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MATHEMATICS

Axiomatic Definition of Small Cancellation Rings

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Abstract—In the present paper, we develop a small cancellation theory for associative algebras with a basis of invertible elements. Namely, we study quotients of a group algebra of a free group and introduce three specific axioms for corresponding defining relations that provide the small cancellation properties of the obtained ring. We show that this ring is nontrivial. It is called a small cancellation ring.

Keywords: small cancellation ring, turn, multi-turn, defining relations in rings, small cancellation group, group algebra

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In this work, we present an axiomatic definition of a small cancellation ring given by generators and defining relations. A theorem on the nontriviality of a small cancellation ring is stated (see Theorem 1). The complete proof of this result takes about 300 pages, and its preliminary version can be found in [3].

1. DESCRIPTION OF THE PROBLEM

The following question is of undoubted interest: If the interactions between defining relations are weak in a certain sense, does the resulting algebra possesses some properties of a free algebra?

In the case of groups, semigroups, and monoids, the small cancellation theory yields the affirmative answer (for more details, see [10, 17]). However, the construction of such a theory for systems with several operations faces significant difficulties. Presented in this paper, the general theory of group-like small cancellation associative rings with a basis of invertible elements was constructed after studying the special case considered in [2]. Theorem 3 shows that the ring introduced in [2] is a particular case of a small cancellation ring. This ring can be viewed as the first step towards in the construction of a division algebra with a finitely generated multiplicative group. Note that this problem goes back to Latyshev, Kaplansky, Lvov and Kurosh the ring situation is essentially more complicated than the group one.

Our motivation is based on the fact that the theory of small cancellation groups and, especially, the iterated small cancellation theory of groups (constructed by Novikov and Adian in solving the Burnside problem) plays a major role in the solution of many classical problems in group theory. This theory provides a powerful method for constructing groups with unusual and even exotic properties, such as infinite Burnside groups [12, 1, 15, 8, 11, 4], Tarski monster group [14], finitely generated infinite divisible groups [7], and many others [13]. Our goal is to develop a similar method for rings. It allows us to take control over defining relations. We mean that the construction of many objects relies on natural, systems of relations. A principle problem is to prevent the appearance of redundant consequences of these relations, in particular, to prevent the degeneracy of the object. In fact the constructed approach will be useful, for example, to construct a division algebra with a finitely generated multiplicative group (which is the same as to be finitely generated as a ring), as well as an analogue of a Tarski monster group and a generalization of Bergman's centralizer theorem for a wider class of rings (in particular, Bergman's centrilizer theorem for the group algebra of a free group).

The classical Novikov–Adian method for solving the Burnside problem [12, 1] (and for constructing a Tarski monster group, i.e., a group generated by any

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two of its noncommuting elements) relies on complicated induction on the rank with a large number of induction hypotheses. The moving force of this method is that different periodic words of the same rank have small interaction period lengths have small common parts. Moreover, after a special reduction, relations of the same rank have small interaction with each other modulo possible transformations of lower ranks. This corresponds to generalized small cancellation conditions, which allows to process the induction step. Note that the concept of turn in a given rank used in Novikov and Adian's work was the starting point for our basic concept of multi-turn (see the definition in Subsection 2.2 below and [2, 3]).

The similar complicated problems are also well known in ring theory, while the corresponding universal approach to them is is not yet known. We believe that the iteration of our construction in the same way as in solving the Burnside problem would yield the desired method for constructing rings and associative algebras with given properties. This paper deals with the case of small cancellations, i.e., when there is only one rank. Specifically, our group-like small cancellation ring is the first step in the construction of a division algebra with a finitely generated multiplicative group.

It is still unknown how a geometric object can be matched to an associative ring. Gromov's program "Infinite groups as geometric objects" (see [5] and also [6]) is preceded by the combinatorial approach. In the group case, a combinatorial object corresponding to a hyperbolic group is a small cancellation group (if each relation is represented as a product of at least seven small pieces). We hope to introduce a definition of a "hyperbolic ring," starting with the small cancellation rings considered in this paper. Apparently, if the definition of a hyperbolic ring is given, then the small cancellation rings considered in this paper should presumably be such rings. We believe that the group rings of hyperbolic groups will present another particular case of hyperbolic rings.

It is natural to assume that, having an appropriate definition of torsion in the ring case, the constructed torsion free rings should without torsion have low cohomological dimension and one-dimensional centralizers of elements, possess the non-amenability property, and answer in the affirmative to Kaplansky's zero divisor conjecture. The development of an iterated small cancellation theory can be useful, for example, for solving the classical problem of constructing a division algebra with a finitely generated multiplicative group.

Note that A. Smoktunovich's approach to the control of relations in rings has led to the construction of a simple nilring and other important examples of nilalgebras (see, e.g., [18, 9]). This approach is based on quite different ideas and is not related to small cancellation theory.

2. SMALL CANCELLATION AXIOMS FOR RINGS

2.1. Small Cancellation Groups

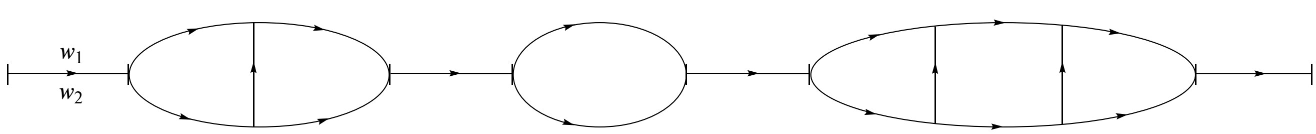
Consider a group G given by generators and defining relations $G = \langle X \mid \mathcal{R} \rangle$. Assume that the set $\mathcal{R}$ of defining relations is closed under taking cyclic shifts and inverses and that all elements of $\mathcal{R}$ are cyclically reduced. The interaction between defining relations is described in terms of small pieces. A word s is called a small piece with respect to $\mathcal{R}$ (in generalized group sense, see [16, 10]) if there are relations of the form sr_1 and sr_2 in $\mathcal{R}$ such that $r_1r_2^{-1} \neq 1$ and $r_1r_2^{-1}$ is not conjugate to a relation from $\mathcal{R}$ in the corresponding free group, even after possible cancellations.

Remark 1. Geometrically small pieces can be treated as words that may appear on the common boundary between two cells in the van Kampen diagram [13, 10]. In particular, if $r_1r_2^{-1} \in \mathcal{R}$, then these cells can be replaced by a single one. Accordingly, we assume from the beginning that $r_1r_2^{-1} \notin \mathcal{R}$.

The small cancellation condition $C(p)$ means that any relation in $\mathcal{R}$ cannot be written as a product of less than p small pieces. For most purposes, seven small pieces suffice, since, under the condition $C(7)$, the discrete Euler characteristic becomes negative [10]. To guarantee this, we can assume that the length of any small piece is less than one-sixth of the length of the relation in which it appears. The main theorem of small cancellation theory can be stated as follows.

Let w_1, w_2 be two words that do not contain occurrences of more than a half of a relation from $\mathcal{R}$. They represent the same element of G if and only if they can be connected by a one-layer map [10, Greendlinger's lemma]. (Specifically, it yields that a small cancellation group is nontrivial.) The transition from w_1 to w_2 can be represented in the form of a sequence of elementary steps, called turns [12]. Each turn reverses only one cell.

For reader's convenience, below we give an example of a one-layer map, where the word w_1 is read on its upper side, the word w_2, on the lower side, and the cells are group relations from $\mathcal{R}$



2.2. Basic Definitions for Rings

Let k be a field. Throughout this paper, small Greek letters are used to denote nonzero elements of k. Let $\mathcal{F}$ be the free group freely generated by an alphabet S. Elements of $\mathcal{F}$ are called monomials or words. Let $k\mathcal{F}$ denote the corresponding group algebra. Elements of $k\mathcal{F}$ are called polynomials. Let $a, b \in \mathcal{F}$. Their product is denoted by $a \cdot b$. We write ab if there are no cancellations between a and b.

Let a finite or infinite set of polynomials $\mathcal{R}$ from $k\mathcal{F}$ be fixed:

$$\mathcal{R} = \left\{ p_i = \sum_{j=1}^{n(i)} \alpha_{ij} m_{ij} \mid \alpha_{ij} \in k, m_{ij} \in \mathcal{F}, i \in I \right\}.$$

We assume that the monomials m_{ij} are reduced, the polynomials p_i are additively reduced, I is an index set, and all coefficients α_{ij} are nonzero.

Let $\langle \mathcal{R} \rangle$ denote the ideal generated (as an ideal) by $\mathcal{R}$. The set of all monomials m_{ij} of $\mathcal{R}$ is denoted by $\mathcal{M}$.

The goal of this work is to define the class of small cancellation rings. Such a ring is represented in the form $\mathcal{A} = k\mathcal{F}/\langle \mathcal{R} \rangle$, and our definition is formulated in the form of three conditions (axioms) on $\mathcal{R}$ (set of relations). Assume that $\mathcal{R}$ is fixed.

Condition 1 (Compatibility Axiom).

(i) If $p = \sum_{j=1}^{n} \alpha_j m_j \in \mathcal{R}$, then $\beta p = \sum_{j=1}^{n} \beta \alpha_j m_j \in \mathcal{R}$ for any $\beta \in k, \beta \neq 0$.

(ii) Let $x \in S \cup S^{-1}$, where S is an alphabet that freely generates $\mathcal{F}$, and $p = \sum_{j=1}^{n} \alpha_j m_j \in \mathcal{R}$. Assume that x^{-1} is the initial symbol of a certain m_i. Then

$$x \cdot p = \sum_{j=1}^{n} \alpha_j x \cdot m_j \in \mathcal{R}$$

(after making the cancelations in the monomials xm_j). We require that a similar condition be satisfied if x^{-1} is the last symbol of the monomial and the multiplication by x in the last equality is on the right side.

From the second condition of the compatibility axiom, it immediately follows that the set $\mathcal{M}$ is closed under taking subwords. In particular, the empty word always belongs to $\mathcal{M}$.

Now we define the concept of a small piece with respect to $\mathcal{R}$ in the algebra $k\mathcal{F}$. This definition plays a central role in our theory.

Definition 1. Let $c \in \mathcal{M}$. Assume that there exist two polynomials

$$p = \sum_{j=1}^{n_1} \alpha_j a_j + \alpha a \in \mathcal{R},$$

$$q = \sum_{j=1}^{n_2} \beta_j b_j + \beta b \in \mathcal{R},$$

such that c is a subword of a and a subword of b, i.e.,

$$a = \hat{a}_1 c \hat{a}_2, \quad b = \hat{b}_1 c \hat{b}_2,$$

where $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, and $\hat{b}_2$ are possibly empty. Assume that

$$\hat{b}_1 \cdot \hat{a}_1^{-1} \cdot p = \hat{b}_1 \cdot \hat{a}_1^{-1} \cdot \left(\sum_{j=1}^{n_1} \alpha_j a_j + \alpha \hat{a}_1 c \hat{a}_2 \right)$$

$$= \sum_{j=1}^{n_1} \alpha_j \hat{b}_1 \cdot \hat{a}_1^{-1} a_j + \alpha \hat{b}_1 c \hat{a}_2 \notin \mathcal{R}$$

(even after making the cancelations) or

$$p \cdot \hat{a}_2^{-1} \cdot \hat{b}_2 = \left(\sum_{j=1}^{n_1} \alpha_j a_j + \alpha \hat{a}_1 c \hat{a}_2 \right) \cdot \hat{a}_2^{-1} \cdot \hat{b}_2$$

$$= \sum_{j=1}^{n_1} \alpha_j a_j \cdot \hat{a}_2^{-1} \cdot \hat{b}_2 + \alpha \hat{a}_1 c \hat{b}_2 \notin \mathcal{R}$$

(even after making the cancelations). Then the monomial c is called a *small piece*.

The set of all small pieces is denoted by $\mathcal{S}$. Clearly, $\mathcal{S} \subseteq \mathcal{M}$. The definition implies that $\mathcal{S}$ is closed under taking subwords. In particular, if $\mathcal{S}$ is nonempty, then the empty word is always a small piece. If $\mathcal{S}$ turns out to be empty, then we still assume that the empty word is a small piece.

Let $u \in \mathcal{M}$. Then either $u = l_1, \dots, l_m$, where $l_1, \dots, l_m$ are small pieces, or u cannot be represented as a product of small pieces. We introduce a measure on monomials of $\mathcal{M}$ (known as the Λ-measure). We say that $\Lambda(u) = m$ if u can be represented as a product of small pieces and the minimum possible number of small pieces in such a representation is equal to m. We say that $\Lambda(u) = \infty$ if u cannot be represented as a product of small pieces.

We fix a constant $\tau \in \mathbb{N}$, $\tau \geq 10$.

Condition 2 (Small Cancellation Axiom with a constant τ). Assume that $p_1, \dots, p_n \in \mathcal{R}$ and a linear combination $\sum_{s=1}^{n} \gamma_s p_s$ is nonzero after making additive cancellations.

Then there exists a monomial a in $\sum_{s=1}^{n} \gamma_s p_s$ with a nonzero coefficient after additive cancellations such that either either a cannot be represented as a product of small pieces or every representation of a as a product of small pieces contains at least $\tau + 1$ small pieces.

That is, $\Lambda(a) \geq \tau + 1$, including $\Lambda(a) = \infty$.

Definition 2. Let $p = \sum_{j=1}^{n} \alpha_j a_j \in \mathcal{R}$. The monomials a_{j_1}, a_{j_2}, $1 \le j_1, j_2 \le n$, are called *incident monomials* (including the case $a_{j_1} = a_{j_2}$). Recall that $\alpha_j \ne 0$, $j = 1, \dots, n$.

Now we introduce the last condition, called Isolation Axiom. In contrast to two preceding axioms, this is an entirely ring-theoretic condition. It makes use of the concept of maximum occurrence of a monomial of $\mathcal{M}$ and the concept of overlap of occurrences.

We consider occurrences of the form $a \in \mathcal{M}$ in a word U, i.e., $U = LaR$, where L and R can be empty. By a maximal occurrence, we mean an occurrence of a monomial of $\mathcal{M}$ that is not contained in a larger occurrence of this kind. Note that the common part of two maximal occurrences is a small piece.

An overlap is defined as a common part of two maximal occurrences.

The third axiom imposes some natural constraints on the rings under consideration. We use its weakest form to cover the largest class of rings. This results in a significant complication of the definition.

Condition 3 (left-sided Isolation Axiom with a constant τ). Let $m_1, m_2, \dots, m_d$ be an arbitrary sequence of monomials of $\mathcal{M}$ such that $m_1 \ne m_d$, $\Lambda(m_i) \ge \tau - 2$ for all $i = 1, \dots, d$, and m_i and m_{i+1} are incident monomials for all $i = 1, \dots, d-1$, and let $a \in \mathcal{M}$ be any monomial with the following properties:

(i) $\Lambda(a) \ge \tau - 2$;

(ii) $am_1, am_d \notin \mathcal{M}$, where am_1 and am_d have no cancellations;

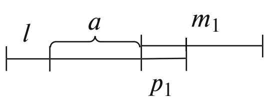
(iii) m_1 is a maximal occurrence in am_1, and m_d is a maximal occurrence in am_d;

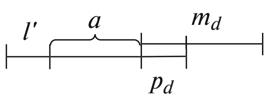
(iv) if ap_1 is a maximal occurrence in am_1 that contains a, and ap_d is a maximal occurrence in am_d that contains a, then there exist monomials $l, l' \in \mathcal{M}$ such that

• l, l' are small pieces,

• $la, l'a \in \mathcal{M}$, where la and $l'a$ have no cancellations,

• there exists a sequence of monomials $b_1, \dots, b_n$ of $\mathcal{M}$ such that $b_1 = lap_1$, $b_n = l'ap_d$, b_i, b_{i+1} are incident monomials for all $i = 1, \dots, n-1$, and $\Lambda(b_i) \ge \tau - 2$





Then $p_1^{-1} \cdot m_1 \ne p_d^{-1} \cdot m_d$

The right-sided Isolation Axiom with a constant τ is formulated symmetrically.

Definition 3. We say that $\mathcal{A} = k\mathcal{F}/\langle\mathcal{R}\rangle$ is a $C(\tau)$-*small cancellation ring* if it satisfies the compatibility axiom, the small cancellation axiom with condition τ, and at least one of the isolation axioms with a constant τ.

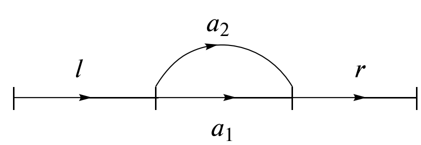
Given a group $G = \langle X \mid \mathcal{R}\rangle$, $m = pq \in \mathcal{R}$, and $w = lpr$, the transformation $w = lpr \mapsto lq^{-1}r = w'$ is called a *turn*.

We replace the concept of a group turn by the concept of a ring multi-turn. Suppose that $\sum_{j=1}^{n} \alpha_j m_j \in \mathcal{R}$, where all $\alpha_j \ne 0$. Assume that v is a monomial of the form $v = lm_h r$ for some h, $1 \le h \le n$. The transition from $v = lm_h r$ to $\sum_{j=1, j\ne h}^{n} (-\alpha_h^{-1}\alpha_j) lm_j r$ is called a *multi-turn*. This transformation extends linearly to βv and then linearly to all polynomials containing monomials of the form βv. The corresponding polynomial $\sum_{j=1}^{n} \alpha_j lm_j r$ is called a *layout* of the given multi-turn.

Examples

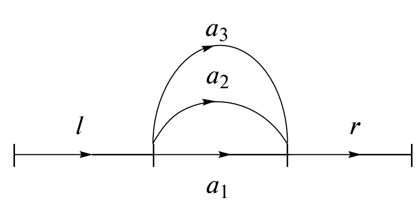
For simplicity, we consider a group algebra over a field of two elements.

Example A. Assume that $v = la_1 r$ and consider a polynomial $a_1 + a_2 \in \mathcal{R}$. In this case, the transition from $la_1 r$ to $la_2 r$ is produced by the corresponding multi-turn:



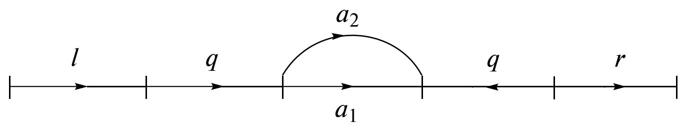
The transition from $la_1 r$ to $la_2 r$ shows that a turn can be treated as a special case of a multi-turn.

Example B. Assume that $v = la_1 r$ and consider a polynomial $a_1 + a_2 + a_3 \in \mathcal{R}$. Then we obtain the transition from $la_1 r$ to $la_2 r + la_3 r$. Graphically, this can be represented as follows:

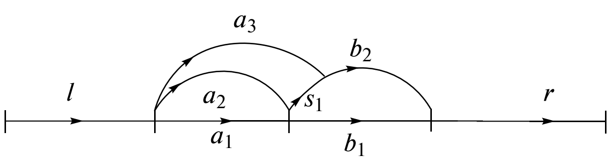


Example C. Assume that $v = lqa_1q^{-1}r$ and consider a polynomial $a_1 + a_2 + 1 \in \mathcal{R}$. Then the corresponding multi-turn is the transition from $lqa_1q^{-1}r$ to

$lqa_2q^{-1}r + l \cdot r$. Note that, after the replacement of a_1 by 1, the subword q cancels out with the subword q^{-1}. The result has the following form (the empty word 1 is not shown):



Example D. Assume that $v = la_1b_1r$ and consider polynomials $a_1 + a_2 + a_3s_1^{-1}$ and $b_1 + s_1b_2 + 1$ of $\mathcal{R}$. Then there are two neighboring multi-turns: a_1 is replaced by $a_2 + a_3s_1^{-1}$ and b_1 is replaced by $s_1b_2 + 1$. This is shown in the following figure:



Assume that a multi-turn on the left is followed by a multi-turn on the right. Then la_1b_1r is replaced by $la_2b_1r + la_3s_1^{-1}b_1r$ and the result is then replaced by $la_2s_1b_2r + la_2 \cdot r + la_3b_2r + la_3s_1^{-1} \cdot r$. Here, cancellations can occur in the monomials la_2r and $la_3s_1^{-1}r$. Now assume that a multi-turn on the right is followed by a multi-turn on the left. Then la_1b_1r is replaced by $la_1s_1b_2r + la_1 \cdot r$ and the result is then replaced by $la_2s_1b_2r + la_3b_2r + la_2 \cdot r + la_3s_1^{-1} \cdot r$ with cancellations made if necessary. Note that the second multi-turn has to be changed in order to apply it. Note also that the final result does not depend on the order of making the multi-turns.

The above examples show that, in contrast to groups, multi-turns in rings give rise to new effects.

We define the following vector space associated with a given monomial and a set of multi-turns. First, we consider the monomial $v = lm_hr$ and one multi-turn generated by the polynomial $\sum_{j=1}^{n} \alpha_j m_j \in \mathcal{R}$. Then the corresponding space is linearly generated by the monomials lm_jr, $j = 1, \dots, n$ (after making possible cancellations) with linear dependence $\sum_{j=1}^{n} \alpha_j lm_jr$. Now we consider the monomial v and several multi-turns. Then the corresponding space is linearly generated by v and all monomials obtained from v with the help of these multi-turns, with linear dependences equal to layouts of these multi-turns.

Consider the above-defined vector space arising in Example D. Example D involves the monomials

$$la_1b_1r, \quad la_2b_1r, \quad la_3s_1^{-1}b_1r,$$

$$la_1s_1b_2r, \quad la_2s_1b_2r, \quad la_3b_2r,$$

$$la_1 \cdot r, \quad la_2 \cdot r, \quad la_3s_1^{-1} \cdot r$$

and linear dependences between them:

$$la_1b_1r + la_2b_1r + la_3s_1^{-1}b_1r = 0,$$

$$la_1s_1b_2r + la_2s_1b_2r + la_3b_2r = 0,$$

$$la_1 \cdot r + la_2 \cdot r + la_3s_1^{-1} \cdot r = 0,$$

$$la_1b_1r + la_1s_1b_2r + la_1 \cdot r = 0,$$

$$la_2b_1r + la_2s_1b_2r + la_2 \cdot r = 0,$$

$$la_3s_1^{-1}b_1r + la_3b_2r + la_3s_1^{-1} \cdot r = 0.$$

Note that the last polynomial is the sum of the previous ones. Thus, we have five (rather than six) linear dependences. Therefore, the dimension of the resulting vector space is at least four (since there are nine pairwise distinct generating monomials). Note that, in the case of groups, the corresponding vector space is always one-dimensional, i.e., this effect degenerates (see the diagram at the end of Subsection 2.1).

3. MAIN THEOREMS

Theorem 1. *Any small cancellation ring is nontrivial.*

Let us give examples of small cancellation rings.

Theorem 2. *The group algebra of a small cancellation group satisfying the condition $C(p)$ for $p \geq 22$ is a small cancellation ring.*

Let $\mathbb{Z}_2\mathcal{F}$ be the group algebra over the field $\mathbb{Z}_2$ of a free group $\mathcal{F}$ with at least four generators. Let $w \in \mathcal{F}$ be a monomial and $|w|$ be its length. Suppose that none of the letters x, x^{-1}, y, y^{-1} occur at the beginning or the end of $w \in \mathcal{F}$. Let m and n be positive integers such that $|w| \ll m \ll n$, and let $v \in \mathcal{F}$ be a monomial given by

$$v = x^m y x^{m+1} y \cdots x^{n-1} y$$

(where the symbol $\ll$ means "much less than").

Suppose that $\mathcal{R}$ consists of the trinomial $1 + v + vw$ and $\mathcal{A} = k\mathcal{F}/\langle\mathcal{R}\rangle$.

Theorem 3. *The ring $\mathcal{A}$ is a small cancellation ring.*

Remark 2. In the algebra $\mathcal{A}$, the element $1 + w$ is invertible, since $v \in \mathcal{F}$ is invertible and if $1 + v + vw = 0$, then $v(1 + w) = (1 + w)v = 1$. A division algebra with a finitely generated multiplicative group is constructed by iterating the following procedure: given a monomial w, we construct a monomial v as above the monomial v and the relation $1 + v + vw = 0$. In the limit, it turns out that the sum of any two monomials is either zero (then they correspond the same element of the algebra) or is equal to the third monomial (then it is invertible). In this way, a division algebra with a finitely generated multiplicative group is constructed. This division ring is finitely generated not only as a ring, but also as a semigroup. The problem is to prove the non-triviality of the ring in question. Rather non-trivial, Theorem 3 makes it possible to take the first step in this procedure.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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