VISUAL BASIC REPRESENTATIONS: AN ATLAS

EUGENE PLOTKIN[*,‡], ANDREI SEMENOV[*,†], NIKOLAI VAVILOV[*,†]

[*] *Fakultät für Mathematik*
Universität Bielefeld
33615 Bielefeld, Deutschland

[†] *Department of Mathematics and Mechanics*
University of Sanct-Petersburg
Petrodvorets, 198904, Russia

[‡] *Department of Mathematics and Computer Science*
Bar Ilan University
Ramat Gan, 52900, Israel

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We depict the weight diagrams (alias, crystal graphs) of basic and adjoint representations of complex simple Lie algebras/algebraic groups and describe some of their uses.

0. Introduction

In this paper we collect the weight diagrams (alias, crystal graphs) of basic representations of complex simple Lie algebras as well as of those adjoint representations, which are not basic. These pictures arise in a number of contexts, but their main significance stems from the fact that they allow the visualization of calculations with root systems, Weyl groups, Lie algebras, Chevalley groups and their representations, to a large extent replacing (or sometimes enhancing) such tools as calculations with matrices or Bruhat decompositions.

These pictures and related combinatorial objects appeared dozens (hundreds?) of times in various contexts, such as representation theory and structure theory of semisimple Lie algebras, algebraic groups and Lie groups, invariant theory, algebraic K-theory, combinatorial geometries, Schubert calculus, Jordan systems, Hermitian symmetric spaces, combinatorics, computer algebra, etc. Our primary emphasis in this paper are the pictures themselves, rather than their uses. In this sense it is a *pendant* to [120, 121] which contain a detailed description of significance of the

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pictures and various numerical invariants connected with them, as well as proofs of some results quoted here *en passant*.

The paper is organized as follows. In Sec. 1 we recall the notation. In Sec. 2 we define weight diagrams, discuss various ways to draw them and explain how the diagrams in the atlas were constructed. Finally, Sec. 3 discusses some of the applications of the pictures referring to [5, 11, 12, 27, 30, 36, 37, 44, 45, 47, 49–51, 55, 81, 85–87, 100, 110, 112, 115, 117, 119–125, 128] for a detailed explanation of these and further examples and many additional references. In the bibliography we cite further papers containing these and similar pictures. Tables 1 and 2 reproduce the numbering of the fundamental roots and the list of basic and adjoint representations. The core of the paper, its *raison d'être*, are the pictures themselves, Figs. 1–28.

Weight diagrams have been around for about half a century (see comments in Sec. 2 below) and first appeared in print a quarter of a century ago in [36]. But during the last few years they gained an entirely new significance. This is primarily due to the fact that — at least for the cases we consider in this paper — they coincide with the corresponding crystal graphs of M. Kashiwara [60–62], which, in turn, are intimately related to canonical bases of G. Lusztig [72–75] and the

Table 1. Numbering of the fundamental roots.

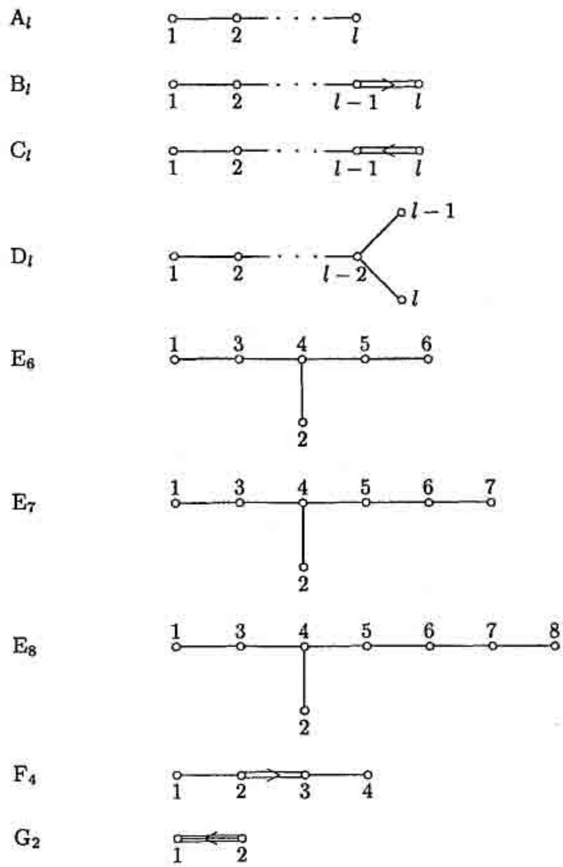


Table 2. Minimal representations.

Φ	μ	Type	dim	Figure
		Microweight representations:		
A_l	$\overline{\omega}_k,\ 1 \leq k \leq l$	$\wedge^k$ (natural)	$\binom{l+1}{k}$	$1, 5, 6$
B_l	$\overline{\omega}_l$	spin	2^l	$7-9$
C_l	$\overline{\omega}_1$	natural	$2l$	3
D_l	$\overline{\omega}_1$	natural	$2l$	4
	$\overline{\omega}_{l-1}, \overline{\omega}_l$	half-spin	2^{l-1}	$10-12$
E_6	$\overline{\omega}_1, \overline{\omega}_6$	minimal	27	20
E_7	$\overline{\omega}_7$	minimal	56	21
		Short-root representations:		
A_l	$\varepsilon_1 - \varepsilon_{l+1}$	adjoint	l^2+2l	13
B_l	$\overline{\omega}_1$	natural	$2l+1$	2
C_l	$\overline{\omega}_2$	short-root	$2l^2-l-1$	16
D_l	$\overline{\omega}_2$	adjoint	$2l^2-l$	$17-19$
E_6	$\overline{\omega}_2$	adjoint	78	$22, 22a$
E_7	$\overline{\omega}_1$	adjoint	133	23
E_8	$\overline{\omega}_8$	adjoint	248	24
F_4	$\overline{\omega}_4$	minimal	26	26
G_2	$\overline{\omega}_1$	minimal	7	27
		Non-basic adjoint representations:		
B_l	$\overline{\omega}_2$	adjoint	$2l^2+l$	14
C_l	$2\overline{\omega}_1$	adjoint	$2l^2+l$	15
F_4	$\overline{\omega}_1$	adjoint	52	25
G_2	$\overline{\omega}_2$	adjoint	14	28

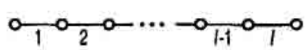


Fig. 1. (A_l, ω_1)

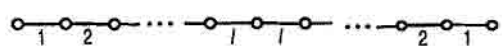


Fig. 2. $(\mathrm{B}_l, \overline{\omega}_1)$

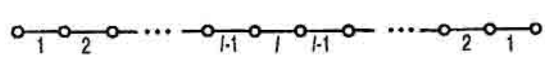


Fig. 3. $(\mathrm{C}_l, \overline{\omega}_1)$

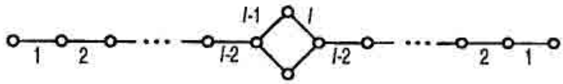


Fig. 4. $(\mathrm{D}_l, \overline{\omega}_1)$

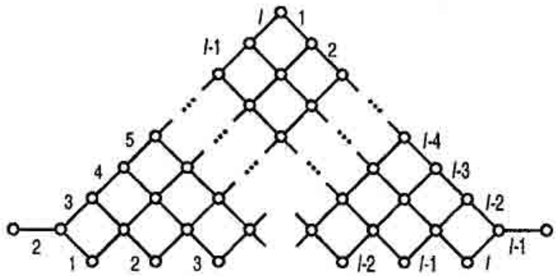


Fig. 5. $(\mathrm{A}_l, \overline{\omega}_2)$

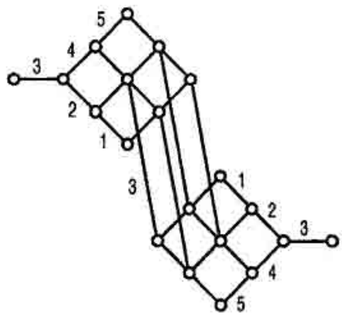


Fig. 6. $(\mathrm{A}_5, \overline{\omega}_3)$

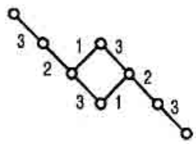


Fig. 7. $(\mathrm{B}_3, \overline{\omega}_3)$

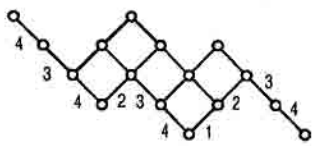


Fig. 8. $(\mathrm{B}_4, \overline{\omega}_4)$

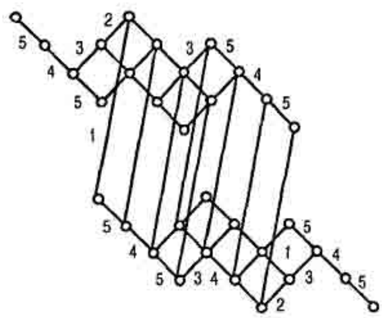


Fig. 9. $(B_5, \overline{\omega}_5)$

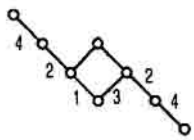


Fig. 10. $(D_4, \overline{\omega}_4)$

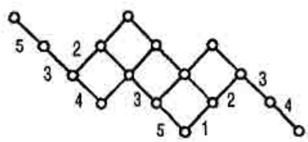


Fig. 11. $(D_5, \overline{\omega}_5)$

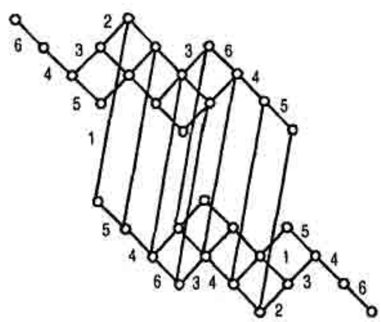


Fig. 12. $(D_6, \overline{\omega}_6)$

path model of P. Littelmann [69–71] (one can find a brilliant presentation of this circle of ideas in [57, 78]). For the microweight cases this is obvious (in the terminology of Kashiwara microweight representations *do not melt*, i.e. they always behave like at temperature 0). For the adjoint representations it is established in the thesis of R. J. Marsh, see [77] for example. Some further related references are listed in the bibliography.

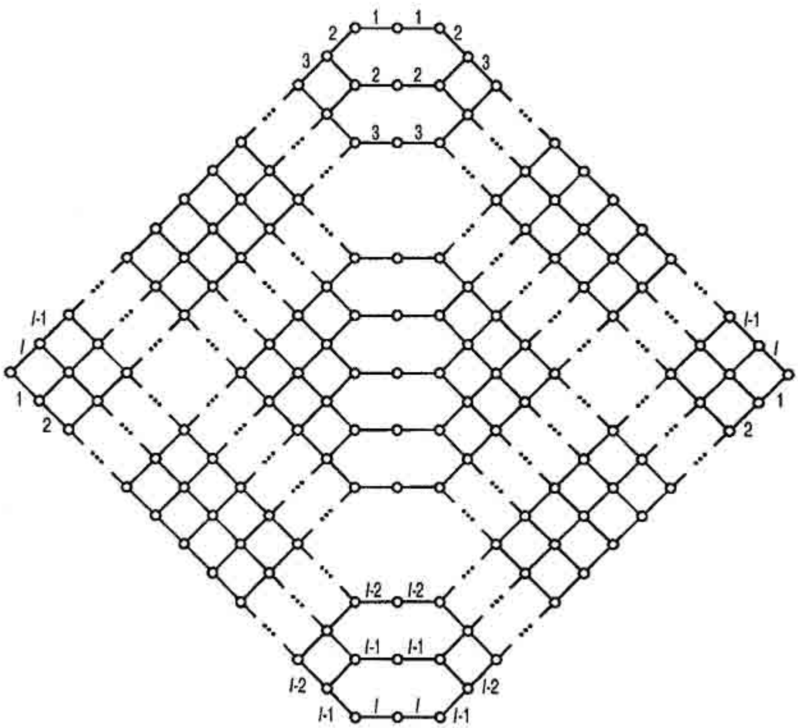


Fig. 13. (A_l, ad)

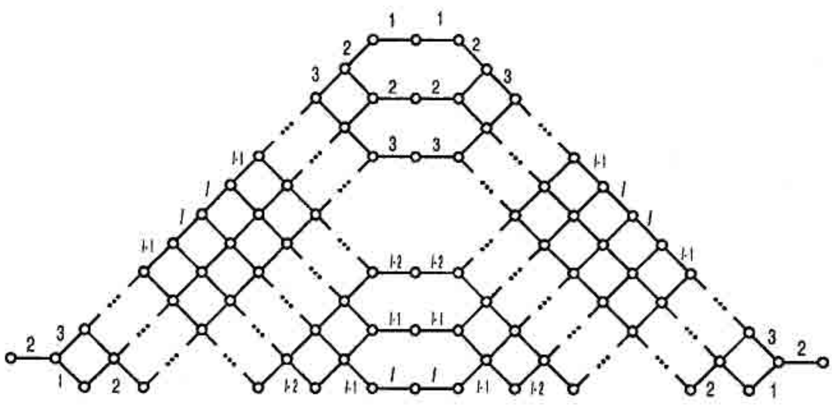


Fig. 14. $(B_l, \overline{\omega}_2)$

This stresses the significance of weight diagrams as at least a powerful mnemotechnical and computational device, completely describing semi-simple Lie algebras/algebraic groups in some small representations in an extremely transparent and compact form. This is, of course, especially relevant for the exceptional algebras and groups, where weight diagrams serve as a substitute of matrix computations. In

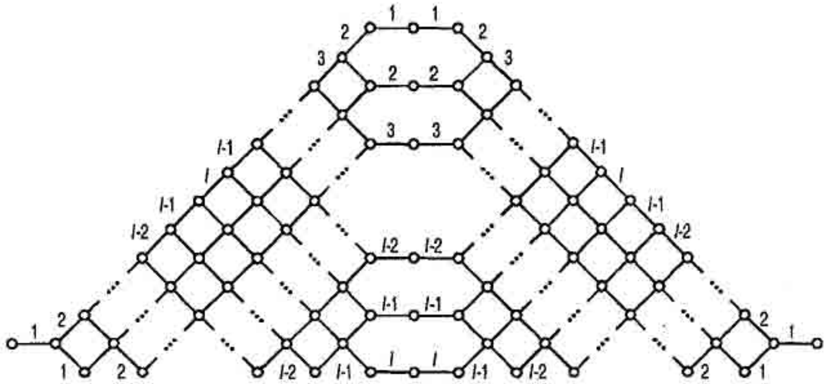


Fig. 15. $(C_l, 2\overline{\omega}_1)$

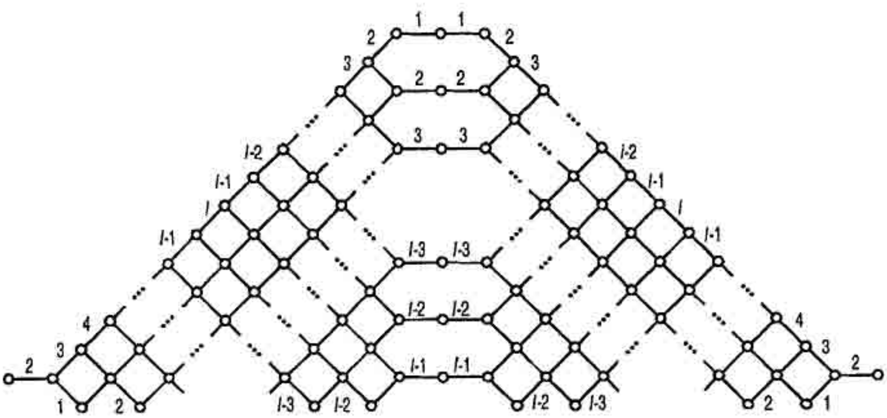


Fig. 16. $(C_l, \overline{\omega}_2)$

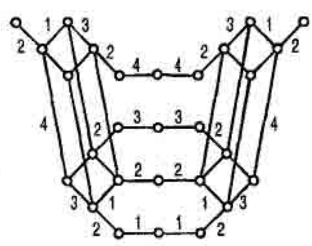


Fig. 17. $(D_4, \overline{\omega}_2)$

fact, in the vast majority of publications weight diagrams were considered as a purely combinatorial object, describing induced Bruhat order on some quotients of the Weyl group ("first look"). At least after [79, 110] it became clear that they are much more, namely that they describe action of a Lie algebra or a Chevalley group at least

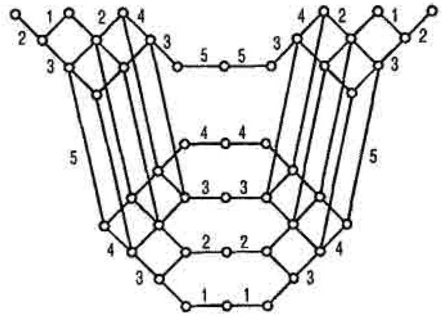


Fig. 18. $(D_5, \overline{\omega}_2)$

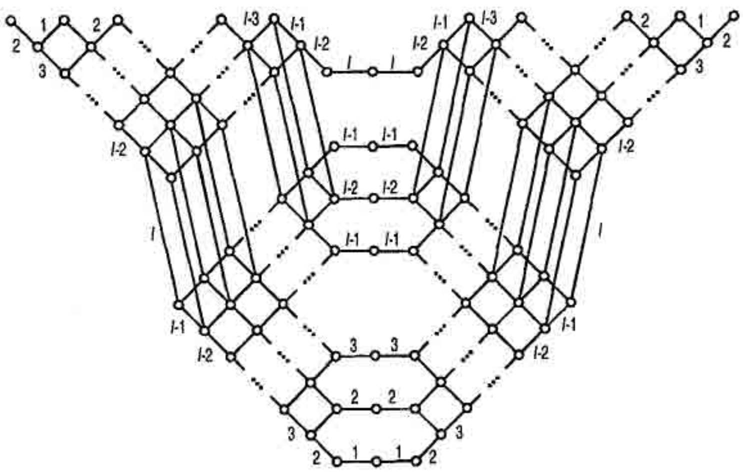


Fig. 19. $(D_l, \overline{\omega}_2)$

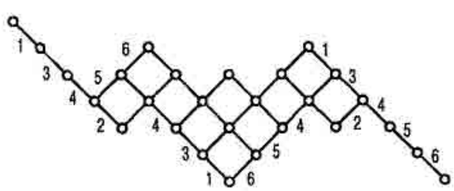


Fig. 20. $(E_6, \overline{\omega}_1)$

up to signs ("second look", see the exposition in [117]). But remarkable properties of crystal bases allow much more, namely it follows that in fact weight diagrams *completely* describe the actions, *including signs* ("third look"). Of course, for small representations it is easy to give direct proofs which make no use whatsoever of quantum deformations (see [120, 121, 125]). In particular, encoding the explicit action of the fundamental/negative fundamental root elements in the 27, the 56

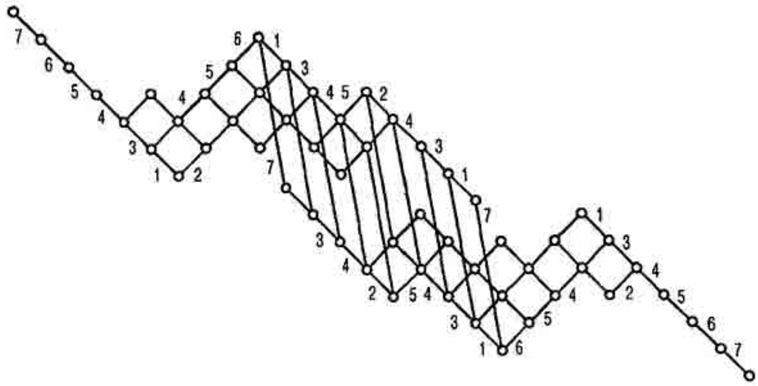


Fig. 21. $(E_7, \overline{\omega}_7)$

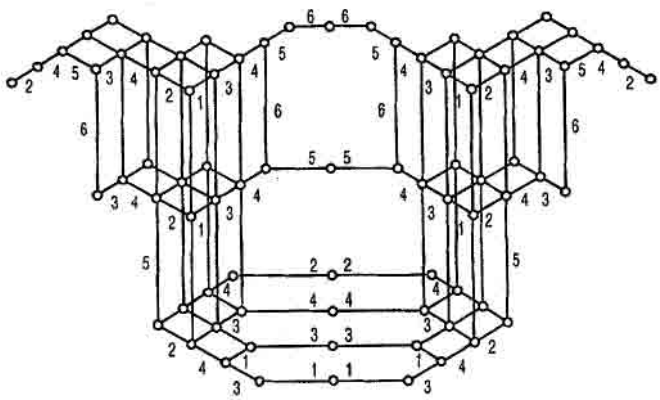


Fig. 22. $(E_6, \overline{\omega}_2)$

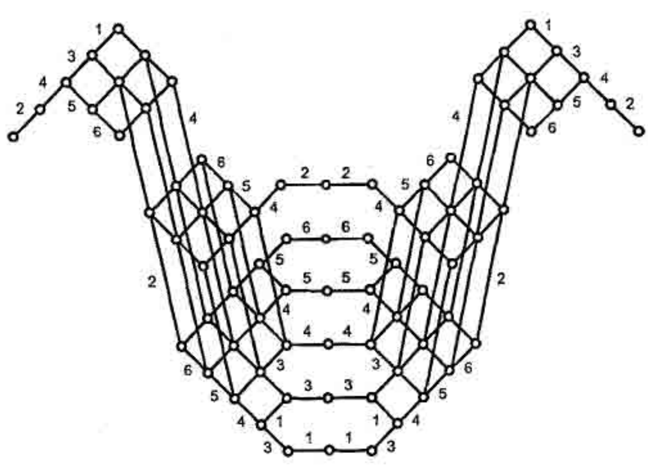


Fig. 22A. $(E_6, \overline{\omega}_2)$

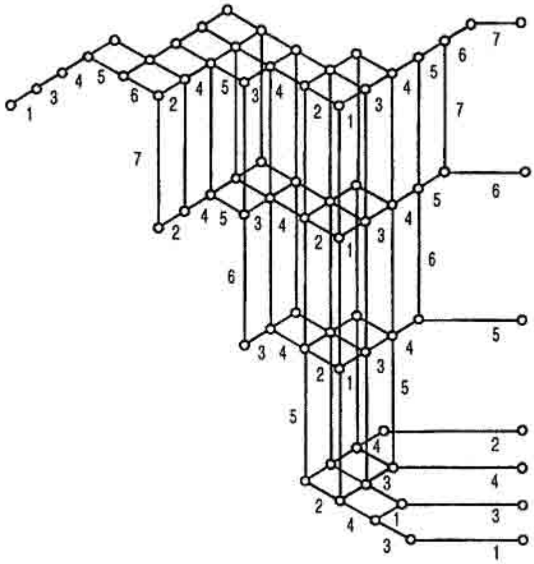


Fig. 23. $(E_7, \overline{\omega}_1)$

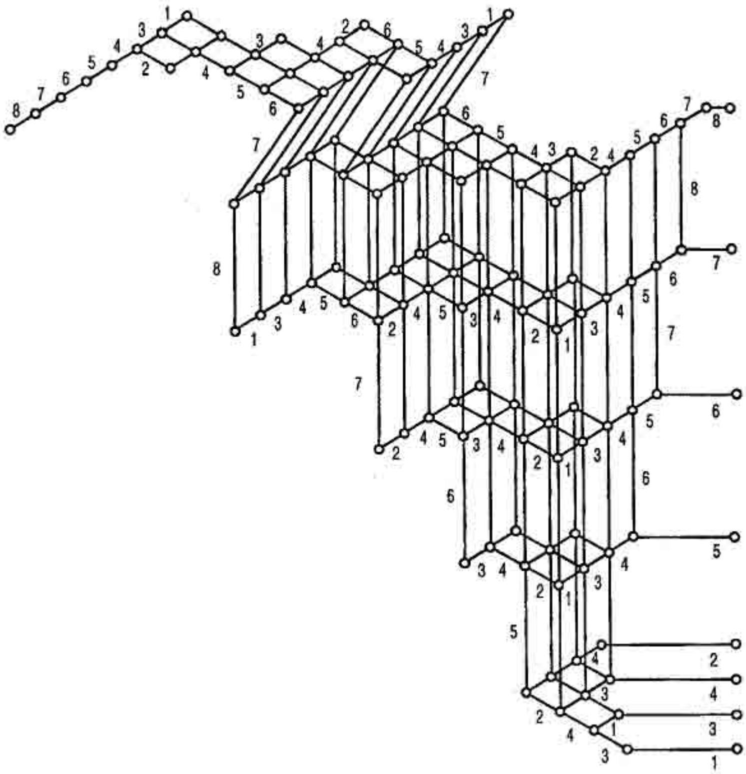


Fig. 24. $(E_8, \overline{\omega}_8)$

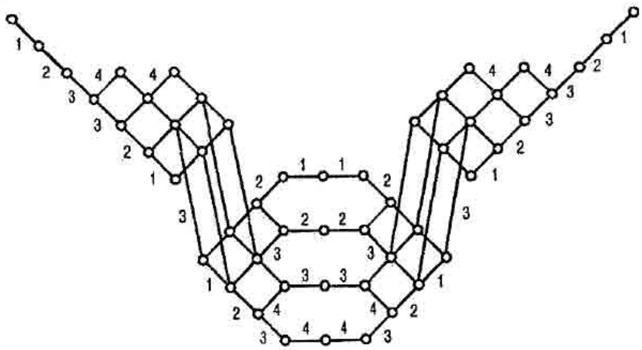


Fig. 25. (F_4, ϖ_2)

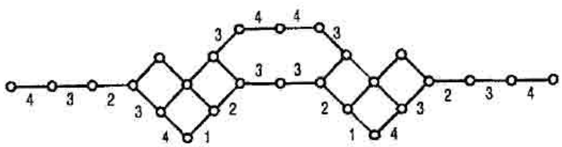
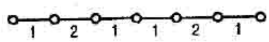


Fig. 26. (F_4, ϖ_4)



Fig, 27. (G_2, ϖ_1)

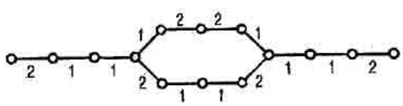


Fig. 28. (G_2, ϖ_2)

and the 248 dimensional representations, our figures 20, 21 and 24 are as complete and elementary definitions of the simply-connected Chevalley groups of types E_6, E_7 and E_8, respectively, as one can imagine.

1. Basic Notions

In this section we briefly recall the notation used in the sequel and the notion of a basic representation. All background information on root systems, Lie algebras, algebraic groups, Chevalley groups and representations may be found in [13, 14, 16, 17, 21, 48, 52–54, 56, 106, 107, 111].

(1) *Root systems and Weyl groups.* Let Φ be a reduced irreducible root system of rank l, $Q(\Phi)$ be the root lattice, $P(\Phi)$ be the weight lattice. Fix an order

on Φ, and let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the sets of positive, negative and fundamental roots respectively. Our numbering of the fundamental roots follows that of [16] (see Table 1). By $\overline{\omega}_1, \dots, \overline{\omega}_l$ one denotes the corresponding fundamental weights.

Let $W = W(\Phi)$ be the Weyl group of the root system Φ, i.e. the group generated by the set of fundamental reflections $w_{\alpha_1}, \dots, w_{\alpha_l}$. For brevity we write $s_i = w_{\alpha_i}$. Denote the set $\{s_i,\ 1 \le i \le l\}$, of fundamental reflections by S. A conjugate t of a fundamental reflection, $t = w w_{\alpha_i} w^{-1}$, is called a reflection and the set of all reflections will be denoted by T. Let l be the length function on W, i.e. $l(w)$ is the length of the shortest expression of $w \in$ W in terms of the fundamental reflections.

There are two partial orders on the group W called the *strong* and the *weak Bruhat orders*. Namely, for $u, v \in W$ we write $u \le v$ and say that u precedes v in the strong Bruhat order (usually called simply the Bruhat order), if there exist reflections $t_1, t_2, \dots, t_k \in T$ such that $l(ut_1t_2\cdots t_m) = l(u) + m$, for $1 \le m \le k$, and $ut_1t_2\cdots t_k = v$. Similarly, the element u precedes v in the weak Bruhat order (sometimes also called the Duflo order), $u \preceq v$, if there exist *fundamental* reflections $t_1, t_2, \dots, t_k \in S$ such that $l(ut_1t_2\cdots t_m) = l(u) + m$, for $1 \le m \le k$, and $ut_1t_2\cdots t_k = v$ (see [7, 8] for additional references).

For any subset $J \subseteq S$ let W_J be the parabolic subgroup of W generated by J. The set of left cosets $W^J = W/W_J$ inherits the structure of a partially ordered set according to both orderings. Indeed, there is a well known characterization of the set W^J in terms of the length function [16, 21]. Let D_J be the set of elements $w \in W$, such that $l(ws) > l(w)$ for all $s \in J$. Then the map $D_J \to W^J$ sending $u \in D_J$ to uW_J is a bijection. Each element $w \in W$ can be uniquely expressed in the form $w = uv$, where $v \in W_J$, $u \in D_J$ and $l(w) = l(u) + l(v)$. In other words, u is the unique shortest representative of the coset wW_J. The Bruhat order (weak Bruhat order) on W/W_J is induced by the corresponding order on the set D_J, namely: $w_1W_J \le w_2W_J$ (respectively $w_1W_J \preceq w_2W_J$) if and only if $u_1 \le u_2$ (respectively $u_1 \preceq u_2$) for the shortest representatives u_1, u_2 of these cosets: $u_1, u_2 \in D_J$, $u_1 \in w_1W_J$, $u_2 \in w_2W_J$.

(2) *Chevalley algebras and groups.* We denote by $L = L_{\mathbb{C}}$ the complex semisimple Lie algebra of type Φ, by $[\ ,\ ]$ the Lie bracket on L and by H its Cartan subalgebra. Then L admits *root decomposition* $L = H \bigoplus \sum L_\alpha$, where L_α are the *root subspaces*, i.e. one-dimensional subspaces, invariant with respect to H. For each root α denote by the same letter the linear functional on H, such that $[h, e_\alpha] = \alpha(h)e_\alpha$. We can identify H and H^* via the Killing form and consider α's as elements in H. However, it is more convenient to consider *coroots* $h_\alpha = 2\alpha/(\alpha,\alpha)$ instead of roots. A choice of non-zero elements $e_\alpha \in L_\alpha$, $\alpha \in \Phi^+$, uniquely determines the elements $e_{-\alpha} \in L_{-\alpha}$, $\alpha \in \Phi^+$, such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Then the set $\{e_\alpha,\ \alpha \in \Phi;\ h_\beta,\ \beta \in \Pi\}$ is a base of the Lie algebra L, called a *Weyl base* of L.

Usually one normalizes a Weyl base in such a way that all the structure constants $N_{\alpha\beta}$, where $[e_\alpha, e_\beta] = N_{\alpha\beta}e_{\alpha+\beta}$, become integers. Such a normalized base is called a *Chevalley base*. Let $L_{\mathbb{Z}}$ be the integral span of a Chevalley base, it is a $\mathbb{Z}$-form of L, called an *admissible $\mathbb{Z}$-form* or a *Chevalley order*. For any commutative ring R we set $L_R = L_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. In other words L_R is the free R-module with the base $e_\alpha = e_\alpha \otimes 1$, $h_\beta = h_\beta \otimes 1$, with the same multiplication table as over $\mathbb{C}$. The algebra L_R is called the *Chevalley algebra* of type Φ over R.

To a reduced irreducible root system Φ and a lattice P, $Q(\Phi) \leq P \leq P(\Phi)$, there corresponds a (unique) affine group scheme $\mathrm{G} = \mathrm{G}_P(\Phi,\ \ )$ over $\mathbb{Z}$, called the *Chevalley–Demazure group scheme* of type (Φ, P), see [23]. The value of the functor G on a commutative ring R with identity is called the *Chevalley group of type* (Φ, P) over R. When $R = K$ a field, this group, i.e. the group of rational points $\mathrm{G}(\Phi, K)$ with coefficients in K, is the split semisimple algebraic group of type (Φ, P) over K.

In the sequel we *always* assume that the group $\mathrm{G}(\Phi, R) = \mathrm{G}_P(\Phi, R)$ is *simply connected*, or, in other words, $P = P(\Phi)$. Fix a split maximal torus $\mathrm{T}(\Phi,\ \ )$ of $\mathrm{G} = \mathrm{G}(\Phi,\ \ )$. Its value $\mathrm{T} = \mathrm{T}(\Phi, R)$ on a ring R will be called the *split maximal torus* of the group $\mathrm{G}(\Phi, R)$. We denote by $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, the elementary root unipotents of the group $\mathrm{G}(\Phi, R)$ with respect to the given torus $\mathrm{T}(\Phi, R)$. The group $\mathrm{E}(\Phi, R)$ generated by all the elementary unipotents $x_\alpha(\xi)$ is called the *elementary subgroup* of $\mathrm{G}(\Phi, R)$. As usual for an $\varepsilon \in R^*$ we set $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$ and $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$.

(3) *Weyl modules.* Let $\pi : L \to \mathrm{gl}(V)$ be a representation of L in a finite dimensional vector space V over $\mathbb{C}$. For an element $\lambda \in H^*$ denote by V^λ the corresponding *weight subspace* of the space V regarded as an H-module, i.e.

$$V^\lambda = \{v \in V \mid \pi(h)v = \lambda(h)v,\ h \in H\}.$$

We say, that λ is a *weight* of the representation π if $V^\lambda \neq 0$. The dimension $m_\lambda = \mathrm{mult}(\lambda)$ of the space V^λ is called the *multiplicity* of the weight λ. Let us denote by $\overline{\Lambda}(\pi)$ the set of *weights of the representation* π, and by $\Lambda(\pi)$ the *set of weights with multiplicities*. This means that all the weights from $\overline{\Lambda}(\pi)$ are distinct, and we assign to each weight $\lambda \in \overline{\Lambda}(\pi)$ a collection of m distinct "weights" $\lambda_1, \dots, \lambda_m \in \Lambda(\pi)$, where $m = \mathrm{mult}(\lambda)$. We denote by $\overline{\Lambda}^*(\pi)$ and by $\Lambda^*(\pi)$ the sets of non-zero weights and non-zero weights with multiplicity, respectively. Let $P = P(\pi)$ be the lattice of weights of the representation π, i.e. the subgroup in $P(\Phi)$ generated by $\overline{\Lambda}(\pi)$. Then, $V = \bigoplus V^\lambda$, $\lambda \in \Lambda(\pi)$. In particular, for the adjoint representation $\pi = \mathrm{ad}$, we have $V = L$, $\Lambda^*(\pi) = \Phi$, $\Lambda(\pi) = \Phi \cup \{0_1, \dots, 0_l\}$, $P = Q(\Phi)$, $V^\alpha = L_\alpha$ for $\alpha \in \Phi$ and $V^0 = H$.

Let $\mu \in \Lambda(\pi)$ and $v^+ \in V$. The weight μ is called the *highest weight* of the representation π and the vector v^+ is called a *highest weight vector* (or a *primitive element*) if $\pi(e_\alpha)v^+ = 0$ for all $\alpha \in \Phi^+$. Of course, this notion depends on the choice of order on the root system Φ. The representation π is irreducible if and only if V is generated as an L-module by a vector of the highest weight. The multiplicity of the highest weight of an irreducible representation is equal to 1, hence a primitive element in this case is uniquely determined up to multiplication by a non-zero scalar. It is well known, that the correspondence between the finite dimensional irreducible modules and their highest weights yields a bijection of the set of isomorphism classes of irreducible finite dimensional L-modules and the set

$$P(\Phi)_{++} = \{\mu \in P(\Phi) \mid (\mu, \alpha) > 0,\ \alpha \in \Pi\}$$

of *dominant* integral weights (with respect to a fixed order).

The *Chevalley–Ree theorem* asserts, that each finite dimensional L-module V contains a $\mathbb{Z}$-lattice M, invariant with respect to all divided powers $\pi(e_\alpha)^m/m!$, $\alpha \in \Phi$, $m \in \mathbb{Z}^+$, and that such a lattice is the direct sum of its weight components $M^\lambda = M \cap V^\lambda$ (see [21, 53, 98, 111]). Such a lattice $M = V_\mathbb{Z}$ is called an *admissible $\mathbb{Z}$-form* of the module V or simply an *admissible lattice*. A base v^λ, $\lambda \in \Lambda(\pi)$, of the space V, consisting of weight vectors, such that $M = \sum \mathbb{Z}v^\lambda$ is an admissible lattice, is called an *admissible base* of V.

Let again R be an arbitrary commutative ring. Set $V_R = V_\mathbb{Z} \otimes_\mathbb{Z} R$. In other words, V_R is the free R-module with the base $v^\lambda = v^\lambda \otimes 1$, $\lambda \in \Lambda(\pi)$. It is clear that V_R is a module over the Chevalley algebra L_R. Indeed, e_α and h_α act on the first component of the product $v \otimes \xi$, $v \in V_\mathbb{Z}$, $\xi \in R$, while the scalars of R act on the second one. If V is an irreducible L-module with the highest weight μ, then V_R is called the *Weyl module* of the Chevalley algebra R with the *highest weight* μ.

Clearly V_R may be considered also as a representation of the (simply connected) Chevalley group $G(\Phi, R)$. It will also be referred to as the Weyl module with the highest weight μ. Even when $R = K$ is a field, this representation does not in general coincide with the irreducible representation with the highest weight μ. In fact it is only indecomposable, not irreducible (of course, it is irreducible for fields of characteristic 0). In the sequel we deal exclusively with the Weyl modules.

(4) *Basic representations.* Let us recall that an irreducible representation π of the complex semi-simple Lie algebra L is called *basic*[1] if the Weyl group $W = W(\Phi)$ acts transitively on the set $\overline{\Lambda}^*(\pi)$ of non-zero weights of the representation π. This is equivalent to saying that if for any two non-zero weights λ, μ their difference is a fundamental root $\alpha = \lambda - \mu$, then $w_\alpha\lambda = \mu$ for the corresponding fundamental reflection $w_\alpha \in W$.

It is straightforward to enumerate basic representations. First of all, it is clear that the non-zero weights of such a representation have multiplicity 1 (they are in the Weyl orbit of the highest weight). Thus $\Lambda^*(\pi) = \overline{\Lambda}^*(\pi)$. It is easy to show [22] that the multiplicity of the zero weight is $m = |\Delta(\pi)|$, where $\Delta(\pi) = \Pi \cap \Lambda^*(\pi)$ is the set of fundamental roots which are weights of the representation π. Therefore we may speak about m "zero-weights" $\widehat{\alpha}$, where $\alpha \in \Delta(\pi)$.

Now if π actually has zero weight then all the remaining weights of π must be the short roots of the root system Φ. Thus every complex semisimple Lie algebra has a unique such representation, called the "short-root representation". Its highest weight μ coincides with the short dominant root of Φ. If there is just one root length

[1]This usage follows [22] and [79]. Many authors, especially in physics, use 'basic' as a substitute of 'fundamental'. In [102] 'basic' refers to a subclass of infinitesimally irreducible representations. However we always use the word 'basic' in the same sense as [79].

then μ is the maximal root and this representation is the adjoint representation of L. If there is no zero weight then $\Lambda(\pi) = \Lambda^*(\pi)$ and all the weights of π form one Weyl orbit. Such a representation is called a *microweight* or *minuscule* representation and of course a list of these representations is very well known (see [17]). Many further details and references concerning these representations may be found in [50, 82, 89, 117, 120, 125]. For the types $\Phi = \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_8$ there are no microweight representations. The total number of basic representations of the Lie algebra L of type Φ equals $|P(\Phi) : Q(\Phi)|$ (see [79]).

In Table 2 we reproduce an explicit list of basic representations for all root systems Φ, giving the corresponding highest weight μ, type and dimension. With the sole exception of the adjoint representation for A_l all these representations are fundamental. The last column refers to the corresponding figures in the atlas. For the classical types we cannot of course draw all of the pictures and in the next section we explain how to construct them (see [82, 120] for details).

Besides basic representations, we include the diagrams of adjoint representations for the root systems B_l, C_l, F_4 and G_2. Of course, these representations are not basic, since the roots have different length and therefore the Weyl group has two orbits on non-zero weights. However, they also have the property that all non-zero weights have multiplicity one, and in this sense they are close to the basic ones. In these cases $\Delta(\pi) = \Pi$ is the set of all fundamental roots and the multiplicity m of the zero weight equals l. We refer to the modules in Table 2 as the *minimal* modules[2].

(5) *Action on a minimal module.* Fix a basic representation π of a Chevalley group $\mathrm{G} = \mathrm{G}(\Phi, R)$ on the free R-module $V = V_R = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. We tend to identify G with its image $\pi(G) = \mathrm{G}_\pi(\Phi, R)$ under this representation and often omit the symbol π in the action of G on V. Thus for an $x \in \mathrm{G}$ and $v \in V$ we write xv for $\pi(x)v$. Decompose the module R into the direct sum of its weight submodules

$$V = \sum V^\lambda \oplus V^0, \quad \lambda \in \Lambda^*(\pi).$$

H. Matsumoto [79, Lemma 2.3], has shown that one may normalize a base of weight vectors $v^\lambda \in V^\lambda$, $\lambda \in \Lambda^*(\pi)$, $v^0_\alpha \in V^0$, $\alpha \in \Delta(\pi)$, in such a way that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by the following very nice formulas:

i. if $\lambda \in \Lambda^*(\pi),\ \lambda + \alpha \notin \Lambda(\pi)$, then $x_\alpha(\xi)v^\lambda = v^\lambda$;

ii. if $\lambda, \lambda + \alpha \in \Lambda^*(\pi)$, then $x_\alpha(\xi)v^\lambda = v^\lambda \pm \xi v^{\lambda+\alpha}$;

iii. if $\alpha \notin \Lambda^*(\pi)$, then $x_\alpha(\xi)v^0 = v^0$, for any $v^0 \in V^0$;

iv. if $\alpha \in \Lambda^*(\pi)$, then $x_\alpha(\xi)v^{-\alpha} = v^{-\alpha} \pm \xi v^0(\alpha) \pm \xi^2 v^\alpha$,
$x_\alpha(\xi)v^0 = v^0 \pm \xi \alpha_*(v^0) v^\alpha$;

[2] There is no consensus on the usage of the word 'minimal' either. In [58] 'minimal' is used for what we call 'basic'. Some authors use 'minimal' as a synonym of 'minuscule', some others reserve it solely for the modules of the smallest dimension.

where α_* is a certain unimodular element of the dual space $(V^0)^* = \mathrm{Hom}_R(V^0, R)$ and $v^0(\alpha)$ is a unimodular element of V^0 (recall that an element v of a free R-module V is *unimodular* if the submodule generated by v is a direct summand of V, or, equivalently, if there exists a $\varphi \in V^* = \mathrm{Hom}_R(V, R)$, such that $\varphi(v) \in R^*$). We refer to this fact as *Matsumoto's Lemma.* In fact since the only basic representations which actually have zero weights come from the adjoint ones, it is easy to give explicit formulas for $v^0(\alpha)$, $\alpha_*(v)$ as well, see [125]. For the sake of brevity we write $v^{\widehat{\alpha}}$ instead of v^0_α. Then our base $\{v^\lambda\}$ of V is indexed by all the weights $\lambda \in \Lambda(\pi)$ *with multiplicities.*

Now we may expand any $v \in V$ in the chosen base, $v = \sum c_\lambda v^\lambda + \sum c^0_\alpha v^0_\alpha$, $\lambda \in \Lambda^*(\pi)$, $\alpha \in \Delta(\pi)$. Usually we suppress distinction between zero and non-zero weights and write simply $v = \sum c_\lambda v^\lambda$, $\lambda \in \Lambda(\pi)$. Then c_λ is called the λ-th *coordinate* of v. Matsumoto's lemma provides explicit formulas for the action of $x_\alpha(\xi)$ on v and on its coordinates.

2. Weight Diagrams

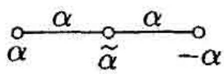
In this section we briefly recall how the *weight diagrams* of the above representations are constructed. Once more our usage dramatically differs from that common in physics (and though less common, still present in mathematics, see [113] as a recent example), where 'weight diagram' denotes the actual configuration of weights in the l-dimensional Euclidean space. Weight diagrams in our sense were first drawn by E. B. Dynkin and his school in mid-fifties (private communication with E. B. Vinberg), but to the best of our knowledge never officially appeared in print. In fact already [42] and the supplement to [43] show that Dynkin was well aware of the combinatorial and geometric properties of weights expressed by weight diagrams. His expression "spindle-shaped" refers to what would nowdays be called "rank symmetry" and "rank unimodality" (see [8, 82, 88–90, 108]). The first appearance of these pictures in print, which we could trace, was in [36]. Weight diagrams were systematically used starting with the paper of M. R. Stein [110]. They preserve most of the essential information about the configuration of weights.

(1) *Weight diagrams.* It is well known that a choice of a fundamental system Π defines a partial order of the weight lattice $P(\Phi)$ as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a linear combination of the fundamental roots with non-negative integral coefficients. Let us associate with a representation π a graph which is *almost* the Hasse diagram of the set $\overline{\Lambda}(\pi)$ of its weights with respect to the above order. Actually, for the representations where all weights have multiplicity one, it will be *precisely* this Hasse diagram. However in general we want the nodes of the diagram to correspond to the base vectors of the corresponding representation space, rather than the weights themselves, so we need $\mathrm{mult}(\mu)$ nodes corresponding to a weight μ. Fortunately for the basic and adjoint representations, all non-zero weights have multiplicity one, so this problem arises only for the zero weight.

We construct a labeled graph (or, in the terminology of M. Kashiwara, a *colored* graph) in the following way. Its vertices correspond to the weights $\lambda \in \Lambda(\pi)$ *with* *multiplicities* of the representation π, and the vertex corresponding to λ is actually marked by λ (usually these labels are omitted). This means that there is *one* node corresponding to each non-zero weight and that to the zero weight there correspond m distinct nodes $\widehat{\alpha}$, $\alpha \in \Delta(\pi)$. We read the diagram from right to left and from bottom to top, which means that a larger weight tends to stand to the left of and higher than a smaller one, with the landscape orientation being primary.

The vertices corresponding to the weights $\lambda, \mu \in \Lambda(\pi)$ are joined by a bond marked α_i (or simply i — in the terminology of M. Kashiwara, of color i) if and only if their difference $\lambda - \mu = \alpha_i \in \Pi$ is a fundamental root[3]. We draw the diagrams in such a way that the marks on the opposite sides of a parallelogram are equal and in that case at least one of them is omitted.

When λ and μ are non-zero weights this definition is unambiguous. It remains to explain how we understand the equality when λ or μ is a zero weight. If $\lambda = \widehat{\alpha}$, where $\alpha \in \Delta(\pi)$, then we stipulate $\mu = -\alpha$ and $\alpha_i = \alpha$, so that $\widehat{\alpha} = (-\alpha) + \alpha$. If $\mu = \widehat{\alpha}$, $\alpha \in \Delta(\pi)$, then $\lambda = \alpha_i = \alpha$ and $\alpha = \widehat{\alpha} + \alpha$. This means that to any root $\alpha \in \Delta(\pi)$ there corresponds the following weight chain of length three:



and $\widehat{\alpha}$ is not adjacent to any other vertex.

The above convention may seem somewhat arbitrary. To really calculate in the presence of zero weights one has to introduce also another sort of bonds, which we used to denote by dotted lines and which join $\widehat{\alpha}$ to $\pm\beta$ if $\alpha, \beta \in \Delta(\pi)$, $\alpha \neq \beta$, are not orthogonal. These bonds have to be read in one direction, from a zero weight to a non-zero one, see [117, 125] for details. (In fact the dotted bonds are precisely the ones which come from those covering relations of the strong Bruhat order which are not present in the weak Bruhat order, see the two following subsections for the explanation and an example). But there is a much deeper explanation of why we draw the weight diagrams the way we do. In this way they describe the action of Kashiwara's raising and lowering operators *at temperature* 0. It is shown in the thesis of R. J. Marsh (see, for example, [77]) that our weight diagrams coincide with crystal graphs for these cases.

(2) *Hasse diagrams of Bruhat order.* For the case of microweight representations there is another natural way to look at these diagrams. Let $\omega = \overline{\omega}_k$ be the highest weight of a microweight representation. Then all the other weights lie in the Weyl orbit of ω and thus correspond bijectively to the cosets W/W_k, where W_k is the Weyl subgroup of the Weyl group $W = W(\Phi)$ generated by reflections in all the fundamental roots except α_k. As recalled in Sec. 1, there is a usual way to introduce a partial order on the set of such cosets, viz. the (*induced*) *Bruhat order*. The

[3] There is another graph, associated with a representation, in which the nodes are as above and two nodes are joined by a bond if their difference is any root, not necessarily fundamental. We refer to this graph as the *weight graph* of a representation, see the next section. One may find discussion of some examples in [15, 31, 33, 99].

following result is essentially contained in [88, 89] (although never stated there in this form, see [121] for a proof).

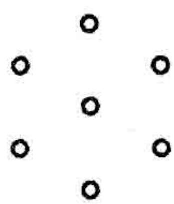
Lemma 1. *Let $\omega = \overline{\omega}_k$ be a microweight and $W_J = W_k$ be its stabilizer in the Weyl group. Then the Bruhat order on $W^J = W/W_J$ coincides with the weak Bruhat order and the poset W^J is anti-isomorphic to the poset $W\omega$ with respect to the usual ordering of weights. In other words*

$$w_1W_J \preceq w_2W_J \qquad \Longleftrightarrow \qquad w_1W_J \leq w_2W_J \qquad \Longleftrightarrow \qquad w_1\omega \geq w_2\omega\,.$$

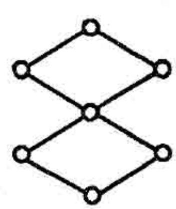
This lemma tells us that for the microweight representations the diagrams described in the preceding subsection are precisely the duals of the Hasse diagrams of the Bruhat order (which in these cases coincides with the weak Bruhat order) on the coset spaces W^J modulo the corresponding parabolic subgroups W_J of W. This explains why precisely the same pictures appear in a variety of contexts, see, for example, [11, 12, 27, 30, 36, 37, 44, 45, 49, 51, 55, 82, 83–87, 89, 90, 100, 110, 115, 117, 119–122, 125, 128].

When there is a zero-weight the dotted lines occur precisely because the corresponding Bruhat order on the non-zero weights is actually stronger then the weak order: a pair of an ordinary and a dotted line with a common vertex corresponds to a bond in the Hasse diagram of the Bruhat order which does not come from a fundamental reflection.

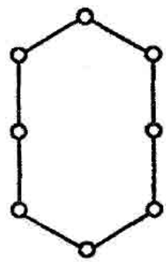
(3) *An example.* In this subsection we compare different ways to draw the weight diagrams and the Hasse diagrams of the Bruhat order for the case of the adjoint representation of a group of type A_2. First, in the literature on physics the 'weight diagram' of this representation would be something like



In M. R. Stein's paper [110] the weight diagram of this representation is depicted as follows (no multiplicities!):

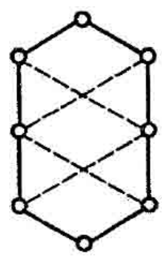


Our way to draw this picture, according to the above convention is:

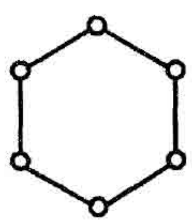


This way was used in the papers by the present authors and A. E. Zalesskii, see [83–87, 102, 114, 115, 117, 119, 123, 128]. This is **the** way how we draw the diagrams in the present paper and it is exactly the one which leads to crystal graphs [77].

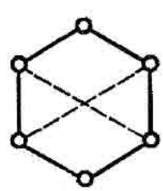
The same diagram with the 'dotted' lines read in one direction, see [117, 125] looks as follows:



It remains to compare these diagrams with the diagrams of the weak Bruhat order



and the strong Bruhat order



on the Weyl group of type A_2 (no zero weight!).

As observed in the preceding subsection, for a microweight all these modes to draw the diagram coincide — what a relief!

(4) *Construction of weight diagrams.* In this subsection we explain how the weight diagrams in this atlas have been drawn and how to draw the diagrams for the classical groups, which are not there. The general idea is to construct the diagrams inductively, i.e. to build them from the weight diagrams corresponding to a proper subsystem. In fact both [62] and [70] describe an algorithm how to construct crystal graphs of all representations of classical groups in terms of Young diagrams, and [70] constructs crystal graphs for some further fundamental representations.

Take $(A_l, \overline{\omega}_k)$ as an example. For this case we construct the weight diagram according to the 'Pascal triangle'. The weights of this representation have the form $e_{i_1} + \ldots + e_{i_k}$, where $1 \leq i_1 < \ldots < i_k \leq l+1$. Clearly, there are $\binom{l+1}{k}$ such weights. Now consider the root subsystem A_{l-1} in A_l, generated by all the fundamental roots, except α_l, and restrict the representation $(A_l, \overline{\omega}_k)$ to this subsystem. Clearly with respect to this subsystem there are two orbits of weights, those without e_{l+1} and those with e_{l+1}. There are $\binom{l}{k}$ weights of the form $e_{i_1} + \ldots + e_{i_k}$, where $1 \leq i_1 < \ldots < i_k \leq l$, and $\binom{l}{k-1}$ weights of the form $e_{i_1} + \ldots + e_{i_{k-1}} + e_{l+1}$, where $1 \leq i_1 < \ldots < i_{k-1} \leq l$, and they form precisely the weight diagrams of the representations $(A_{l-1}, \overline{\omega}_k)$ and $(A_{l-1}, \overline{\omega}_{k-1})$ respectively. All the bonds of these diagrams are marked with the fundamental roots $\alpha_1, \ldots, \alpha_{l-1}$ and it remains only to fit α_l in the picture. This is done as follows. Clearly, the only weights from which one can subtract $\alpha_l = e_l - e_{l+1}$ are the weights of the form $e_{i_1} + \ldots + e_{i_{k-1}} + e_l$, where $1 \leq i_1 < \ldots < i_{k-1} \leq l-1$. There are $\binom{l-1}{k-1}$ such weights. Thus we have described an inductive procedure to construct the weight diagram of the representation $(A_l, \overline{\omega}_k)$: we have to take the diagrams of the representations $(A_{l-1}, \overline{\omega}_k)$ and $(A_{l-1}, \overline{\omega}_{k-1})$ and glue them with the bonds with label l along the weight diagram of $(A_{l-2}, \overline{\omega}_{k-1})$. In all other cases we proceed similarly.

Consider the spin and half-spin representations as another example. In this case one has to restrict to the subsystem generated by all the fundamental roots except α_1. For example, a weight of the spin representation for B_l has the form $\frac{1}{2}(\pm e_1 \pm \ldots \pm e_l)$ and will be denoted in the sequel simply by the sequence of signs $(\pm, \pm, \ldots, \pm)$. There are 2^l such weights. Clearly, restricting to the subsystem B_{l-1} generated by the fundamental roots $\alpha_2, \ldots, \alpha_l$, we fix the first component to be $+$ or $-$. There are 2^{l-1} weights which start with $+$ and 2^{l-1} weights which start with $-$. Thus the weight diagram $(B_l, \overline{\omega}_l)$ consists of two weight diagrams of type $(B_{l-1}, \overline{\omega}_l)$. It remains to establish how they are glued together by the root α_1. Clearly, the only weights, from which one can subtract $\alpha_1 = e_1 - e_2$ have the form $(+, -, \pm, \ldots, \pm)$ and there are 2^{l-2} such weights. Thus we glue the two copies of $(B_{l-1}, \overline{\omega}_l)$ by the bonds with label 1 along the weight diagram of $(B_{l-2}, \overline{\omega}_l)$.

The same procedure works for the half-spin $(D_l, \overline{\omega}_{l-1})$ and $(D_l, \overline{\omega}_l)$. Their weights may be presented by the same sequences of signs $(\pm, \pm, \ldots, \pm)$, with the number of pluses even in one case and odd in another one. As graphs without labels they are isomorphic to the weight diagrams of type $(B_{l-1}, \overline{\omega}_{l-1})$. To construct, say, $(D_l, \overline{\omega}_{l-1})$ one has to take a copy of $(D_{l-1}, \overline{\omega}_{l-2})$ and a copy of $(D_{l-1}, \overline{\omega}_{l-1})$ and glue them along $(D_{l-2}, \overline{\omega}_{l-2})$.

Exactly the same procedure has been applied to construct the weight diagrams in all other cases. For example, the weight diagram for the adjoint representation of the Lie algebra of type E_8 was constructed as follows. The dimension of this representation equals 248. Its restriction to E_7 clearly gives a copy of the adjoint representation of the Lie algebra of type E_7, two copies of the minuscule module of type E_7 (in positive and negative roots respectively) and three copies of the trivial representation (the maximal and the negative maximal root and the one dimensional toral subalgebra, corresponding to α_8): $248 = 133 + 56 + 56 + 1 + 1 + 1$. Now we do the same with the representations of E_7, considering their branching with respect to E_6. Thus the restriction of the adjoint representation of type E_7 to E_6 decomposes into the direct sum of the adjoint representation of E_6, the two minuscule modules (the one with the highest weight $\overline{\omega}_1$ in the positive roots and its dual with the highest weight $\overline{\omega}_6$ in the negative ones) and a trivial summand (the toral subalgebra, corresponding to α_7): $133 = 78 + 27 + 27 + 1$. Further we restrict the adjoint representation of the Lie algebra of type E_6 to a subalgebra of type D_5, etc. By the same token we restrict the 56-dimensional representation of E_7 to E_6 to get the two minuscule and two trivial summands, then we restrict the 27-dimensional representations of E_6 to D_5, etc. Looking at the weights we find out in each case how the pieces are glued together. The details of the inductive procedure should be clear in each case from the way we draw the diagrams.

For three cases, namely for $(A_l, \overline{\omega}_k)$, $k \geq 3$, spin and half-spin representations, we do not draw general patterns, since they are too messy. In these cases we restrict ourselves to few examples, Figs. 5–12, which should explain how to construct the diagrams in the general case. On the other hand for the adjoint representation of type E_6 we draw the diagram in two different ways, acoording to D_5 and according to A_5. The diagrams for the adjoint representations of types E_7 and E_8 do not fit into a page. For these cases we draw one half of the diagram, representing the positive roots and how they are joined to the zero weight. Most of the pictures were contained in our theses [84, 102, 114].

3. Some Applications

In this section we sketch some of the uses of the weight diagrams. Many more details and additional references may be found in [5, 11, 12, 27, 30, 36, 37, 44, 45, 47, 49–51, 55, 81, 82, 85–87, 100, 110, 115, 117, 119–125, 128]. No attempt to be complete is being made here. We know of many similar recipies and it might be useful to collect them in one place, but this is far beyond the scope of the present paper.

(1) *Root systems.* Even at the level of the root systems weight diagrams may be very useful, especially for the exceptional types. In fact, the diagrams of the adjoint representations visualize the order relation at the set of roots. This may be very helpful, for example, for calculations in the maximal unipotent subgroup $U(\Phi, R)$ of a Chevalley group or the maximal nilpotent subalgebra $\mathfrak{n}(\Phi, R)$ of the corresponding Chevalley algebra.

More precisely, for the adjoint representation of the group of type Φ the non-zero weights are precisely the roots of Φ. The bonds represent *covering relations* with respect to the usual partial ordering of the roots, associated with the choice of a fundamental system Π:

$$\alpha \geq \beta \qquad \Longleftrightarrow \qquad \alpha - \beta = \sum m_i \alpha_i,\ m_i \geq 0\,.$$

This means that α covers β if $\alpha - \beta$ is a fundamental root, the mark i at the bond tells us that in fact $\alpha - \beta = \alpha_i$.

Another visualization of roots which works for *all* diagrams is via equivalence classes of paths. As we know, the bonds represent *fundamental* roots. A positive/negative root $\alpha = \sum m_i \alpha_i$ is represented by strictly increasing/decreasing paths which has $|m_1|$ bonds with the label 1, $|m_2|$ bonds with the label 2, ..., $|m_l|$ bonds with the label l. For example, the sum of a weight λ with a given positive root $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_m}$ is a weight if and only if there is a strictly increasing path with the origin λ and the labels $i_1, \ldots, i_m$ (in any order). All directed paths with the same origin λ and the same terminus μ are equivalent.

After some practice with the weight diagrams a combination of the two above interpretations makes calculations in E_8 not much more complicated than the calculations in G_2, based on the usual two-dimensional picture. For example, the weight diagram replaces tables of roots. Thus, to recall the coefficients of the maximal root one has simply to count the number of labels in a path from zero to the leftmost node of the diagram.

(2) *Weight graphs.* With every representation one can associate other graphs, where the nodes are again the weights of this representation — or sometimes the *extremal weights* — and two weights are joined by a bond if their difference is a root. These graphs often have extremely strong symmetry properties. They arise in a number of contexts, for instance, as regular graphs [15, 19]; as adjacency graphs of regular polytopes [35]; as kissing graphs for sphere packings and in coding theory [31]; and in finite geometries [34, 99]. Many of these graphs have special names. For example, the weight graph of $(E_6, \overline{\omega}_1)$ is called the *Schläfli graph*, whereas the weight graph of $(E_7, \overline{\omega}_7)$ is the *Gosset graph*.

An attempt to draw these graphs produces a mess and they are usually depicted by various shorthand pictures, showing some of their subgraphs and the way how they are glued together. However a weight diagram together with the table of roots (or, what is the same, together with the weight diagram of the adjoint representation of the corresponding type) contains all the information necessary to reconstruct the graph completely. In fact the weight diagrams are very faithful and practical graphical presentation of the graphs.

For example, Fig. 24 represents the 240 non-intersecting equal spheres in the 8-dimensional Euclidean space kissing a central sphere of the same radius (one should drop the zero weights, corresponding to the central sphere). The whole configuration of spheres is made extremely transparent by contemplating the picture. Two spheres kiss each other exactly when their difference is a root. Thus, the sphere represented by the leftmost node kisses 56 further spheres, apart from the central one, and these are exactly all other spheres lying in the two upper layers of the diagram. In turn, these spheres form a configuration presented at Fig. 21 and it is easy to see, that there are exactly 27 spheres kissing the central one and two further kissing spheres, see Fig. 20, etc.

(3) *Weyl groups.* The pictures provide also a very convenient vizualization of some permutation actions of the Weyl groups, as well as very powerful tools for calculations in these groups. Namely, the pictures contain detailed information about the action of the Weyl group on the extremal weights of a minimal representation. The Weyl group $W = W(\Phi)$ is generated by the fundamental reflections $s_1, \dots, s_l$. For a microweight representation a fundamental reflection s_i transposes the pairs of nodes joined by a bond marked i and leaves all other nodes invariant. The only other possibility which may occur for any minimal representation is a chain of two consecutive bonds marked i which passes through a zero weight. In this case s_i transposes the non-zero nodes of such a chain.

In general, for an arbitrary element $w \in W$ one proceeds as follows: One decomposes w as a product of the fundamental reflections $s_{i_1}, \dots, s_{i_m}$ and looks at the paths whose bonds have labels $i_1, \dots, i_m$. The paths do not have to be monotonous this time, but the order of labels is important, $s_i s_j$ does not in general coincide with $s_j s_i$. This interpretation of W is especially convenient when one is given two weights λ and μ and wants to find an element w of the Weyl group sending λ to μ. To do this one has only to find a path from λ to μ and then to take $w = s_{i_1} \dots s_{i_m}$, where $i_1, \dots, i_m$ are the labels at the path in the *inverse order*.

As we know from the previous section, for a microweight representation with the highest weight ω our diagram is *anti-isomorphic* to the Hasse diagram of the induced Bruhat order on W/W_J, where W_J is the stabilizer of ω in W. In other words, the nodes of the diagram are the cosets wW_J of W modulo W_J and two cosets $w_1 W_J$ and $w_2 W_J$ are joined by a bond marked i if $s_i w_1 W_J = w_2 W_J$. However now the leftmost node represents the coset W_J. This interpretation may be used, for example, to find the shortest element in a coset (one has to find a shortest path from the leftmost node to the node representing a given coset and then to multiply the corresponding fundamental reflections) or to find the decomposition of W into (W_K, W_J)-double cosets (say, if K is a maximal subset of Π obtained by dropping α_h, one has simply to cut the diagram through the bonds marked h, see [36, 82, 100] and the subsection "Branching rules" below).

(4) *Action constants.* In Secs. 1°–5° we described the action of $G(\Phi, R)$ on a minimal module (V, π). This action is most suggestively described in the following way. Conceive a vector $v = \sum a_\lambda v^\lambda \in V$ as the marked graph which is obtained by putting marks a_λ to the corresponding vertices of the weight diagram of type (Φ, π). Thus, the components of a vector are *partially ordered* and not linearly ordered. Expand a root $\alpha \in \Phi$ in the fixed base of the root system: $\alpha = \sum m_i \alpha_i$, as in Sec. 2. Then the action of $x_\alpha(\xi)$ on v looks as follows: it adds the λ-th coordinate of v multiplied by $\pm\xi$ to the coordinate standing in the vertex μ such that there is a directed path (we go in the positive/negative direction if m_i are positive/negative)

from λ to μ having precisely $|m_i|$ bonds with the label i for any $i = 1, \ldots, l$. There are slightly more complicated rules if the path starts/stops at zero and the path which has $2|m_i|$ bonds with the label i has to be taken into account too (see [79, 110, 117] for details).

Let for simplicity V be a minuscule representation. Then the above paragraph shows that the weight diagram completely determines the structure constants $c_{\lambda\alpha}$,

$$x_\alpha(1)v^\lambda = v^\lambda + c_{\lambda\alpha}v^{\lambda+\alpha},$$

of the action $G \times V \longrightarrow V$ *up to sign*. A slightly more delicate analysis allows to read off the signs of these constants as well, see [120, 125]. For example, look at Fig. 20, representing the action of $G(E_6, R)$ on a 27-dimensional module. In can be easily checked, that with respect to an appropriate base all the $c_{\lambda,\alpha}$ for a fundamental or a negative fundamental root α take values 0 or 1 (this is exactly the *crystal base* of this module). In this base there is a simple rule, which allows to read off the signs of $c_{\lambda,\alpha}$ for all α by looking at the *order* of labels in the paths representing α. For example, let $\alpha = \alpha_1 + \alpha_3$. Then an inspection of the diagram shows that out of the six paths representing α three have the labels $(1,3)$, whereas the remaining three have the labels $(3,1)$ (as read in the positive direction, from right to left). This means precisely, that in the first three cases $c_{\lambda,\alpha} = 1$, while in the remaining three cases $c_{\lambda,\alpha} = -1$, where λ is the origin of the corresponding path.

This interpretation has been used, for example, to calculate the orbits of these representations, stabilizers of vectors, etc. In many instances this is very important to understand the structure of the groups, both over fields (in the study of *internal* modules, see the next subsection) and over rings, where the study of groups in specific representations may be the only approach that works (see Sec. 10 below).

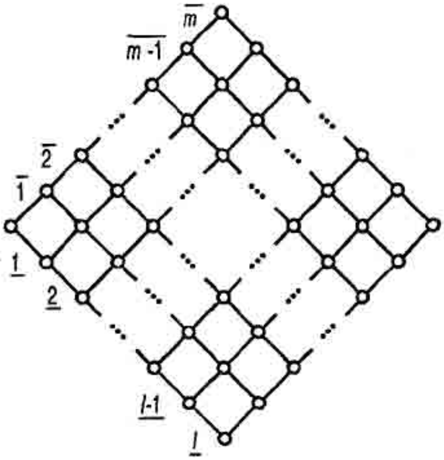
(5) *Internal Chevalley modules.* One of the most important applications of the fact that we can explicitly control the action constants by the weight diagram is to the structure theory of the group G itself. First, the structure constants appearing in the Chevalley commutator formula may be themselves interpreted as a special case of the action constants and may be read off from the weight diagram. Second, many important calculations have to be performed not in the whole group but in one of its parabolic subgroups, usually a maximal one.

The representations occurring as the conjugation action of the Levi factor L_P of a parabolic subgroup P on the consecutive factors of the descending central series of its unipotent radical U_P are called *internal Chevalley modules*. These modules have been extensively studied [4, 91, 96, 97]. For a maximal parabolic subgroup they are usually basic [4]. This means that the pictures collected in this atlas give an important tool for vizualizing the structure of parabolic subgroups of G.

Let $u \in \Pi x_\alpha(a_\alpha)$ be an element of the unipotent radical U_P of a parabolic subgroup P. Look at its projection $\overline{u}$ to $U_P/[U_P, U_P]$. In many calculations (computation of Bruhat decomposition, classification of conjugacy classes, etc.) the only thing that matters is whether some entries can be made zeros or not by the action of the Levi factor or its Borel subgroup. Weight diagrams are very suitable for such calculations. For example, if we know that the entry a_α is non-zero, we may assume that all entries a_β, such that $\alpha - \beta$ is a root of L_P, are zeros. Very often such easy arguments allow to reduce to groups of smaller rank, where everything can be done by hand. Roughly speaking, this procedure stands to the usual calculations with the Chevalley commutator formula as the calculations with block matrices do to the calculations with ordinary matrices.

It is particularly efficient when $[U_P, U_P]$ is very small, for example, for the cases when U_P is *abelian* (these are exactly the parabolic subgroups corresponding to the microweights of the *dual* root system and there is an *a priori* explanation due to R. Steinberg [89] for the coincidence of the usual order on the roots in the unipotent radical with the induced Bruhat order) or *extraspecial*, see [92, 98, 115, 119] and references there. Thus, in the case of an extraspecial parabolic subgroup the above argument immediately reduces analysis to the case of D_4. Analogous arguments work more generally, but then one has to iterate them in consecutive layers of U_P.

(6) *Tensor products.* It is very easy to construct the weight diagram of a tensor product of two distinct groups. It is simply the direct product of the weight diagrams of the factors. Thus, for example, the weight diagram of the natural representation of A_l is a chain of length $l+1$. This means that the weight diagram of the tensor product of the natural representations of A_l and A_m looks as follows

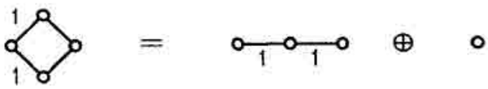


where the subscribed indices refer to the fundamental roots of A_l, whereas the superscribed ones refer to the fundamental roots of A_m. As one can expect, this diagram often arises as a subdiagram of the diagrams for other types.

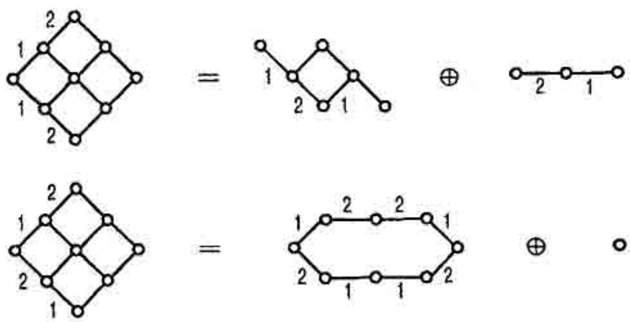
Quite remarkably, the diagrams allow the visualization of the decomposition of the tensor product of two representations of the *same* group into irreducible/ indecomposable summands[4]. It is easy to describe a simple transformation rule, which breaks such a tensor product into indecomposable summands (see [60, 78]).

[4]Here, as in the next subsection discussing branching rules, one should be cautious about characteristic. Microweight representation remains irreducible after reduction. Otherwise to be on the safe side one should assume that $R = K$ is a field of characteristic 0.

We do not intend to go into details here, restricting ourselves to the following self-explanatory example. The simplest case of the basic rule is presented below:



Already this simple rule suffices to see the difference in the decomposition of $\overline{\omega}_1 \times \overline{\omega}_1$ and $\overline{\omega}_1 \times \overline{\omega}_2$ for A_2:



Indeed, in the first case the left and the right squares must be contracted, whereas in the second case the top and the bottom ones. The general algorithm uses Kashiwara's definition of the product of coloured graphs, see [60, 78].

(7) *Branching rules.* From the weight diagram it is immediate to read off the branching of the corresponding representation with respect to a subsystem subgroup. In the case when $\Delta = \langle \Pi \setminus \{\alpha_h\} \rangle$ is the symmetric part of the maximal parabolic subset obtained by dropping the hth fundamental root the procedure is particularly easy. Then the restriction of π to $G(\Delta, R)$ looks as follows: One has to cut the diagram of π through the bonds with the label h. In general, when $\Delta = \langle J \rangle$, $J \subseteq \Pi$, is the symmetric part of a parabolic subset, one has to simply cut the diagram through all the bonds having labels not in J.

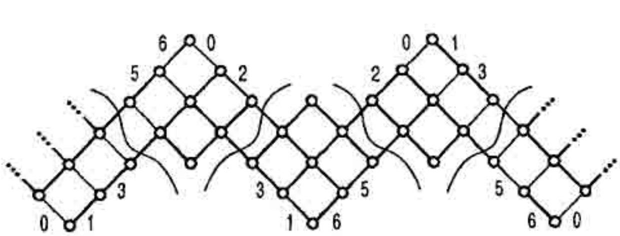
For example, to restrict a 27-dimensional module of type E_6 to D_5 or to A_5 one has to cut the weight diagram at Fig. 20 through the bonds marked with 1 or with 2, respectively[5]. In this way one gets summands of degrees 1,16 and 10, or, respectively, 6, 15 and 6. One may go one step further and restrict to D_4, cutting both 1 and 6 and getting three summands of degrees 8 and three summands of degree 1; or to A_4, cutting both 2 and 6 and getting three summands of degree 5, one summand of degree 10 and 2 summands of degree 1.

For the remaining subsystem subgroups the procedure is only slightly more complicated. It is classically known that any subsystem of Φ is obtained by spanning a subsystem by a subset of the *extended* fundamental system $\overline{\Pi} = \Pi \cup \{\alpha_0\}$

[5] As shown in [82] for microweight representations there is an *a priori* correspondence between the irreducible constituents of the restriction to a subsystem subgroups and the corresponding double cosets of the Weyl group, see Sec. 3 above.

and then repeating this procedure for every irreducible component of the resulting systems, etc. In other words, the negative maximal root has to be introduced in the picture as well. As observed by C. Parker, an advanced way to do this is to draw the corresponding weight diagram for the *affine* Weyl group, cut it along the bonds labelled i, where $\Delta = \langle \overline{\Pi} \setminus \{\alpha_i\} \rangle$, and then to identify the weights in one w_0-orbit again. (The genuine Russian approach was, of course, to roll a sheet with the weight diagram in the form of a cylinder, glue it up and cut it elsewhere.)

Not to go into details here, we illustrate the idea of Parker's method by the following self-explanatory example:

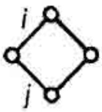


This picture shows that the restriction of the 27-dimensional module for E_6 to $3\,A_2$ is the direct sum of three 9-dimensional modules, each of which is the tensor product of the natural module for one copy of A_2 with the dual natural module for another copy.

(8) *Orbit of the highest weight vector.* It is well known that the orbit $G\,v^+$ of the highest weight vector $v^+ \in V$ is an intersection of quadrics [68]. For example, when $(\Phi, \omega) = (A_l, \overline{\omega}_k)$, the equations defining these quadrics are precisely the *Plücker equations*. At a seminar C. M. Ringel asked one of us "Does the fact that 10 out of the 27 quadratic equations defining the orbit of the highest weight vector for the 27-dimensional representation of E_6 are algebraically independent anything to do with the fact that the corresponding weight diagram (Fig. 20) has 10 squares?". It does indeed, and the answer is called the theory of standard monomials [27, 63–65, 104, 105].

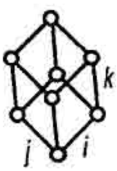
Let $R = K$ be a field. Then any unimodular row of length $l+1$ over K can be the first row of $SL(l+1, K)$ in the natural representation (for a commutative ring there are further K-theoretical obstructions which depend on a ring and which we do not discuss here). The only other representation which has this property is the natural representation of $Sp(2l, K)$. In the diagrams of these representations (Figs. 1 and 3) this is expressed by the fact that they do not have neither squares, nor two consecutive bonds with the same label. The orbit of the highest weight vector in the natural representation of an orthogonal group is a *quadric*, or, in other words, it is defined by one homogeneous quadratic equation. This equation is visible in the corresponding diagrams (Figs. 2 and 4). Namely the diagram for the case B_l has two consecutive bonds in the middle labeled by l. On the other hand, the diagram for the case D_l is not a chain anymore, it has a square in the middle.

In general, any occurrence of either of these situations drops the dimension of the orbit of the highest weight vector by 1. For example, a square



corresponds to the tensor product of the natural representations of $A_1 = \langle\alpha_i\rangle$ and $A_1 = \langle\alpha_j\rangle$. To be the first column of a matrix of $SL(2,K) \times SL(2,K)$ in such a representation, a column $(x_1, x_2, x_3, x_4)^t$ must satisfy the equation $x_1x_4 = x_2x_3$. In general, counting the squares in the weight diagram of the tensor product of the natural representations of A_l and A_m convinces us that in this case the dimension of the orbit of the highest weight vector is $l+m+1$ (there are $(l+1)(m+1)$ vertices and lm small squares). This is, of course, classically known (*Segre embedding*). As another example, Fig. 8 suggests that the dimension of the orbit of the highest weight vector in the spin representation of B_4 equals $16 - 5 = 11$.

The same argument works for more complicated pictures. For example, it is easy to check that a cube:



(which corresponds to the tensor product of the natural representations of $A_1 = \langle\alpha_i\rangle$, $A_1 = \langle\alpha_j\rangle$ and $A_1 = \langle\alpha_k\rangle$) gives us 4 independent quadratic equations. Now Fig. 21 convinces us that the orbit of the highest weight vector in $(E_7, \overline{\omega}_7)$ has dimension $56 - 24 - 4 = 28$.

However weight diagrams contain *much* more information than just the dimensions of the orbits. Figure 20 has 10 small squares, but even more remarkably, it has 27 rectangles. Each of these rectangles represents an equation and in fact a slightly more delicate analysis shows that one can read off the shape of the equation and the corresponding signs from the weight diagram [120, 122]. No wonder, since the theory of standard monomials [63–65, 104, 105] tells us that for microweight representations the equations come from subsystems of type D_m, in other words they have the shape

$$\pm x_{\lambda_1} x_{\mu_1} \pm \ldots \pm x_{\lambda_m} x_{\mu_m} = 0\,,$$

where $\lambda_1 + \mu_1 = \ldots = \lambda_m + \mu_m$ and there is exactly one pair (λ_i, μ_i) such that λ_i and μ_i are not comparable.

(9) *Multilinear invariants.* The quadratic equations described in the preceding subsection are a part of a more general problem: to describe all equations among matrix entries of a matrix representing an element of a Chevalley group G in a representation (V, π). Indeed, we seldom think of the split classical groups as being

generated by the root unipotents $x_\alpha(\xi)$. For most mathematicians they are rather the isometry groups of certain bilinear/quadratic forms.

Analogous realizations of some of the exceptional groups in terms of multilinear forms/forms of higher degree were known already to L. E. Dickson. Later in the fifties and early sixties such realizations were extensively studied by H. Freudenthal, C. Chevalley, T. A. Springer, J. Tits, F. Veldkamp, N. Jacobson and others. However in the mid sixties this theory went out of fashion.

Now after the works of M. Aschbacher, A. M. Cohen and others [1–3, 26–29, 117, 124], etc., it is gradually becoming clear that the approach of H. Freudenthal was the correct approach to the exceptional groups. These realizations are extremely useful even to study exceptional groups over finite fields, where there are a variety of other methods that work. For example, the corresponding geometries are much richer than buildings and allow to construct subgroups of the exceptional groups more easily (it has been used by M. Aschbacher and others to classify the maximal subgroups of the finite exceptional groups). Over rings this might be even the *only* reasonable approach.

The characteristic free multilinear invariants for the exceptional groups may be very easily reconstructed from the diagram. Once more, this applies not only to their shape, but also to the corresponding signs. This has been demonstrated in [117] in an easy example of $(\Phi, \pi) = (\mathrm{E}_6, \overline{\omega}_1)$. In this case the monomials appearing in the cubic form invariant under the action of $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ form one orbit under the action of the *extended* Weyl group $\widetilde{W} = \langle w_\alpha(1),\ \alpha \in \Phi\rangle$. Thus one fixes the sign of a single monomial, say $x_\lambda x_\mu x_\nu$, where λ is the leftmost node of the diagram, μ is the upper middle node and ν is the rightmost node, and applies the elements of $\widetilde{W}$ to it to get the signs of other monomials. As a permutation of the one-dimensional subspaces $\langle v^\lambda\rangle$ the preimage $w_i(1)$ of a fundamental reflection s_i acts exactly as does s_i, but it also changes sign of v^λ if $\lambda - \alpha_i$ is a weight of π, see [117, 120] for a detailed analysis in this case and [122] in general.

(10) *Matrices for Chevalley groups.* For the *natural* representations of the classical Chevalley groups (that is $\mathrm{SL}(l+1, R)$, $\mathrm{SO}(2l+1, R)$, $\mathrm{Sp}(2l, R)$ and $\mathrm{SO}(2l, R)$ for types A_l, B_l, C_l and D_l respectively, Figs. 1–4) it is easy to perform calculations involving the whole matrix $\pi(g)$. The use of matrices is especially important, when everything else does not work, i.e. for groups over rings of large dimension, which have few units. In these cases the corresponding groups do not admit anything like Bruhat or Gauß decomposition. This inhibits efficient use of elementary calculations.

It is our belief that the easiest and one of the most efficient ways to think about the groups of types G_2, F_4, E_6, E_7 and E_8 over rings is precisely to think of them as certain groups of 7×7, 26×26, 27×27, 56×56 or 248×248 matrices. Let (V, π) be a representation of $G = \mathrm{G}(\Phi, R)$. Then an element $g \in G$ may be represented by a matrix $\pi(g) = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda(\pi)$, with respect to an admissible base (recall, that we always consider weights with multiplicities). It is crucial here, that the rows and columns of the matrices $\pi(g)$ are *partially* ordered by the corresponding weight diagram and not linearly ordered.

For example, the μth column $g_{*,\mu}$ of the matrix $\pi(g)$ consists of the coefficients in the expansion of $\pi(g)v^{\mu}$ with respect to v^{λ}. In other words, columns above are obtained by freezing the second index in $(g_{\lambda\mu})$. Such columns may be identified with the corresponding elements of V. Analogously the rows $g_{\lambda,*}$ are obtained by freezing the first index and correspond to the vectors from the dual module V^*. As we know from Secs. 4 and 8, one can very efficiently calculate with such columns and rows using the corresponding weight diagrams.

As has been observed in [126], many usual calculations with Chevalley groups over rings may be reorganized in such a way, that they would involve only *elementary calculations* (the ones based on the *Steinberg relations* among the elementary root unipotents, [21, 111]) and the *stable calculations* (i.e. calculations involving only one row or one column of a matrix at a time, [79, 110]). In particular, this applies to the calculations needed to prove the main structure theorems for Chevalley groups over commutative rings and stability of K-functors. See [117, 124] for a detailed description of the whole project and exhaustive references. Further papers are forthcoming. The same techniques have been applied by L. Di Martino and N. A. Vavilov to study generation of finite Chevalley groups [41].

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