j]$$
$$= \sum_{i,j,k} \sigma(\alpha_i) \sigma(\beta_j) n_k^{i,j} u_k^{i,j}.$$

We verified that the pair (σ, σ_F) defines a semi-automorphism of the algebra F. Note that σ_F does not change variables from X and does not change all commutators constructed from these variables.

Now we are able to define the automorphism $\hat{\sigma}$ of the category Θ^0. This automorphism does not change objects and $\hat{\sigma}(\nu) = \sigma_{F_2} \nu \sigma_{F_1}^{-1} : F_1 \to F_2$ for every morphism $\nu : F_1 \to F_2$. It is easy to check that if φ is an arbitrary semi-inner automorphism of Θ^0 for the given $\sigma \in \mathrm{Aut}(P)$ then there is a factorisation $\varphi = \varphi_0 \hat{\sigma}$, where φ_0 is an inner automorphism.

The same scheme which was applied for the definition of semi-inner automorphisms of the category Θ^0 works for the definition of semi-inner automorphisms of the semigroup $\mathrm{End}(F)$, where F is a free finitely generated Lie algebra. An automorphism φ of $\mathrm{End}(F)$ is called a *semi-inner* automorphism if there exists a semi-automorphism $(\sigma, s) : F \to F$ such that $\varphi(\nu) = s \nu s^{-1}$ for every $\nu \in \mathrm{End}(F)$. The factorization $\varphi = \varphi_0 \hat{\sigma}$, where φ_0 is inner holds also in this case.

2. Notations and preliminaries

Let X be a finite set. Denote by $F(X)$ the free Lie algebra over an infinite field P generated by the set X of free generators. The Lie operation is denoted by [,]. Denote the group of all non-zero elements of P by P^*. We denote the semigroup of all endomorphisms of $F(X)$ by $\mathrm{End}(F(X))$. Any endomorphism of $F(X)$ is uniquely determined by a mapping $X \to F(X)$. Therefore, we define an endomorphism φ of $F(X)$ by defining $\varphi(x)$ for all $x \in X$. Denote the group of all automorphisms of $F(X)$ by $\mathrm{Aut}(F(X))$.

We fix a basis, say the Hall basis in $F(X)$ and consider the presentations of elements of $F(X)$ in this basis. Denote the length of a monomial $u \in F(X)$ by $|u|$, we call it also the degree of u. Denote the set of all elements of X included in u by $\chi(u)$. The set $\chi(u)$ is a support of element u, which is uniquely defined by the presentation of u in the fixed basis. Denote the number of occurrences of a letter x in u by $l_x(u)$. Let $p \in F(X)$. Denote the degree of the polynomial p by $\deg(p)$. Denote the cardinality of the set X by $|X|$.

Let us denote the semigroup of all endomorphisms $\varphi \in \mathrm{End}(F(X))$ which assign a linear polynomial from $F(X)$ to any $x \in X$ by $\mathrm{End}_l(F(X))$. Denote the group of the linear automorphisms of $F(X)$ by $\mathrm{Aut}_l(F(X))$.

Let $X = \{x_1 \dots x_n\}$. If φ is linear then $\varphi(x_i) = a_{i1}x_1 + \dots + a_{in}x_n$, where $a_{ij} \in P$. This means that a linear automorphism is defined by its matrix of coefficients. The multiplication of linear automorphisms corresponds to the multiplication of their matrices. Thus, the semigroup $\mathrm{End}_l(F(X))$ is isomorphic to the matrix semigroup $M_n(P)$. The scalar matrix corresponds to the linear automorphism, defined by $f_a(x_i) = ax_i$. Therefore, the automorphism f_a commutes with all linear endomorphisms. However, it does not commute with an arbitrary endomorphism.

Denote the endomorphism of $F(X)$ which assigns the same $p \in F(X)$ to any $x \in X$ by c_p. All endomorphisms of the type c_p form a subsemigroup C_F of $\mathrm{End}(F(X))$. Let us denote the subsemigroup of C_F consisting of all endomorphisms of the type c_u where $u \in X$ by C_X. Let $C_l = C_F \cap \mathrm{End}_l(F(X))$. C_l consists of all endomorphisms of the type $c_p \in C_F$ where p is a linear polynomial.

Definition 2.1. An automorphism $\xi \in \mathrm{Aut}(\mathrm{End}(F(X)))$ which acts identically on $\mathrm{Aut}_l(F(X))$, is called a linearly stable automorphism of $\mathrm{End}(F(X))$.

3. Linearly stable automorphisms of End($F(X)$) are inner

In this section we prove that the semigroup C_F of all constant endomorphisms is invariant in respect to the action of any linearly stable automorphism of $\mathrm{End}(F(X))$ and that any linearly stable automorphism acts identically on the semigroup $\mathrm{End}_l(F(X))$ of all linear endomorphisms. Then we prove that any linearly stable automorphism of $\mathrm{End}(F(X))$ is an inner automorphism.

Lemma 3.1. *Let ξ be a linearly stable automorphism of* $\mathrm{End}(F(X))$. *Then* $\xi(C_F) = C_F$.

Proof. Take $u \in F(X)$, $c_u \in C_F$. Consider $g \in \mathrm{Aut}_l(F(X))$ such that $g(x) = y$, $g(y) = x$, and $g(z) = z$ for any $z \in X$ distinct from x and y. Then $c_u g = c_u$ and $\xi(g) = g$. Hence, $\xi(c_u)(x) = \xi(c_u g)(x) = \xi(c_u)\xi(g)(x) = \xi(c_u)(y) = v$. Thus, $\xi(c_u) = c_v \in C_F$. Therefore, $\xi(C_F) \subset C_F$. Similarly, $\xi^{-1}(C_F) \subset C_F$. Hence, $\xi(C_F) = C_F$. □

Lemma 3.2. *Let* $\xi \in \mathrm{Aut}(\mathrm{End}(F(X)))$ *be a linearly stable automorphism. Let* $\xi(c_p) = c_q$. *Then* $\chi(p) = \chi(q)$ (*polynomials* p *and* q *are constructed from the same variables*).

Proof. Suppose that $x \in \chi(p) \setminus \chi(q)$. Let $a \in P$ be a torsion free element. Consider $g_a \in \mathrm{Aut}_l(F(X))$, which assigns ax to x and assigns z to z for any $z \in X$ different from x. Then $g_a(p) = w \neq p$ (indeed all elements of the basis are linearly independent and a is torsion free) and $g_a(q) = q$. Therefore, $g_a c_p = c_w \neq c_p$. On the other hand,

$\xi(g_a c_p) = \xi(g_a)\xi(c_p) = g_a c_q = c_q = \xi(c_p)$. It contradicts to injectivity of ξ. The similar reasoning can be applied to $x \in \chi(q) \setminus \chi(p)$ as well. Thus, $\chi(q) = \chi(p)$. □

Lemma 3.3. *Any linearly stable automorphism ξ of $\mathrm{End}(F(X))$ acts identically on C_X.*

Proof. It is obvious that any $c_x \in C_X$ is a right identity of C_F. Therefore, $\xi(c_x) = c_p \in C_F$ (Lemma 3.1) and $\xi(c_x)$ is a right identity of C_F. Hence, $c_x c_p = c_x$. It follows from Lemma 3.2 and identity $[xx] = 0$ that $p = ax$, $a \in P$. Therefore, $c_x c_p = c_p$. Thus, $\xi(c_x) = c_p = c_x$. □

Lemma 3.4. *For any linearly stable automorphism ξ of $\mathrm{End}(F(X))$ and for any $\varphi \in \mathrm{End}(F(X))$, $\xi(c_{\varphi(x)}) = c_{\xi(\varphi)(x)}$.*

Proof. We have $\varphi c_x = c_{\varphi(x)}$. Then $\xi(c_{\varphi(x)}) = \xi(\varphi c_x) = \xi(\varphi) c_x = c_{\xi(\varphi)(x)}$. □

Lemma 3.5. *Any linearly stable automorphism ξ of $\mathrm{End}(F(X))$ acts identically on $\mathrm{End}_l(F(X))$.*

Proof. Let us prove first that ξ acts identically on C_l. Let $c_p \in C_l$, where p is a linear polynomial. Let $g \in \mathrm{Aut}_l(F(X))$ be defined by $g(x) = p$, $x \in \chi(p)$, and $g(y) = y$ for any other $y \in X$. It follows from Lemma 3.3 that $\xi(c_x) = c_x$. Therefore, $g c_x = c_p$ and $\xi(c_p) = \xi(g)\xi(c_x) = g c_x = c_p$.

Hence, $c_{\varphi(x)} = \xi(c_{\varphi(x)}) = \xi(\varphi) c_x = c_{\xi(\varphi)(x)}$, for any $\varphi \in \mathrm{End}_l(F(X))$. Thus, $\xi(\varphi)(x) = \varphi(x)$. □

Lemma 3.6. *Let $\xi \in \mathrm{Aut}(\mathrm{End}(F(X)))$ be a linearly stable automorphism. Let $\xi(c_{p_i}) = c_{q_i}$ and $a_i \in P$ for $i = 1, 2, \ldots, k$. Then $\xi(c_{a_1 p_1 + \cdots + a_k p_k}) = c_{a_1 q_1 + \cdots + a_k q_k}$.*

Proof. It follows from Lemma 3.5 that $\xi(c_{ax}) = c_{ax}$. Therefore, $\xi(c_{ap_i}) = \xi(c_{p_i} c_{ax}) = c_{q_i} c_{ax} = c_{aq_i}$.

Define $\varphi \in \mathrm{End}(F(X))$ as follows $\varphi(x) = p_1$, $\varphi(y) = p_2$. Then $c_{p_1+p_2} = \varphi c_{x+y}$. It follows from Lemma 3.4 that $\xi(\varphi)(x) = q_1$ and $\xi(\varphi)(y) = q_2$. It follows from Lemma 3.5 that $\xi(c_{x+y}) = c_{x+y}$. Therefore, $\xi(c_{p_1+p_2}) = \xi(\varphi) c_{x+y} = c_{q_1+q_2}$.

Let us prove that $\xi(c_{a_1 p_1 + \cdots + a_k p_k}) = c_{a_1 q_1 + \cdots + a_k q_k}$ by induction on k. We have proved above the basis of the induction: $\xi(c_{a_1 p_1}) = c_{a_1 q_1}$. Assume that the statement is true for $k < t$. It follows from the assumption of the induction that $\xi(c_{a_1 p_1 + \cdots + a_{t-1} p_{t-1}}) = c_{a_1 q_1 + \cdots + a_{t-1} q_{t-1}}$ and $\xi(c_{a_t p_t}) = c_{a_t q_t}$. $\xi(c_{p+p'}) = c_{q+q'}$. Therefore, $\xi(c_{a_1 p_1 + \cdots + a_t p_t}) = c_{a_1 q_1 + \cdots + a_t q_t}$. □

Lemma 3.7. *Let $\xi \in \mathrm{Aut}(\mathrm{End}(F(X)))$ be a linearly stable automorphism. Then $\xi(c_{[x_1,x_2]}) = c_{a[x_1,x_2]}$, where $a \in P$ and $a \neq 0$.*

Proof. Suppose that $\xi(c_{[x_1,x_2]}) = c_f$.

Denote the endomorphism generated by the mapping $x_i \to a_i x_i$ by $\tau_{a_1 \ldots a_n}$. It follows from Lemma 3.5 that $\xi(\tau_{a_1 \ldots a_n}) = \tau_{a_1 \ldots a_n}$. Therefore, $\xi(\tau_{a_1 \ldots a_n} c_{[x_1,x_2]}) = \tau_{a_1 \ldots a_n} c_f$. Hence,

$\xi(c_{a_1a_2[x_1,x_2]}) = c_{\tau_{a_1\dots a_n}(f)}$. It follows from Lemma 3.6 that $\xi(c_{a_1a_2[x_1,x_2]}) = c_{a_1a_2f}$. Thus, $a_1a_2f = \tau_{a_1\dots a_n}(f)$.

Suppose that $f = \alpha_1 f_1 + \cdots + \alpha_t f_t$ is the decomposition of f with respect to the basis of Hall. Observe that the decomposition of $\tau_{a_1\dots a_n}(f)$ contains the same elements of the basis as the decomposition of f but with different coefficients. Elements of the basis are linearly independent over P. Therefore, we obtain a system of equations of the form

$$\left(a_1^{k_1^i}\dots a_n^{k_n^i} - a_1a_2\right)\alpha_i = 0$$

from the equation $a_1a_2f = \tau_{a_1\dots a_n}(f)$. For the element $[x_1, x_2]$ of the basis we get the equation $(a_1a_2 - a_1a_2)\alpha = 0$. In all other cases it is easy to find $a_1,\dots,a_n$ such that

$$\left(a_1^{k_1^i}\dots a_n^{k_n^i} - a_1a_2\right) \neq 0$$

(for example, if $\mathrm{char}(P) \neq 2$, take $a_1 = a_2 = \cdots = a_n = 2$ for the monomials with the number of multipliers different from 2 and take $a_1 = a_2 = 2$, $a_3 = \cdots = a_n = 1$ for monomials $[x_i, x_j]$, where $\{i, j\} \neq \{1, 2\}$). Hence, all coefficients α_i are equal to zero except the coefficient of the monomial $[x_1, x_2]$. □

Let $a \in P^*$. Consider a scalar automorphism f_a of $F(X)$ defined by the rule $f_a(x) = ax$, for every $x \in X$. It defines an inner automorphism $\hat{f}_a$ of the semigroup $\mathrm{End}(F(X))$ by the rule $\hat{f}_a(\varphi) = f_a\varphi f_a^{-1}$. $\hat{f}_a$ acts trivially on $\mathrm{End}_l(F(X))$. Thus, $\hat{f}_a$ is linearly stable.

Proposition 1. *Let $\xi \in \mathrm{Aut}(\mathrm{End}(F(X)))$ be a linearly stable automorphism. Then there exists $a \in K$ such that $\xi = \hat{f}_a$.*

Proof. Let $\xi(c_{[x_1,x_2]}) = c_{a[x_1,x_2]}$. Take the scalar automorphism f_a corresponding to the element a. Consider the bijection $F(X) \to F(X)$ which multiplies a monomial of the length n on a^{n-1}. Suppose p is a polynomial presented as the sum of its homogeneous components: $p = p_1 + \cdots + p_s$, where $\deg(p_i) = i$ or $p_i = 0$. Denote $p_1 + ap_2 + \cdots + a^{s-1}p_s$ by $\bar{p}$. Let us prove that $\xi(c_p) = c_{\bar{p}}$ for any $c_p \in C_F$ by induction on the number r of monomials of the polynomial p.

Let us prove the base of the induction for $r = 1$ by induction on the degree of the monomial $p = u$. The base of this induction follows from the Lemma 3.5. Suppose that $\xi(c_u) = c_{a^{l-1}u}$ for any u such that $|u| = l < k$. Suppose now that $|u| = k$ and $u = [u_1, u_2]$, where $|u_1| = k_1$ and $|u_2| = k_2$. Let $\varphi(x_1) = u_1$, $\varphi(x_2) = u_2$ and $\varphi(x) = x$ for any other $x \in X$. Then $\varphi c_{[x_1,x_2]} = c_u$. It follows from the assumption of the induction that $\xi(c_{u_1}) = c_{a^{k_1-1}u_1}$, $\xi(c_{u_2}) = c_{a^{k_2-1}u_2}$ and $\xi(c_x) = c_x$ for any $x \in X$. Therefore, it follows from Lemma 3.4 that $\xi(\varphi)(x) = \overline{\varphi(x)}$ for any $x \in X$. Hence,

$$\xi(c_u) = \xi(\varphi c_{[x_1,x_2]}) = \xi(\varphi)c_{a[x_1,x_2]} = c_{\xi(\varphi)(a[x_1,x_2])} = c_{a[a^{k_1-1}u_1, a^{k_2-1}u_2]} = c_{a^{k-1}u} = c_{\bar{u}}.$$

Thus, we proved the basis of the first induction for $r = 1$.

Suppose that $\xi(c_p) = c_{\bar{p}}$ for any $p \in F(X)$ which contains less than k monomials. Suppose now that p contains k monomials and $p = q + g$ where each of the polynomials q

and g contains less than k monomials. It follows from the assumption of the induction that $\xi(c_q) = c_{\bar q}$, $\xi(c_g) = c_{\bar g}$ and $\xi(c_x) = c_x$ for any $x \in X$. Hence, it follows from Lemma 3.6 that $\xi(c_p) = \xi(c_{q+g}) = c_{\bar q + \bar g} = c_{\bar p}$. Thus, $\xi(c_p) = c_{\bar p}$ for any $p \in F(X)$.

In particular, $c_{\xi(\varphi)(x)} = \xi(\varphi c_x) = \xi(c_{\varphi(x)}) = c_{\overline{\varphi(x)}}$ for every $\varphi \in \mathrm{End}(F(X))$. Thus, $\xi(\varphi)(x) = \overline{\varphi(x)}$. On the other hand it is easy to see that $\hat f_a(\varphi)(x) = \overline{\varphi(x)}$. Therefore, $\xi(\varphi)(x) = \hat f_a(\varphi)(x)$. This equality holds for every $x \in X$ and every $\varphi \in \mathrm{End}(F(X))$. Thus, $\xi = \hat f_a$. □

Corollary 3.8. *Any linearly stable automorphism of* $\mathrm{End}(F(X))$ *is inner.*

Remark 3.9. (1) Let $|X| > 2$. Consider the automorphism $\varphi \in \mathrm{Aut}(F(X))$ defined by $\varphi(x) = x + [y,z]$, φ acts identically on the rest of variables from X. Then $\hat f_a(\varphi)(x) = x + a[y,z]$. Therefore, the linearly stable automorphism $\hat f_a$ for $a \neq 1$ does not act identically on the group $\mathrm{Aut}(F(X))$.

(2) Let $X = \{x, y\}$. Then $\mathrm{Aut}_l(F(X)) = \mathrm{Aut}(F(X))$ [8]. In this case the automorphism $\hat f_a$ acts identically on the group $\mathrm{Aut}(F(X))$. However, if $a \neq 1$ then $\hat f_a$ does not act identically on the semigroup $\mathrm{End}(F(X))$. Indeed, take an endomorphism φ such that $\varphi(x) = [x,y]$. Then $\hat f_a(\varphi)(x) = a[x,y]$.

Proposition 2. *Let* $|X| > 2$. *If* ξ *is a stable automorphism* (*acts identically on* $\mathrm{Aut}(F(X))$) *then* ξ *acts identically on* $\mathrm{End}(F(X))$.

Proof. It follows from Remark 3.9 that if $|X| > 2$ then $\xi = \hat f_a$ is a stable automorphism if and only if $a = 1$. This proves the proposition. □

We say that an automorphism f of $\mathrm{Aut}(F(X))$ is an extendable automorphism if there exists an automorphism g of $\mathrm{End}(F(X))$ whose restriction to $\mathrm{Aut}(F(X))$ is f. It is obvious that all extendable automorphisms of $\mathrm{Aut}(F(X))$ form a subgroup of $\mathrm{Aut}(\mathrm{Aut}(F(X)))$. This subgroup is the image $\mathrm{Im}(\tau)$ of a homomorphism τ defined in the introduction.

Corollary 3.10. *If* $|X| > 2$ *then the group* $\mathrm{Aut}(\mathrm{End}(F(X)))$ *is isomorphic to the subgroup of* $\mathrm{Aut}(\mathrm{Aut}(F(X)))$ *consisting of all extendable automorphisms.*

4. Quasi-stable automorphisms of End($F(X)$)

In this section we define the notion of the quasi-stable automorphism of $\mathrm{End}(F(X))$ and prove that any quasi-stable automorphism of $\mathrm{End}(F(X))$ is inner.

Define, first, the diagonal automorphisms of the group $\mathrm{Aut}_l(F(X))$. We proceed from the canonical isomorphism $\delta : \mathrm{Aut}_l(F(X)) \to \mathrm{GL}_n(P)$, $n = |X|$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_l\big(F(X)\big) & \xrightarrow{\delta} & \mathrm{GL}_n(P) \\ & \searrow^{h} & \downarrow{h_1} \\ & & P^*. \end{array}$$

Define $\tilde{h}_1(A) = h_1(A)A$, for every matrix $A \in \mathrm{GL}_n(P)$. Then $\tilde{h}_1$ is an automorphism of $\mathrm{GL}_n(P)$. It corresponds to the automorphism $\tilde{h}$ of $\mathrm{Aut}_l(F(X))$ defined by $\tilde{h} = \delta^{-1}\tilde{h}_1\delta$. It is easy to see that $\tilde{h}(g)(x) = h(g)g(x)$ for every $g \in \mathrm{Aut}_l(F(X))$ and every $x \in X$. The automorphisms $\tilde{h}$ and $\tilde{h}_1$ are called diagonal automorphisms.

Definition 4.1. An automorphism $\xi \in \mathrm{Aut}(\mathrm{End}(F(X)))$ is called quasi-stable if the group $\mathrm{Aut}_l(F(X))$ is invariant in respect to ξ and the restriction of ξ to $\mathrm{Aut}_l(F(X))$ is a diagonal automorphism.

Denote the restriction of ξ to $\mathrm{Aut}_l(F(X))$ by $\tau_0(\xi)$. Then $\tau_0(\xi) = \tilde{h}$ for some homomorphism $h : \mathrm{Aut}_l(F(X)) \to P^*$.

Consider the case $X = \{x, y\}$. In this case $\mathrm{Aut}(F(X)) = \mathrm{Aut}_l(F(X))$ (see [8]). Thus, we do not need the assumption that ξ leaves $\mathrm{Aut}_l(F(X))$ invariant, and $\tau_0(\xi) = \tau(\xi)$ where $\tau : \mathrm{Aut}(\mathrm{End}(F)) \to \mathrm{Aut}(\mathrm{Aut}(F))$ is the homomorphism defined in the introduction.

Remark 4.2. For the case $|X| > 2$ we also could proceed from the homomorphism $h : \mathrm{Aut}(F(X)) \to P^*$. However, in this case the corresponding $\tilde{h}$ is not an automorphism of $\mathrm{Aut}(F(X))$.

Let ξ be a quasi-stable automorphism of $\mathrm{End}(F(X))$. Let $\tau_0(\xi)$ be equal to $\tilde{h}$. It means that for any $g \in \mathrm{Aut}_l(F(X))$ we have $\xi(g) = h(g)g$.

Let $x \in X$. Denote the automorphism defined by the mapping $g(x) = y$, $g(y) = x$ and $g(z) = z$ for the rest of elements of X by g_{xy}. Denote $h(g_{xy}) = a_{xy}$. Notice that $g_{xy}^2 = e$. Therefore, $a_{xy}^2 = 1$. Since P does not contain zero divisors $a_{xy} = \pm 1$. Denote $\varphi \in \mathrm{End}_l(F(X))$ such that $\varphi(x) = x$ and $\varphi(y) = a_{xy}x$ for any other $y \in X$ by l.

Lemma 4.3. *Let ξ be a quasi-stable automorphism of* $\mathrm{End}(F(X))$. *Then $\xi(c_u) = c_v l$, where $l \in \mathrm{End}_l(F(X)$ is defined above. Thus, $\xi(C_F) = C_F l$.*

Proof. Let $g_{xy} \in \mathrm{Aut}(F)$ and $\xi(g_{xy}) = a_{xy}g_{xy}$, $c_u = c_u g_{xy}$, $\xi(c_u) = \xi(c_u)a_{xy}g_{xy}$. Let $\xi(c_u)(x) = v$. Therefore, for any $y \in X$, $\xi(c_u)(y) = a_{xy}\xi(c_u)(x) = c_v l(y)$. Thus, $\xi(c_u) = c_v l$. $\square$

Lemma 4.4. *Let ξ be a quasi-stable automorphism of* $\mathrm{End}(F(X))$. *Then for any $y \in X$ we obtain that $\xi(c_y) = c_{a_{xy}^{-1}y} l$. In particular, $\xi(c_y)(z) = a_{xz}a_{xy}^{-1}y$.*

Proof. Let $c_y \in C_X$. It follows from the previous lemma that $\xi(c_y) = c_p l$ for some $p \in F(X)$. Let $p = ay + b_1 y_1 + \cdots + b_k y_k + p_1$, where p_1 is a non-linear polynomial. Since c_y is a right identity in the semigroup C_F for any $c_y \in C_X$ we obtain that $\xi(c_y)$ is a right identity for $C_F l$. Hence, $c_y l \xi(c_y) = c_y l$. In particular $(a_{xy} a + a_{xy_1} b_1 + \cdots + a_{xy_k} b_k) y = c_y l \xi(c_y)(x) = c_y l(x) = y$. Therefore $a_{xy} a + a_{xy_1} b_1 + \cdots + a_{xy_k} b_k = 1$.

Suppose that $a \neq 0$. Let g_{my} be the automorphism defined by the mapping $g_{my}(y) = y$ and $g_{my}(z) = mz$ for $z \neq y$, $z \in X$ and let $h(g_{my}) = d_m \neq 0$. Since $g_{my} c_y = c_y$ we obtain $d_m g_{my} c_p(x) = \xi(g_{my} c_y)(x) = \xi(c_y)(x) = c_p(x)$. Hence, $d_m g_{my}(p) = p$. Remind that we present p in the basis of Hall of linearly independent elements. Thus, we obtain a system of equalities for corresponding coefficients. $d_m a = a, d_m m b_i = b_i, \ldots$. Since $a \neq 0$ we obtain that $d_m = 1$. Choosing m of infinite order we obtain that all other coefficients of p are equal to 0. Hence $p = ay$ and $a_{xy} a = 1$. Thus, in this case $\xi(c_y) = c_{1/a_{xy} y} l$.

Suppose now that $a = 0$. Then $a_{xy_1} b_1 + \cdots + a_{xy_k} b_k = 1$ and $p = b_1 y_1 + \cdots + b_k y_k + p_1$. Since $a_{xy_1} b_1 + \cdots + a_{xy_k} b_k = 1$ at least one of the coefficients $b_1, \ldots, b_k$ is non zero. Let $m, a_1, \ldots, a_k \in P$. Let g'_{my} be the automorphism generated by the mapping $g'_{my}(y) = y$, $g'_{my}(y_i) = m y_i + a_i y$ and $g'_{my}(z) = mz$ for other $z \in X$ and let $h(g'_{my}) = d'_m \neq 0$. Then $g'_{my} c_y = c_y$. Therefore, $d'_m g'_{my} c_p(x) = \xi(g'_{my} c_y)(x) = \xi(c_y)(x) = c_p(x)$. Hence, $d'_m g'_{my}(p) = p$. Comparing the coefficients of y we obtain that $d'_m(b_1 a_1 + \cdots + b_k a_k) = 0$. Since we always can choose $a_1, \ldots, a_k \in P$ such that $b_1 a_1 + \cdots + b_k a_k \neq 0$ we obtain that $d'_m = 0$. But this is impossible. Thus, $\xi(c_y) = c_{1/a_{xy} y} l$.

In particular $\xi(c_y)(z) = c_{1/a_{xy} y} l(z) = a_{xz} a_{xy}^{-1} y$. □

Proposition 3. *Any quasi-stable automorphism ξ of* $\mathrm{End}(F(X))$ *is inner.*

Proof. Remind that $l : X \to X$ is a function, assigning x to x, $a_{xz} z$ to z for $x, z \in X$, $a_{xz} \in P^*$. l defines the automorphism α of $F(X)$. α defines the inner automorphism ξ_1 of $\mathrm{End}(F(X))$. Denote $\xi_2 = \xi \xi_1^{-1}$. To prove the proposition it is enough to prove that ξ_2 acts identically on $\mathrm{End}_l(F(X))$ and then use Corollary 3.8.

$\xi_1(c_y)(z) = \alpha^{-1} c_y \alpha(z) = \alpha^{-1}(a_{xz} y) = a_{xz} a_{xy}^{-1} y = \xi(c_y)(z)$ (Lemma 4.4). Hence, $\xi_2 = \xi \xi_1^{-1}$ acts identically on C_X.

Suppose that $g \in \mathrm{Aut}_l(F(X))$ is presented by a diagonal matrix that is $g(y) = a_y y$ for any $y \in X$ and $a_y \in P^*$. Suppose that $\xi(\alpha) = s\alpha$ and $\xi(g) = tg$, where $s, t \in P^*$. Then $\xi_2(g)(y) = \xi \xi_1^{-1}(g)(y) = \xi(\alpha g \alpha^{-1})(y) = s a_{xy} t a_y s^{-1} a_{xy}^{-1} y = t a_y y = \xi(g)(y)$. Therefore $\xi_2(g) = \xi(g)$. Since we use below in the proof of proposition only linear automorphisms presented by a diagonal matrix we refer to this equality without explanation.

Let p be a linear polynomial. Let g_p be the automorphism defined by the mapping $g(x) = p$ and $g(y) = y$ for $y \neq x$, $y \in X$. Then $\xi_2(c_p) = \xi_2(g c_x) = \xi_2(g) \xi_2(c_x) = \xi(g) c_x = h(g) g c_x = c_{h(g) g(x)} = c_{h(g) p}$. Thus $\xi_2(c_p) = c_{kp}$, where $k \in P^*$.

Let g_{mx} be the automorphism generated by the mapping $g_{mx}(x) = x$ and $g_{mx}(y) = my$ for $y \neq x$, $y \in X$ and let $h(g_{mx}) = d_m$. Since $g_{mx} c_x = c_x$ we obtain that $c_x = \xi_2(c_x) = \xi_2(g_{mx} c_x) = \xi(g_{mx}) c_x = d_m g_{mx} c_x = c_{d_m g_{mx}(x)} = c_{d_m x}$. Therefore $d_m = 1$. Let p be a linear polynomial. Let a be the sum of all coefficients of p. Then $\xi_2(c_x c_p) = \xi_2(c_{ax}) = \xi_2(c_x g_{ax} c_y) = c_x g_{ax} c_y = c_{ax}$. On the other hand $\xi_2(c_x c_p) = \xi_2(c_x) \xi_2(c_p) = c_x c_{kp} = c_{akx}$. If $a \neq 0$ then $k = 1$, that is $\xi_2(c_p) = c_p$. Thus ξ_2 acts identically on each c_p, where p is a linear polynomial with non-zero sum of coefficients.

Now consider the case when the sum of all coefficients a of a linear polynomial p is zero. Let $p = ax + b_1y_1 + \cdots + b_ky_k$. Let $g = g_{a_1,\dots,a_k}$ be the automorphism generated by the mapping $g(x) = x$, $g(y_1) = a_1y_1, \dots, g(y_k) = a_ky_k$ and $g(z) = z$ for other $z \in X$ and let $h(g) = h(g_{a_1,\dots,a_k}) = d_{a_1,\dots,a_k}$. Choose $a_1, \dots, a_k$ such that the sum of all coefficients b of the linear polynomial $g_{a_1,\dots,a_k}(p)$ is non zero. We have proved above that $\xi_2(g_{a_1,\dots,a_k}(p)) = g_{a_1,\dots,a_k}(p)$. Since $g_{a_1,\dots,a_k}c_x = c_x$ we obtain that

$$c_x = \xi_2(c_x) = \xi_2(g_{a_1,\dots,a_k}c_x) = \xi(g_{a_1,\dots,a_k})\xi_2(c_x) = d_{a_1,\dots,a_k}g_{a_1,\dots,a_k}c_x = c_{d_{a_1,\dots,a_k}x}.$$

Therefore, $d_{a_1,\dots,a_k} = 1$. Hence, $\xi_2(g_{a_1,\dots,a_k}) = g_{a_1,\dots,a_k}$. Consequently $g_{a_1,\dots,a_k}c_p = \xi_2(g_{a_1,\dots,a_k}c_p) = g_{a_1,\dots,a_k}\xi_2(c_p)$. Since $g_{a_1,\dots,a_k}$ is an automorphism of $F(X)$ we obtain that $\xi_2(c_p) = c_p$.

Thus, ξ_2 acts identically on C_l. Let $\varphi \in \mathrm{End}_l(F(X))$. $c_{\varphi(x)} = \xi_2(c_{\varphi(x)}) = \xi_2(\varphi c_x) = \xi_2(\varphi)\xi_2(c_x) = \xi_2(\varphi)c_x = c_{\xi_2(\varphi)(x)}$. Thus, $\xi_2(\varphi)(x) = \varphi(x)$. Hence, $\xi_2(\varphi) = \varphi$. $\square$

Regarding the material of Section 4 see also [13].

5. Automorphisms of End($F(X)$)

In this section we prove that any automorphism of $\mathrm{End}(F(x, y))$ is semi-inner.

Theorem 2. *Any automorphism of* $\mathrm{End}(F(x, y))$ *is a semi-inner automorphism.*

Proof. P. Cohn [7] proved that the group $\mathrm{Aut}(F(X))$ is generated by linear and triangular automorphisms. Triangular automorphisms assign $ax_i + f(x_1, \dots, x_{i-1})$ to x_i. Hence, for $X = \{x, y\}$ (X consists of two elements), a triangular automorphism assigns $ay + f(x)$ to y. A Lie polynomial of one variable is a linear polynomial. Therefore, the group $\mathrm{Aut}(F(x, y))$ consists of linear automorphisms only. Thus, $\mathrm{Aut}(F(x, y))$ is isomorphic to $\mathrm{GL}_2(P)$. Let $\delta : \mathrm{Aut}(F(x, y)) \to \mathrm{GL}_2(P)$ be an isomorphism. Then $\nu \to \delta^{-1}\nu\delta$ defines an isomorphism $\mathrm{Aut}(\mathrm{GL}_2(P)) \to \mathrm{Aut}(\mathrm{Aut}(F(x, y)))$. If $\nu \in \mathrm{Aut}(\mathrm{GL}_2(P))$ is semi-inner then $\delta^{-1}\nu\delta$ is a semi-inner automorphism of $\mathrm{Aut}(F(x, y))$. If $\nu \in \mathrm{Aut}(\mathrm{GL}_2(P))$ is diagonal then $\delta^{-1}\nu\delta$ is a diagonal automorphism of $\mathrm{Aut}(F(x, y))$. It is well known [24] that the group of automorphisms of $\mathrm{GL}_2(P)$ is generated by semi-inner and diagonal automorphisms. Hence, the group of automorphisms of $\mathrm{Aut}(F(x, y))$ is generated by semi-inner and diagonal automorphisms.

Let ξ be an automorphism of $\mathrm{End}(F(X))$. In the introduction we defined a homomorphism $\tau : \mathrm{Aut}(\mathrm{End}(F(X))) \to \mathrm{Aut}(\mathrm{Aut}(F(X)))$, where $\tau(\xi) = \xi^\tau$ is the restriction of ξ to $\mathrm{Aut}(F(X))$. Hence, $\tau(\xi)$ is a product of semi-inner and diagonal automorphisms of $\mathrm{Aut}(F(X))$. Since a diagonal automorphism commutes with a semi-inner automorphism we obtain $\tau(\xi) = \xi_s\xi_d$, where ξ_s is a semi-inner automorphism and ξ_d is a diagonal automorphism of $\mathrm{Aut}(F(X))$. ξ_s is a semi-inner automorphism of $\mathrm{Aut}(F(X))$ defined by a semi-automorphism f of $F(X)$. f defines an automorphism ξ_1 of $\mathrm{End}(F(X))$. Thus, $\xi_s = \xi_1^\tau$. Then $\xi_1^{-1}\xi = \xi_2$ is an automorphism of $\mathrm{End}(F(X))$ and $\xi_2^\tau = \xi_d$. Thus, ξ_2 is a quasi-stable automorphism. It follows from Proposition 3 that ξ_2 is inner. Consequently $\xi = \xi_1\xi_2$ is semi-inner as a product of semi-inner automorphisms. □

Now we formulate an application of Theorem 2. First, we consider two problems. Let F_1 and F_2 be free Lie algebras over a field P. Suppose that the semigroups of endomorphisms $\mathrm{End}(F_1)$ and $\mathrm{End}(F_2)$ are isomorphic. Does this imply the isomorphism of algebras F_1 and F_2? Let ξ be an isomorphism of $\mathrm{End}(F_1)$ and $\mathrm{End}(F_2)$. In which cases one can state that there exists an isomorphism or a semi-isomorphism $f : F_1 \to F_2$ which induces ξ, i.e., $\xi(\varphi) = f\varphi f^-$ for every $\varphi \in \mathrm{End}(F_1)$? If f induces ξ then the pair (f, ξ) defines the isomorphism or semi-isomorphism of the actions of the semigroup $\mathrm{End}(F_1)$ on F_1 and $\mathrm{End}(F_2)$ on F_2.

Proposition 4. *If $F_1 = F(x, y)$ then for any isomorphism $\xi : \mathrm{End}(F_1) \to \mathrm{End}(F_2)$ there exists a semi-isomorphism $f : F_1 \to F_2$ which induces ξ.*

Proof. Show that there exists an isomorphism $F_1 \to F_2$. We have to show that if F_2 is freely generated by a set Y then $|Y| = 2$. Isomorphism ξ induces an automorphism of groups $\mathrm{Aut}(F_1)$ and $\mathrm{Aut}(F_2)$. The group $\mathrm{Aut}(F_1)$ contains a non-trivial center consisting of scalar automorphisms. If $|Y| \neq 2$ then the group $\mathrm{Aut}(F_2)$ does not possess such a property. Thus, $|Y| = 2$ and there is an isomorphism $f_1 : F_1 \to F_2$. We change this f_1 in order to get a desired semi-isomorphism f. Define $\hat{f}_1 : \mathrm{End}(F_1) \to \mathrm{End}(F_2)$ by the rule $\hat{f}_1(\varphi) = f_1\varphi f_1^{-1}$ for every $\varphi \in \mathrm{End}(F_1)$. The product $\hat{f}_1^{-1}\xi$ is an automorphism of the semigroup $\mathrm{End}(F_1)$. Using Theorem 2 we get that this automorphism is semi-inner. Thus, $\hat{f}_1^{-1}\xi = \hat{g}$, where g is a semi-automorphism of the algebra F_1. Now $\xi = \hat{f}_1\hat{g}$. Semi-isomorphism $f = f_1 g : F_1 \to F_2$ induces the initial ξ. □

Problem 5.1. Does Theorem 2 admit a generalization for the case of arbitrary X, $|X| \geqslant 2$?

Proposition 5. *The following conditions on a free Lie algebra $F(X)$ are equivalent*

(1) *Any automorphism of* $\mathrm{End}(F(X))$ *is semi-inner.*
(2) *For any automorphism ξ of* $\mathrm{End}(F(X))$ *the group* $\xi(\mathrm{Aut}_l(F(X)))$ *is conjugated to* $\mathrm{Aut}_l(F(X))$ (*in the group* $\mathrm{Aut}(F(X))$).

Proof. $1 \Rightarrow 2$. There exists a semi-inner automorphism (σ, g) of $F(X)$ such that for any $\varphi \in \mathrm{End}(F(X))$ $\xi(\varphi) = g\sigma\varphi\sigma^{-1}g^{-1}$. For any $\alpha \in \mathrm{Aut}_l(F(X))$ we have $\sigma\alpha\sigma^{-1} \in \mathrm{Aut}_l(F(X))$. Therefore, $\mathrm{Aut}_l(F(X))$ and $\xi(\mathrm{Aut}_l(F(X)))$ are conjugated by $g \in \mathrm{Aut}(F(X))$.

$2 \Rightarrow 1$. Let ξ be an automorphism of $\mathrm{End}(F(X))$. $\mathrm{Aut}_l(F(X))$ and $\xi(\mathrm{Aut}_l(F(X)))$ are conjugated by $g \in \mathrm{Aut}(F(X))$. g defines an inner automorphism $\hat{g}$ of $\mathrm{End}(F(X))$. $\xi\hat{g}^{-1} = \xi_1$ is an automorphism of $\mathrm{End}(F(X))$ which induces an automorphism ξ_2 of $\mathrm{Aut}_l(F(X))$. ξ_2 is semi-inner and, therefore, it is extended to semi-inner automorphism $\hat{\xi}_2$ of $\mathrm{End}(F(X))$. Automorphism $\xi_1\hat{\xi}_2^{-1} = \xi_3$ is a linearly stable automorphism of $\mathrm{End}(F(X))$. Therefore, it is inner (see Corollary 3.8). Hence, $\xi = \xi_1\hat{g} = \xi_3\hat{\xi}_2\hat{g}$ is semi-inner. □

6. Reduction theorem

In this section we prove the general reduction theorem for a large class of varieties Θ. This theorem allows to reduce the problem of the description of automorphisms of the category Θ^0 to the same problem for a much simpler category (consisting of two objects).

We assume that the variety Θ satisfies the following 3 conditions.

(1) The variety Θ is hopfian. This means that every object $F = F(X)$ of the category Θ^0 is hopfian, i.e., every surjective endomorphism $\nu : F \to F$ is an automorphism.
(2) If $X = \{x_0\}$ is a one element set and $F_0 = F(x_0)$ is the cyclic free algebra then for every automorphism φ of the category Θ^0 we require $\varphi(F_0)$ is also a cyclic free algebra $F(y_0)$.
(3) We assume that there exists a finitely generated free algebra $F^0 = F(X^0)$, $X^0 = \{x_1, \ldots, x_k\}$, generating the whole variety Θ, i.e., $\Theta = \mathrm{Var}(F^0)$.

For the sake of convenience in this paper we call a variety, satisfying these conditions, a hereditary variety.

We fix F^0 and F_0.

Proposition 6 ([5], see also Appendix). *The conditions* (1) *and* (2) *imply that for every* $F = F(X)$ *and every* $\varphi : \Theta^0 \to \Theta^0$ *the algebras* F *and* $\varphi(F)$ *are isomorphic.*

Lemma 6.1 [5]. *Any automorphism* $\varphi : \Theta^0 \to \Theta^0$ *such that algebras* F *and* $\varphi(F)$ *are isomorphic has the form*

$$\varphi = \varphi_0 \varphi_1,$$

where φ_0 *is an inner automorphism of* Θ^0 *and* φ_1 *does not change objects.*

Consider a constant morphism $\nu_0 : F^0 \to F_0$ such that $\nu_0(x) = x_0$ for every $x \in X^0$.

Theorem 3 (Reduction Theorem). *Let* φ *be an automorphism of the category* Θ^0 *which does not change objects, and let* φ *induce the identity automorphism of the semigroup* $\mathrm{End}(F^0)$ *and* $\varphi(\nu_0) = \nu_0$. *Then* φ *is an inner automorphism.*

Note that for the variety Θ of all commutative associative algebras with 1 over a field this theorem has been proved by A. Berzins in [5].

The proof of the theorem consists of several steps.

(1) It will be convenient to attach to the category Θ^0 the category of affine spaces $K^0_\Theta(H)$ over the algebra $H = F^0$ [31]. The objects of $K^0_\Theta(H)$ have the form $\mathrm{Hom}((F(X), H)$, where F is an object of the category Θ^0. Morphisms

$$\tilde{s} : \mathrm{Hom}\big(F(X), H\big) \to \mathrm{Hom}\big(F(Y), H\big)$$

are defined by morphisms $s : F(Y) \to F(X)$ by the rule $\tilde{s}(\nu) = \nu s$ for every $\nu : F(X) \to H$. We have a contravariant functor $\Phi : \Theta^0 \to K^0_\Theta(H)$. The condition $\mathrm{Var}(H) = \Theta$ implies that this functor yields the duality of the categories Θ^0 and $K_\Theta^0(H)$ (see [31] and Appendix). Consider the automorphism φ^H of the category of affine spaces which is the image of φ under the duality above. Functor $\varphi^H : K_\Theta^0(H) \to K_\Theta^0(H)$ does not change objects and for $s : F(Y) \to F(X)$ we define $\varphi^H(\tilde{s}) = \widetilde{\varphi(s)}$. This definition is correct, since $\tilde{s}_1 = \tilde{s}_2$ implies $s_1 = s_2$.

We will show that φ^H is in a certain sense a quasi-inner automorphism. First of all, φ defines a substitution on each set $\mathrm{Hom}(F(X), H)$. Indeed, $\nu : F(X) \to H$ and $\varphi(\nu) : F(X) \to H$ give rise to a substitution μ_X defined by $\mu_X(\nu) = \varphi(\nu)$. The following proposition explains the transition from Θ^0 to $K_\Theta^0(H)$.

Proposition 7. *Let $s : F(Y) \to F(X)$. Then*

$$\varphi^H(\tilde{s}) = \mu_Y \tilde{s} \mu_X^{-1} : \mathrm{Hom}(F(X), H) \to \mathrm{Hom}(F(Y), H).$$

Proof. For every $s : F(Y) \to F(X)$ and every $\nu : F(X) \to H$ the equality $\varphi^H(\tilde{s})(\nu) = \widetilde{\varphi(s)}(\nu) = \nu\varphi(s)$ holds. Therefore, we have

$$\mu_Y \tilde{s} \mu_X^{-1}(\nu) = \mu_Y\big(\mu_X^{-1}(\nu)s\big) = \varphi\big(\varphi^{-1}(\nu)s\big) = \nu\varphi(s) = \varphi^H(\tilde{s})(\nu). \qquad \square$$

Remark 6.2. If the automorphism φ^H has a presentation above we call it quasi-inner.

Consider separately the case $X = X^0$ and take the substitution $\mu_{X^0} : \mathrm{Hom}(F^0, H) \to \mathrm{Hom}(F^0, H)$. By the condition of the theorem the equality $\mu_{X^0}(\nu) = \varphi(\nu) = \nu$ holds for any $\nu : F^0 \to H = F^0$. This means that $\mu_{X^0} = 1$. Then for $s : F(Y) \to F(X^0)$ we have

$$\varphi^H(\tilde{s}) = \mu_Y \tilde{s} \mu_{X^0}^{-1} = \mu_Y \tilde{s} = \widetilde{\varphi(s)}.$$

For $s : F(X^0) \to F(Y)$ we get

$$\varphi^H(\tilde{s}) = \mu_{X^0} \tilde{s} \mu_Y^{-1} = \tilde{s}\mu_Y^{-1} = \widetilde{\varphi(s)}.$$

Therefore, $\tilde{s} = \widetilde{\varphi(s)}\mu_Y$.

(2) Now we use the category of polynomial maps $\mathrm{Pol}_\Theta(H)$. Objects of this category have the form H^n, where n changes and H is fixed. Morphisms are represented by polynomial maps $s^\alpha : H^n \to H^m$ defined below. Take a set $X = \{x_1, \ldots, x_n\}$. Denote $\alpha_X : \mathrm{Hom}(F(X), H) \to H^n$ the canonical bijection defined by $\alpha_X(\nu) = (\nu(x_1), \ldots, \nu(x_n))$ for every $\nu : F(X) \to H$. Let now $s : F(Y) \to F(X)$ be given and $X = \{x_1, \ldots, x_n\}$ $Y = \{y_1, \ldots, y_m\}$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}\big(F(X), H\big) & \xrightarrow{\ \tilde{s}\ } & \mathrm{Hom}\big(F(Y), H\big) \\ \Big\downarrow{\scriptstyle \alpha_X} & & \Big\downarrow{\scriptstyle \alpha_Y} \\ H^n & \xrightarrow{\quad s^\alpha \quad} & H^m, \end{array}$$

where $s^\alpha = \alpha_Y \tilde{s} \alpha_X^{-1}$; $\tilde{s} = \alpha_Y^{-1} s^\alpha \alpha_X$. Then $s^\alpha(a_1, \ldots, a_n) = \alpha_Y \tilde{s} \alpha_X^{-1}(a_1, \ldots, a_n)$. Take a point $\nu = \alpha_X^{-1}(a_1, \ldots, a_n) : F(X) \to H$. Then

$$s^\alpha(a_1, \ldots, a_n) = \alpha_Y \tilde{s}(\nu) = \alpha_Y(\nu s) = \big(\nu s(y_1), \ldots, \nu s(y_m)\big).$$

Denote $s(y_i) = w_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$. We have got

$$s^\alpha(a_1, \ldots, a_n) = \big(w_1(a_1, \ldots, a_n), \ldots, w_m(a_1, \ldots, a_n)\big).$$

Indeed,

$$\begin{aligned} s^\alpha(a_1, \ldots, a_n) &= \big(\nu\big(w_1(x_1, \ldots, x_n)\big), \ldots, \nu\big(w_m(x_1, \ldots, x_n)\big)\big) \\ &= \big(w_1\big(x_1^\nu, \ldots, x_n^\nu\big), \ldots, w_m\big(x_1^\nu, \ldots, x_n^\nu\big)\big) \\ &= \big(w_1(a_1, \ldots, a_n), \ldots, w_m(a_1, \ldots, a_n)\big). \end{aligned}$$

Thus, we defined morphisms $s^\alpha : H^n \to H^m$ in the category $\mathrm{Pol}_\Theta(H)$.

Consider constant morphisms in the category Θ^0. First, take morphisms of the form: $\nu = \nu_a : F_0 \to F(X)$ defined by $\nu_a(x_0) = a$, $a \in F(X)$. Recall that the constant morphism $\nu_0 : F^0 \to F(x_0)$ is defined by $\nu_0(x) = x_0$ for every $x \in X^0$.

Take $\nu = \nu_a \nu_0 : F(X^0) \to F(X)$. Then $\nu(x) = a$ for every $x \in X^0$, and ν is a constant we will be dealing with.

Let, further, φ be an automorphism of Θ^0 which does not change objects. This φ induces a substitution on each set $\mathrm{Hom}(F(X), F(Y))$ denoted by $\mu_{X,Y}$. In particular, $\mu_{X,X^0} = \mu_X$. The substitution $\mu_{x_0,X}$ on the set $\mathrm{Hom}(F_0, F(X))$ induces the substitution σ_X on the algebra $F(X)$ defined by the rule $\varphi(\nu_a) = \nu_{\sigma_X(a)}$. It is proved [31] that for every $\mu : F(X) \to F(Y)$ the formula

$$\varphi(\mu) = \sigma_Y \mu \sigma_X^{-1}$$

holds. In this sense the automorphism φ is said to be a quasi-inner automorphism in the category Θ^0. Take now $\nu = \nu_a \nu_0$. Then

$$\varphi(\nu) = \varphi(\nu_a)\varphi(\nu_0) = \nu_{\sigma_X(a)}\varphi(\nu_0).$$

If φ does not change ν_0 then

$$\varphi(\nu) = \nu_{\sigma_X(a)}\nu_0.$$

For every $x \in X^0$ we get $\varphi(\nu)(x) = \sigma_X(a)$, here $\varphi(\nu)$ is also a constant.

Now we are in the position to make the next step. We return to the category of polynomial maps and consider how the constant maps defined above in Θ^0 look like in $\mathrm{Pol}_\Theta(H)$. Take $s : H = F(X^0) = F^0 \to F(X)$ defined by $s = s_w \nu_0$, $w \in F(X)$. Then $s(x) = w$ for every $x \in X^0$. Let $X^0 = \{x_1, \ldots, x_k\}$. We get the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\big(F(X), H\big) & \xrightarrow{\tilde{s}} & \mathrm{Hom}\big(F\big(X^0\big), H\big) \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_{X^0}} \\ H^n & \xrightarrow{s^\alpha} & H^k. \end{array}$$

Then $s^\alpha(a_1, \ldots, a_n) = (w(a_1, \ldots, a_n), \ldots, w(a_1, \ldots, a_n))$, where $w(a_1, \ldots, a_n)$ is taken k times. Considering the projection $\pi : H^k \to H$, $\pi(b_1, \ldots, b_k) = b_1$ we get $\pi s^\alpha(a_1, \ldots, a_n) = w(a_1, \ldots, a_n)$.

Take an arbitrary $s : F(Y) \to F(X)$. Let $X = \{x_1, \ldots, x_n\}$, and $Y = \{y_1, \ldots, y_m\}$. Let $s(y_i) = w_i(x_1, \ldots, x_n) = w_i$. Take a constant map $s_i = \nu_{w_i} \nu_0 : F(X^0) \to F(X)$. The sequence $s_1, \ldots, s_m$ depends on s and on the basis of Y. In this situation we denote

$$s =_Y (s_1, \ldots, s_m).$$

We have also $s^\alpha : H^n \to H^m$, and $s_i^\alpha : H^n \to H^k$. There is a relation between s^α and s_i^α, $i = 1, \ldots, m$:

$$s^\alpha(a_1, \ldots, a_n) = \big(\pi s_1^\alpha(a_1, \ldots, a_n), \ldots, \pi s_m^\alpha(a_1, \ldots, a_n)\big).$$

Indeed,

$$\begin{aligned} s^\alpha(a_1, \ldots, a_n) &= \big(w_1(a_1, \ldots, a_n), \ldots, w_m(a_1, \ldots, a_n)\big) \\ &= \big(\pi s_1^\alpha(a_1, \ldots, a_n), \ldots, \pi s_m^\alpha(a_1, \ldots, a_n)\big). \end{aligned}$$

This formula is a key working tool for the proof of the theorem. It was the reason to replace the category of affine spaces by the category of polynomial maps.

Now we are able to prove the reduction theorem.

Proof of Theorem 3. Let us return to the automorphism $\varphi : \Theta^0 \to \Theta^0$. For every algebra $F = F(X)$, $X = \{x_1, \ldots, x_n\}$ we will construct an automorphism $\sigma_X : F \to F$ depending on φ. The collection of such automorphisms will define φ as an inner automorphism.

Consider morphisms $\varepsilon_i = \nu_{x_i} \nu_0 : F^0 \to F$, $i = 1, \ldots, n$. We have $\varphi(\varepsilon_i) = \varphi(\nu_{x_i})\nu_0$, and let $\varphi(\nu_{x_i})(x_0) = y_i = \varphi(\varepsilon_i)(x)$ for every $x \in X^0$. Let $Y = \{y_1, \ldots, y_n\}$. From the proof of Proposition 6 follows that if the variety Θ is hopfian then Y is also a basis in F.

Define the automorphism $\sigma_X : F \to F$ by the rule $\sigma_X(x_i) = y_i$.

Let s be an automorphism of the algebra $F = F(X)$, and let $s(x_i) = w_i(x_1, \ldots, x_n) = w_i$. Take $\nu_{w_i} : F_0 \to F$, $\nu_{w_i} = s\nu_{x_i}$ and let $s_i = \nu_{w_i} \nu_0 = s\nu_{x_i} \nu_0 = s\varepsilon_i$.

We have $s =_X (s_1, \ldots, s_n)$. We will check that $\varphi(s) =_Y (\varphi(s_1), \ldots, \varphi(s_n))$.

We have to verify that if $\varphi(s)(y_i) = w_i'$ then $\varphi(s_i)(x) = w_i'$ for every $x \in X^0$. Compute

$$\varphi(s)(y_i) = \varphi(s)\varphi(\nu_{x_i})(x_0) = \varphi(s\nu_{x_i})(x_0) = \varphi(\nu_{w_i})(x_0)$$
$$= \varphi(\nu_{w_i})\nu_0(x) = \varphi(\nu_{w_i})\varphi(\nu_0)(x) = \varphi(\nu_{w_i}\nu_0)(x) = \varphi(s_i)(x)$$

for every $x \in X^0$. Thus, $\varphi(s)(y_i) = w_i' = \varphi(s_i)(x)$ for every $x \in X^0$.

This implies $\varphi(s)\sigma_X =_X (\varphi(s_1), \dots, \varphi(s_n))$. Indeed, $\varphi(s)\sigma_X(x_i) = \varphi(s)(y_i) = \varphi(s_i)(x)$ for every $x \in X^0$.

Consider the image of the formula $\tilde{s_i} = \widetilde{\varphi(s_i)}\mu_X$, $i = 1, \dots, n$, in the category of polynomial maps $\mathrm{Pol}_\Theta(H)$.

Take the diagram

$$\begin{array}{ccc} \mathrm{Hom}\big(F(X), H\big) & \xrightarrow{\mu_X} & \mathrm{Hom}\big(F(X), H\big) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_X} \\ H^n & \xrightarrow{\mu_X^\alpha} & H^n \end{array}$$

We have got a map $\mu_X^\alpha = \alpha_X\mu_X\alpha_X^{-1} : H^n \to H^n$. In particular, $\mu_{X^0}^\alpha = \alpha_{X^0}\mu_X\alpha_{X^0}^{-1}$. By the condition of theorem, $\mu_{X^0} = 1$ and $\mu_{X^0}^\alpha = 1$.

Since, $\varphi^H(\tilde{s}) = \mu_X\tilde{s}\mu_X^{-1} = \widetilde{\varphi(s)}$ for $\varphi(s) : F(X) \to F(X)$ the following equality holds

$$\varphi(s)^\alpha = \alpha_X\widetilde{\varphi(s)}\alpha_X^{-1} = \alpha_X\mu_X\alpha_X^{-1}\alpha_X\tilde{s}\alpha_X^{-1}\alpha_X\mu_X^{-1}\alpha_X^{-1} = \mu_X^\alpha s^\alpha\big(\mu_X^\alpha\big)^{-1}.$$

For $s_i : F^0 \to F$ we get $\tilde{s_i} = \widetilde{\varphi(s_i)}\mu_X$ and hence,

$$s_i^\alpha = \alpha_{X^0}\tilde{s}_i\alpha_X^{-1} = \alpha_{X^0}\widetilde{\varphi(s_i)}\alpha_X^{-1}\alpha_X\mu_X\alpha_X^{-1} = \varphi(s_i)^\alpha\mu_X^\alpha,$$

where $s_i^\alpha = \varphi(s_i)^\alpha\mu_X^\alpha$ are polynomial mappings from H^n to H^k. For $a = (a_1, \dots, a_n) \in H^n$ we have

$$s^\alpha(a_1, \dots, a_n) = \big(\pi s_1^\alpha(a_1, \dots, a_n), \dots, \pi s_n^\alpha(a_1, \dots, a_n)\big)$$
$$= \big(\pi\varphi(s_1)^\alpha\mu_X^\alpha(a_1, \dots, a_n), \dots, \pi\varphi(s_n)^\alpha\mu_X^\alpha(a_1, \dots, a_n)\big).$$

Take

$$\varphi(s)\sigma_X =_X \big(\varphi(s_1), \dots, \varphi(s_n)\big)$$

and apply this formula to the point $\mu_X^\alpha(a_1, \dots, a_n)$. Then

$$\big(\varphi(s)\sigma_X\big)^\alpha\big(\mu_X^\alpha(a_1, \dots, a_n)\big) = \big(\pi\varphi(s_1)^\alpha\mu_X^\alpha(a_1, \dots, a_n), \dots, \pi\varphi(s_n)^\alpha\mu_X^\alpha(a_1, \dots, a_n)\big)$$
$$= s^\alpha(a).$$

Thus, $s^\alpha(a) = (\varphi(s)\sigma_X)^\alpha \mu_X^\alpha(a)$. Since this formula holds in every point a then

$$s^\alpha = \big(\varphi(s)\sigma_X\big)^\alpha \mu_X^\alpha = \sigma_X^\alpha \varphi(s)^\alpha \mu_X^\alpha.$$

Hence, $\mu_X^\alpha = (\varphi(s)^{-1})^\alpha (\sigma_X^{-1})^\alpha s^\alpha = (s\sigma_X^{-1}\varphi(s)^{-1})^\alpha$.

Denote $\xi_X = s\sigma_X^{-1}\varphi(s)^{-1}$. This is an automorphism of the algebra $F = F(X)$ and $\mu_X^\alpha = \xi_X^\alpha$. Therefore, $\tilde{\xi_X} = \mu_X$. In particular, ξ_X does not depend on the choice of the automorphism s.

Let now an arbitrary $\delta : F(X) \to F(Y)$ be given. Then we have

$$\varphi^H\big(\tilde{\delta}\big) = \mu_X \tilde{\delta} \mu_Y^{-1} = \tilde{\xi_X}\tilde{\delta}\tilde{\xi}_Y^{-1} = \widetilde{\xi_Y^{-1}\delta\xi_X} = \widetilde{\varphi(\delta)}.$$

This gives $\varphi(\delta) = \xi_Y^{-1}\delta\xi_X$.

Since our initial s is arbitrary, one can take $s = 1$. Then $\xi_X = \sigma_X^{-1}$.

Finally we get

$$\varphi(\delta) = \sigma_Y \delta \sigma_X^{-1}. \qquad \square$$

7. Automorphisms of the category of free Lie algebras. The proof of the main theorem

Return to the variety Θ of all Lie algebras over an infinite field. We want to prove that every automorphism of the category Θ^0 is semi-inner.

Proof of Theorem 1. This variety Θ is hopfian and is generated by the free Lie algebra $F^0 = F(x, y)$ [6]. It is clear that condition 2 is also valid. Thus, the conditions from Section 6 are fulfilled. Therefore, the variety Θ is hereditary.

It is enough to consider automorphisms φ which do not change objects [19]. Take such a φ and induce the automorphism φ_{F^0} of the semigroup $\mathrm{End}(F^0)$. According to Theorem 2 such an automorphism is semi-inner and is defined by the semi-automorphism $(\sigma, s_{F^0}) : F^0 \to F^0$. For every algebra $F = F(X)$, which is distinct from F^0 take a semi-automorphism $(\sigma, \sigma_F) : F \to F$. Semi-automorphisms $(\sigma, s)_{F^0} = (\sigma, s_{F_0})$ and $(\sigma, s)_F = (\sigma, \sigma_F)$ define a semi-inner automorphism ψ of the category Θ^0. This ψ does not change objects. Automorphisms φ and ψ act in the same way on the semigroup $\mathrm{End}(F^0)$. Thus, the automorphism $\varphi_1 = \psi^{-1}\varphi$ acts on $\mathrm{End}(F^0)$ identically.

Take a constant morphism $\nu_0 : F^0 \to F_0$ with $\nu_0(x) = \nu_0(y) = x_0$. Let us verify that $\varphi_1(\nu_0)$ is also a constant. Take an automorphism η of the algebra F^0 defined by $\eta(x) = y$, $\eta(y) = x$. We have $\nu_0\eta = \nu_0$. Therefore $\varphi_1(\nu_0\eta) = \varphi_1(\nu_0)\eta = \varphi_1(\nu_0)$. Hence, $\varphi_1(\nu_0)(x) = \varphi_1(\nu_0)\eta(x) = \varphi_1(\nu_0)(y) = ax_0$ for $a \neq 0$.

Automorphisms of free Lie algebras: $f_{F_0}(x_0) = ax_0$ and $f_F(x) = x$ for $x \in X$, $F = F(X) \neq F_0$, define an inner automorphism $\hat{f}$ of the category Θ^0, which does not change objects. Observe that isomorphism f_F acts trivially on F^0. We have $\hat{f}(\nu_0) = f_{F_0}\nu_0 f_{F^0}^{-1}$ and

$$\hat{f}(\nu_0)(x) = f_{F_0}\nu_0 f_{F^0}^{-1}(x) = f_{F_0}\nu_0(x) = f_{F_0}(x_0) = ax_0.$$

Thus, $\varphi_1(\nu_0)$ and $\hat{f}(\nu_0)$ coincide. Therefore $\hat{f}^{-1}\varphi_1(\nu_0) = \nu_0$. Denote $\hat{f}^{-1}\psi^{-1}\varphi = \psi_0$. Then $\psi_0(\nu_0) = \nu_0$ and ψ_0 acts trivially on $\mathrm{End}(F(x, y))$. By Reduction Theorem the automorphism ψ_0 is inner. We have got $\varphi = \psi\hat{f}\psi_0 = \hat{\sigma}\psi_1\hat{f}\psi_0$.

Thus, φ is semi-inner and the theorem is proved. □

Along with the automorphisms of categories of free algebras of varieties it is natural to consider also the autoequivalences of these categories (see Section 1). Let (φ, ψ) be an autoequivalence of the category of free Lie algebras. We call it semi-inner if the functors φ and ψ are semi-isomorphic to the identity functor.

It was proved in [38], that for every Θ and every autoequivalence (φ, ψ) of the category Θ^0 there are factorizations

$$\varphi = \varphi_0\varphi_1, \qquad \psi = \varphi_1^{-1}\psi_0,$$

where φ_0 and ψ_0 are isomorphic to the identity functor and φ_1 is an automorphism.

This means that every autoequivalence is isomorphic to an automorphism. This remark and Theorem 1 lead to the following statement:

Theorem 4. *Every autoequivalence of the category of free Lie algebras is semi-inner.*

In the introduction we discussed the categories of algebraic sets $K_\Theta(H)$, $H \in \Theta$. The two problems were pointed out, namely, the problem of isomorphism of categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ and the problem of equivalence of the same categories. For the variety of Lie algebras and algebras H_1 and H_2, satisfying $\mathrm{Var}(H_1) = \mathrm{Var}(H_2) = \Theta$, the first problem is solved in [32] with the aid of Theorem 1, while the solution of the second problem also in [32] requires arguments from Theorem 4.

Appendix

In this section we prove two propositions we have referred to in the text.

Remind that the contravariant functor $\Phi : \Theta^0 \to K_\Theta^0(H)$ assign $\mathrm{Hom}((F(X), H)$ to any $F(X) \in \Theta^0$ and for any $s : F(Y) \to F(X)$ it assigns $\tilde{s} : \mathrm{Hom}(F(X), H) \to \mathrm{Hom}(F(Y), H)$ defined by the rule $\tilde{s}(\nu) = \nu s$.

Proposition 8. *The functor $\Phi : \Theta^0 \to K_\Theta^0(H)$ defines a duality of categories if and only if* $\mathrm{Var}(H) = \Theta$.

Proof. In our case the duality of categories means that if s_1, s_2 morphisms $F(Y) \to F(X)$ then $s_1 \neq s_2$ implies $\tilde{s}_1 \neq \tilde{s}_2$. Let $s_1 \neq s_2$ and assume $\tilde{s}_1 = \tilde{s}_2$. Take $y \in Y$ such that $s_1(y) =$

w_1, $s_2(y) = w_2$, and $w_1 \neq w_2$. Show that the non-trivial identity $w_1 \equiv w_2$ is fulfilled in algebra H. Take an arbitrary $\nu : F(X) \to H$. The equality $\tilde{s}_1 = \tilde{s}_2$ yields $\tilde{s}_1(\nu) = \tilde{s}_2(\nu)$. We have also $\nu s_1 = \nu s_2$. Apply this equality to y. We get $\nu s_1(y) = \nu(w_1) = \nu s_2(y) = \nu(w_2)$. Since ν is arbitrary, we get that $\tilde{s}_1 = \tilde{s}_2$ implies $w_1 \equiv w_2$ in H.

Assume that $\mathrm{Var}(H) = \Theta$. Then there are no non-trivial identities in H. This means that the equality $\tilde{s}_1 = \tilde{s}_2$ does not hold in $K^0_\Theta(H)$. We proved that if $\mathrm{Var}(H) = \Theta$ then $s_1 \neq s_2$ implies $\tilde{s}_1 \neq \tilde{s}_2$ and we get a duality of categories.

Conversely, let us prove that if $\mathrm{Var}(H) \neq \Theta$ then there is no duality. Since $\mathrm{Var}(H) \neq \Theta$ there exists a non-trivial identity $w_1 \equiv w_2$ in H, where w_1, w_2 in some $F(X)$. Take $Y = \{y_0\}$. Consider s_1 and s_2 from $F(Y)$ to $F(X)$ defined by the rule: $s_1(y_0) = w_1$, $s_2(y_0) = w_2$. Show that $\tilde{s}_1 = \tilde{s}_2$. This will mean that there is no duality. Take an arbitrary $\nu : F(X) \to H$. Then $\tilde{s}_1(\nu) = \nu s_1$, $\tilde{s}_2(\nu) = \nu s_2$, both $F(Y) \to H$. Take y_0. Then $\nu s_1(y_0) = \nu(w_1)$, $\nu s_1(y_0) = \nu(w_2)$. Since $w_1 \equiv w_2$ is an identity in H then $\nu(w_1) = \nu(w_2)$ and correspondingly, $\nu s_1(y_0) = \nu s_2(y_0)$. Since the set Y consists of one element y_0 then $\nu s_1 = \nu s_2$. This equality holds for every ν and therefore, $\tilde{s}_1 = \tilde{s}_2$. □

Proposition 9. *Let the variety Θ be hopfian, φ an automorphism of Θ^0 and $\varphi(F_0)$ be isomorphic to F_0. Then $\varphi(F)$ is isomorphic to F for every $F = F(X)$.*

We use some new notions to prove Proposition 9.

Definition 7.1. Let X be a set in a free algebra $F = F(Y)$. We say that X defines freely algebra F if every map $\mu_0 : X \to F$ can be extended uniquely up to endomorphism $\mu : F \to F$.

Remark 7.2. In many cases the notions "to define freely" and "to generate freely" coincide. For instance, this is true for the variety of all groups (E. Rips, unpublished). If this is true for the variety of Lie algebras we do not know.

Lemma 7.3. *Let the variety Θ be hopfian. Let $|X| \geqslant |Y|$. Then X defines freely $F = F(Y)$ if and only if X is a basis in F and $|X| = |Y|$.*

Proof. Take an arbitrary surjection $\mu_0 : X \to Y$. If X defines freely F then there exists surjective endomorphism $\mu : F \to F$. Since F is hopfian, μ is automorphism. Then μ_0 is a bijection. The inverse bijection defines the inverse automorphism. Therefore, $|X| = |Y|$ and X is a basis in F. The "only if" part is evident. □

For the sake of self-completeness we repeat some material from [5]. Take a free algebra $F = F(X)$ and consider a system of morphisms $\varepsilon_i : F_0 \to F$, $i = 1, \dots, n$.

Definition 7.4. A system of morphisms $(\varepsilon_1, \dots, \varepsilon_n)$ defines freely an algebra F if for every sequence of homomorphisms $f_1, \dots, f_n$, $f_i : F_0 \to F$ there exists a unique endomorphism $s : F \to F$ such that $f_i = s\varepsilon_i$ where $i = 1, \dots, n$.

It is proved in [5] that the system $(\varepsilon_1, \dots, \varepsilon_n)$ defines freely F if and only if the system of elements $(\varepsilon_1(x_0), \dots, \varepsilon_n(x_0))$ defines freely the algebra F. It is obvious, that if the system $(\varepsilon_1, \dots, \varepsilon_n)$ defines freely F then the system $(\varphi(\varepsilon_1), \dots, \varphi(\varepsilon_n))$ defines freely $\varphi(F)$ if $\varphi(F_0) = F(y_0)$.

Proof of Proposition 9. Let the variety Θ be hopfian, $\varphi(F_0) = F(y_0)$. Let $\varphi(F) = F(Y)$ where $F = F(X)$. We prove that algebras $F(X)$ and $F(Y)$ are isomorphic.

Let, first, $|X| \geqslant |Y|$ and $X = \{x_1, \dots, x_n\}$. Define the system $(\varepsilon_1, \dots, \varepsilon_n)$ by the condition $\varepsilon_i(x_0) = x_i$, for every i. This system defines freely algebra F. Then the system $(\varphi(\varepsilon_1), \dots, \varphi(\varepsilon_n))$ defines freely algebra $\varphi(F) = F(Y)$. This means that the set Y' of elements $y_i' = \varphi(\varepsilon_i)(y_0)$ defines freely algebra $F(Y)$. Since $|Y'| = |X| \geqslant |Y|$ then the system Y' is a basis in $F(Y)$ and $|Y'| = |X| = |Y|$. The map $x_i \to y_i'$ defines the isomorphism of algebras $F(X)$ and $F(Y)$.

Let now $|X| < |Y|$. Then $F(X) = \varphi(F(Y))^{-1}$. Applying the same method to the automorphism φ^{-1} we get the contradiction. □

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