A NON EXPLICIT COUNTEREXAMPLE TO A PROBLEM OF QUASI-NORMALITY

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Abstract. In 1986, S.Y. Li and H. Xie proved the following theorem: *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic in some domain D, all of whose zeros are of multiplicity at least k. Then $\mathcal{F}$ is normal if and only if the family $\left\{\frac{f^{(k)}}{1+|f|^{k+1}} : f \in \mathcal{F}\right\}$ is locally uniformly bounded in D.*

Here we give, in the case $k = 2$, a counterexample to show that if the condition on the multiplicities of the zeros is omitted, then the local uniform boundedness of $\mathcal{F}_2$ does not imply even quasi-normality. In addition, we give a simpler proof for the Li-Xie theorem (and an extension of it) that does not use Nevanlinna's Theory which was used in the original proof.

1. Introduction

Marty's Theorem characterizes normality by using the first derivative and it has an obvious geometrical meaning.

H.L. Royden [6] extended one direction of Marty's Theorem and proved

Theorem R. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, with the property that for each compact set $K \subset D$, there is a positive increasing function $h_K : [0, \infty) \to [0, \infty)$ such that*

$$|f'(z)| \leq h_K(|f(z)|) \tag{1}$$

for all $f \in \mathcal{F}$ and $z \in K$. Then $\mathcal{F}$ is normal in D.

This result was extended further in various directions. In [2], (1) is limited to only 5 values. In [7, Thm.2], h_K is replaced by a nonnegative function that needs to be bounded in a neighborhood of some x_0, $0 \leq x_0 < \infty$. Then, in [10] it was shown that it is enough that h_K be finite only in a single point $0 < x_0 < \infty$. Moreover, in [7, Thm.3], this result is extended further to higher derivatives, i.e., (1) is replaced by

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$|f^{(\ell)}(z)| \le h_K(|f(z)|)$, $f \in \mathcal{F}$, $z \in K$, where $\ell \ge 2$ and the members of $\mathcal{F}$ have zeros of multiplicity $\ge \ell$. The following generalization of Marty's Theorem due to S.Y. Li and H. Xie also deals with higher derivatives.

Theorem LX. [3] *Let $k \ge 1$ be an integer and let $\mathcal{F}$ be a family of functions meromorphic on a domain D such that each $f \in \mathcal{F}$ has zeros only of multiplicity $\ge k$. Then $\mathcal{F}$ is normal in D if and only if the family*

$$\mathcal{F}_k = \left\{ \frac{f^{(k)}}{1+|f|^{k+1}} : f \in \mathcal{F} \right\} \quad \textit{is locally uniformly bounded in } D. \tag{2}$$

The direction ($\Rightarrow$) in Theorem LX is true even without the assumption that the zeros of each $f \in \mathcal{F}$ are of multiplicity at least k. In Section 2, we give a simpler proof for the direction ($\Rightarrow$) of (an extension of) Theorem LX, without using Nevanlinna's Theory. The condition on the multiplicities of $f \in \mathcal{F}$ is essential in the direction ($\Leftarrow$). Indeed, let $\hat{\mathcal{F}}_k$ be the family of all polynomials of degree at most $k-1$ in some domain $D \subset \mathbb{C}$. Then $\frac{f^{(k)}}{1+|f|^{k+1}} \equiv 0$ for each $f \in \hat{\mathcal{F}}_k$, but $\hat{\mathcal{F}}_k$ is not normal in D. However, $\hat{\mathcal{F}}_k$ is a quasi-normal family in D (of order $k-1$).

Here, we'd like to remind the reader of the definition of quasi-normality: A family $\mathcal{F}$ of meromorphic functions in a domain $D \subseteq \mathbb{C}$ is said to be *quasi-normal* if from each sequence $\{f_n\}_n \subset \mathcal{F}$ one can extract a subsequence which converges locally uniformly (with respect to the spherical metric) on $D \setminus E$ where the set E (which may depend on $\{f_n\}_n$) has no accumulation point in D. If the exceptional set E can always be chosen to have at most q elements, we say that $\mathcal{F}$ is *quasi-normal of order at most* q. Finally, $\mathcal{F}$ is said to be *quasinormal of (exact) order* q if it is quasi-normal of order at most q, but not quasi-normal of order at most $q-1$. Now the question that naturally arises is whether the condition (2) implies quasi-normality.

The conjecture that (2) implies quasi-normality (without the assumption on the multiplicities of the zeros) gets support also from another direction.

First let us set some notation. For $z_0 \in \mathbb{C}$ and $r > 0$, we set $\Delta(z_0, r) := \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) := \Delta(z_0, r) \setminus \{z_0\}$. Furthermore, we denote the open unit disk by $\Delta := \Delta(0,1)$. We write $f_n \overset{\chi}{\Longrightarrow} f$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ on D if the convergence is in the Euclidean metric.

Let us recall the well-known result of L. Zalcman.

Lemma 1 (Zalcman's Lemma)**.** [9] *A family $\mathcal{F}$ of functions meromorphic in some domain D is not normal at $z_0 \in D$ if and only if there exist points z_n in D, $z_n \to z_0$; numbers $\varrho_n \to 0^+$, and functions $f_n \in \mathcal{F}$ such that*

$$f_n(z_n + \varrho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad \textit{in} \quad \mathbb{C}, \tag{3}$$

where g is a nonconstant meromorphic function in $\mathbb{C}$.

Now, suppose that g is a limit function from (3), and we have some $C > 0$ and $r > 0$ such that

$$\frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} \le C \quad \text{for every} \quad z \in \Delta(z_0, r) \quad \text{and} \quad n \in \mathbb{N}. \tag{4}$$

Let us denote the poles of g by P_g (P_g might be the empty set). Then

$$f_n(z_n + \varrho_n\zeta) \Rightarrow g(\zeta) \quad \text{on} \quad \mathbb{C} \setminus P_g. \tag{5}$$

(Here we substitute "$\overset{\chi}{\Rightarrow}$" by "$\Rightarrow$" since in every compact subset of $\mathbb{C} \setminus P_g$, $f_n(z_n + \varrho_n\zeta)$ is holomorphic for large enough n).

Differentiating (5) k times gives

$$\varrho_n^k f_n^{(k)}(z_n + \varrho_n\zeta) \Rightarrow g^{(k)}(\zeta) \quad \text{in} \quad \mathbb{C} \setminus P_g.$$

But then by (3) and (4), we get that $g^{(k)} \equiv 0$ in $\mathbb{C} \setminus P_g$ and so $g^{(k)} \equiv 0$ in $\mathbb{C}$. This implies that g is a polynomial of degree at most $k-1$. Hence, we get that the collection of all limit functions obtained by (3) is a quasi-normal family.

However, it turns out that without the condition on the multiplicities of the zeros, the family $\mathcal{F}$ of Theorem LX is not quasi-normal.

We construct a detailed counterexample for the case $k = 2$. This is the content of Section 3.

2. Proof of an extension of Theorem LX

As mentioned above, we want to show how Theorem LX can be proved without Nevanlinna's theory. This also gives us the opportunity to slightly extend this result by replacing the exponent $k + 1$ in the denominator with an arbitrary $\alpha > 0$ resp. $\alpha \ge k + 1$.

Theorem 1. *Let $k \geq 1$ be an integer, $\alpha > 0$ be a real number and let $\mathcal{F}$ be a family of functions meromorphic on a domain D. Then the following holds.*

(a) *If each $f \in \mathcal{F}$ has zeros only of multiplicity $\geq k$ and if the family*
$$\mathcal{F}_{k,\alpha} := \left\{ \frac{|f^{(k)}|}{1+|f|^\alpha} : f \in \mathcal{F} \right\}$$
is locally uniformly bounded in D, then $\mathcal{F}$ is normal.

(b) *If $\alpha \geq k+1$ and if $\mathcal{F}$ is normal, then $\mathcal{F}_{k,\alpha}$ is locally uniformly bounded.*

(c) *Assume that $\alpha > 1$ and that all functions in $\mathcal{F}$ are holomorphic. Then the normality of $\mathcal{F}$ implies that $\mathcal{F}_{k,\alpha}$ is locally uniformly bounded.*

For $\alpha = k+1$, (a) and (b) are just Theorem LX. For $k = 1$, (c) was already proven in [4]. For $a < k+1$, (b) clearly fails as the consideration of the single function $f(z) = \frac{1}{z}$ near its pole shows. That the condition $\alpha > 1$ in (c) is best possible is shown by the family of the functions $f_n(z) := (z-3)^n$ which is normal in the unit disk Δ and satisfies
$$\begin{aligned} \frac{|f_n^{(k)}(z)|}{1+|f_n(z)|^\alpha} &= n(n-1)\cdot \ldots \cdot (n-k+1) \cdot \frac{|z-3|^{n-k}}{1+|z-3|^{\alpha n}} \\ &\geq \frac{1}{2} \cdot (n-k)^k \cdot |z-3|^{n(1-\alpha)-k} \longrightarrow \infty \qquad (n \to \infty) \end{aligned}$$
for all $z \in \Delta$.

For the proof of (b), we need the following lemma.

Lemma 2. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of meromorphic functions in a domain D, satisfying $f_n \overset{\chi}{\Rightarrow} \infty$ in D. Then for every $\ell \in \mathbb{N}$, $\frac{f_n^{(\ell)}}{f_n^{\ell+1}} \Rightarrow 0$ in D.*

Proof. We apply induction. Since $\frac{1}{f_n(z)} \Rightarrow 0$ in D, we can differentiate it and obtain that $\frac{f_n'(z)}{f_n^2(z)} \Rightarrow 0$ in D, and this proves the case $\ell = 1$.

Assume that the lemma holds for $m \leq \ell$. We prove it now for the case $m = \ell+1$. We have $\frac{f_n^{(\ell)}}{f_n^{\ell+1}}(z) \Rightarrow 0$ in D, and hence, since $f_n(z) \Rightarrow \infty$ in D, also $\frac{f_n^{(\ell)}(z)}{f_n(z)^{\ell+2}} \Rightarrow 0$ in D. Differentiating the last convergence gives
$$\frac{f_n^{(\ell+1)}}{f_n^{\ell+2}}(z) - (\ell+2)\frac{f_n'}{f_n^2}\frac{f_n^{(\ell)}}{f_n^{\ell+1}}(z) \Rightarrow 0 \quad \text{in} \quad D.$$
The induction assumption for $m = 1$ and $m = \ell$ implies that the right term in the left hand above converges uniformly to 0 on compacta of D, and thus also $\frac{f_n^{(\ell+1)}}{f_n^{\ell+2}}(z) \Rightarrow 0$ in D, as required. ☐

For holomorphic functions a similar result has been proved by H. Chen and X. Hua [1]. Here the exponent $\ell+1$ is replaced by an arbitrary $\alpha > 1$. We need this result in the proof of (c).

Lemma 3. [1] *Let* $\{f_n\}_n$ *be a sequence of functions holomorphic in a disk* $\Delta(z_0, r)$ *which converges to* ∞ *uniformly in* $\Delta(z_0, r)$. *Then for all* $k \in \mathbb{N}$ *and all* $\alpha > 1$ *the sequence* $\left\{\frac{f_n^{(k)}}{f_n^{\alpha}}\right\}_n$ *converges to 0 locally uniformly in* $\Delta(z_0, r)$.

Proof of Theorem 1

(a) The proof given in [3] for $\alpha = k+1$ remains valid for arbitrary $\alpha > 0$. For completeness we give the details.

Assume that $\mathcal{F}_{k,\alpha}$ is locally uniformly bounded in D, and suppose by negation that $\mathcal{F}$ is not normal at some $z_0 \in D$. Then similarly to (3) we get the existence of f_n, z_n, ϱ_n and g such that $f_n(z_n + \varrho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta)$ in $\mathbb{C}$. With the same reasoning as above (see (4), (5) and the following lines), we deduce that g is a polynomial of degree at most $k-1$. But now according to the condition on the multiplicities of the zeros of each f_n, we get that the zeros of g also must be of multiplicity at least k. This implies that g has no zeros and thus g is a constant function, a contradiction.

(b) It suffices to consider the case $\alpha = k+1$. The case $\alpha > k+1$ then follows from the boundedness of $x \mapsto \frac{1+x^{k+1}}{1+x^{\alpha}}$ in $[0, \infty[$ which ensures that with a certain constant $M > 0$ we have

$$\frac{|f^{(k)}(z)|}{1+|f(z)|^{\alpha}}(z) \le M \cdot \frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}}$$

for all f meromorphic in D and all $z \in D$.

Assume that $\mathcal{F}$ is normal in D, and suppose by negation that $\mathcal{F}_{k,\alpha}$ (where $\alpha = k+1$) is not locally uniformly bounded in any neighborhood of some $z_0 \in D$. Thus, there exist functions $f_n \in \mathcal{F}$, and points $z_n \to z_0$ such that

$$\frac{f_n^{(k)}(z_n)}{1+|f_n^{k+1}(z_n)|} \underset{n\to\infty}{\longrightarrow} \infty. \tag{6}$$

By the normality of $\mathcal{F}$, $\{f_n\}_{n=1}^{\infty}$ has a subsequence that, without loss of generality, we also denote by $\{f_n\}_{n=1}^{\infty}$, such that $f_n \overset{\chi}{\Rightarrow} f$ in D.

We separate now into several cases according to the nature of f.

Case 1. $f \not\equiv \infty$.

Case 1.1 $f(z_0) \in \mathbb{C}$.

For small enough $r > 0$, by Weierstraß's theorem $f_n^{(k)}(z) \Rightarrow f^{(k)}(z)$ in $\Delta(z_0, r)$, and also $1 + |f_n^{k+1}(z)| \Rightarrow 1 + |f(z)|^{k+1}$ in $\Delta(z_0, r)$. Since $1 + |f_n(z)|^{k+1} \geq 1$, we get that

$$\frac{f_n^{(k)}(z)}{1 + |f_n(z)|^{k+1}} \Longrightarrow \frac{f^{(k)}(z)}{1 + |f(z)|^{k+1}} \qquad \text{in } \Delta(z_0, r),$$

a contradiction to (6).

Case 1.2 $f(z_0) = \infty$.

Here, for small enough $r > 0$, f is holomorphic in $\Delta'(z_0, r)$ and in addition $|f(z)| \geq 2$ and $|f_n(z)| \geq 2$ for all $z \in \Delta(z_0, r)$ and large enough n. Thus $\frac{f_n^{(k)}(z)}{1+f_n(z)^{k+1}}$ are holomorphic in $\Delta(z_0, r)$ for large enough n. We then get by the maximum principle that

$$\frac{f_n^{(k)}(z)}{1 + f_n(z)^{k+1}} \Rightarrow \frac{f^{(k)}(z)}{1 + f(z)^{k+1}} \quad \text{in} \quad \Delta(z_0, r)$$

and then for large enough n,

$$\max_{|z-z_0| \leq r/2} \frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} \leq \max_{|z-z_0| \leq r/2} \frac{|f_n^{(k)}(z)|}{|1 + f_n(z)^{k+1}|} \leq \max_{|z-z_0| \leq r/2} \frac{|f^{(k)}(z)|}{|1 + f(z)^{k+1}|} + 1.$$

The last expression is a positive constant that does not depend on n and this is a contradiction to (6).

Case 2. $f \equiv \infty$.

In this case, we get by Lemma 2 that $\frac{f_n^{(k)}(z)}{f_n(z)^{k+1}} \Rightarrow 0$ in D, and this is a contradiction to (6).

(c) Now we consider the case that $\alpha > 1$ and that all functions in $\mathcal{F}$ are holomorphic.

We assume that $\mathcal{F}$ is normal but that $\mathcal{F}_{k,\alpha}$ is not locally uniformly bounded in D. Then as in the proof of (b) we find a $z_0 \in D$, functions $f_n \in \mathcal{F}$ and points $z_n \in D$ such that $\lim_{n\to\infty} z_n = z_0$ and

$$\frac{|f_n^{(k)}|}{1 + |f_n|^\alpha}(z_n) \underset{n\to\infty}{\longrightarrow} \infty.$$

Since $\mathcal{F}$ is normal, after extracting a suitable subsequence we may assume that $\{f_n\}_n$ converges locally uniformly to some limit function f, possibly $f \equiv \infty$.

If f is holomorphic in D, then as in Case 1.1 of (b) by Weierstraß' theorem we obtain

$$\lim_{n\to\infty} \frac{|f_n^{(k)}|}{1 + |f_n|^\alpha}(z_n) = \frac{|f^{(k)}|}{1 + |f|^\alpha}(z_0) \neq \infty,$$

a contradiction.

If $f \equiv \infty$, then from Lemma 3 we deduce that $\left\{\frac{f_n^{(k)}}{f_n^\alpha}\right\}_n$ converges to 0 uniformly in a certain neighbourhood U of z_0. This contradicts our choice of $\{z_n\}_n$ and $\{f_n\}_n$.

3. Constructing the counterexample

We construct a sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$, such that for every $n \geq 1$ and $z \in \Delta(0,2)$, $\frac{|f_n''(z)|}{1+|f_n(z)|^3} \leq 1$ and $\{f_n\}_{n=1}^{\infty}$ is not quasi-normal in $\Delta(0,2)$.

Let $g_n(z) = z^n - 1$, $n \geq 1$. The zeros of g_n are all simple, $g_n(z_\ell^{(n)}) = 0$, $0 \leq \ell \leq n-1$, where $z_\ell^{(n)}$ is the ℓ-th root of unity of order n. Define for every $n \geq 1$, $h_n = g_n e^{p_n}$, where p_n is a polynomial to be determined. We have $h_n' = (g_n' + g_n p_n')e^{p_n}$, and $g_n'(z_\ell^{(n)}) \neq 0$, $0 \leq \ell \leq n-1$. We want that

$$p_n'(z_\ell^{(n)}) = -g_n''(z_\ell^{(n)})/2g_n'(z_\ell^{(n)}), \quad 0 \leq \ell \leq n-1 \tag{7}$$

to get that $h_n''(z_\ell^{(n)}) = 0$.

We have

$$h_n^{(3)} = e^{p_n}\left(g_n^{(3)} + 3g_n''p_n' + \boldsymbol{3g_n'p_n''} + g_n p_n^{(3)} + 3g_n'{p_n'}^2 + 3g_n p_n' p_n'' + g_n {p_n'}^3\right)$$

We want that

$$p_n''(z_\ell^{(n)}) = -(g_n^{(3)} + 3g_n''p_n' + 3g_n'{p_n'}^2)/3g_n'\Big|_{z=z_\ell^{(n)}}, \quad 0 \leq \ell \leq n-1 \tag{8}$$

to get $h_n^{(3)}(z_\ell^{(n)}) = 0$.

Observe that when (7) is satisfied to determine $p_n'(z_\ell^{(n)})$, then as in (7), condition (8) is in fact a condition that depends only on the values of g_n and its derivatives at the points $z_\ell^{(n)}$, $0 \leq \ell \leq n-1$.

We have

$$\begin{aligned} h_n^{(4)} = e^{p_n}\big(&g_n^{(4)} + 4g_n^{(3)}p_n' + 6g_n''p_n'' + \boldsymbol{4g_n'p_n^{(3)}} + g_n p_n^{(4)} + 6g_n''{p_n'}^2 + 12g_n'p_n'p_n'' + 3g_n {p_n''}^2 \\ &+ 2g_n p_n' p_n^{(3)} + 4g_n'{p_n'}^3 + 6g_n {p_n'}^2 p_n'' + g_n {p_n'}^4\big), \end{aligned}$$

we want that

$$\begin{aligned} p_n^{(3)}(z_\ell^{(n)}) = -\big(g_n^{(4)} + 4g_n^{(3)}p_n' + 6g_n''p_n'' + 6g_n''{p_n'}^2 + 12g_n'p_n'p_n'' + 4g_n'{p_n'}^3\big)/4g_n'\Big|_{z=z_\ell^{(n)}}, \\ 0 \leq \ell \leq n-1 \end{aligned} \tag{9}$$

to get $h_n^{(4)}(z_\ell^{(n)}) = 0$. Observe that when (7) and (8) are satisfied to determine $p_n'(z_\ell^{(n)})$ and $p_n''(z_\ell^{(n)})$, then also (9) is in fact a condition that depends only on the values of g_n and its derivatives at the points $z_\ell^{(n)}$, $0 \leq \ell \leq n-1$. By the theory of interpolation [8, p. 52], for every $n \geq 1$ the conditions (7), (8) and (9) can be achieved with a polynomial p_n of degree at most $4n-1$.

Now, by our construction, for every $n \geq 1$, h_n'' has a zero of multiplicity at least 3 at each point $z_\ell^{(n)}$, $0 \leq \ell \leq n-1$, and so $\frac{h_n''}{h_n^3}$ is holomorphic (in fact, entire) in $\Delta(0,2)$. Thus we have $\max\limits_{z \in \overline{\Delta}(0,2)} |h_n''(z)/h_n^3(z)| = c_n > 0$.

Define now for every $n \geq 1$, $f_n := a_n \cdot h_n$, where $|a_n|$ is a large enough constant such that $\left|\frac{c_n}{a_n^2}\right| \leq 1$ and such that every subsequence of $\{f_n\}_{n=1}^\infty$ is not normal at any point of $\partial\Delta = \{z : |z| = 1\}$. In fact, we can take $|a_n|$ to be so large such that $f_n \to \infty$ locally uniformly in $\mathbb{C} \setminus \partial\Delta$.

Now, for $z = z_\ell^{(n)}$, $0 \leq \ell \leq n-1$, $f_n''(z_\ell^{(n)}) = 0$ and thus the left hand side of (2) is zero. If $z \neq z_\ell^{(n)}$, $z \in \Delta(0,2)$, then $f_n(z) \neq 0$ and

$$\frac{|f_n''(z)|}{1 + |f_n(z)|^3} \leq \frac{|f_n''(z)|}{|f_n(z)|^3} = \frac{1}{|a_n|^2} \frac{|h_n''(z)|}{|h_n(z)|^3} \leq \frac{c_n}{|a_n|^2} \leq 1$$

and (2) is satisfied (uniformly in $\Delta(0,2)$). This completes the proof that $\{f_n\}_{n=1}^\infty$ has the desired properties to be a counterexample.

4. Some Remarks

Remark 1. We have not obtained an explicit formula for f_n, and this explains the title of this paper.

Remark 2. We have shown in fact a stronger counterexample: The condition that $\left\{\frac{f''}{f^3} : f \in \mathcal{F}\right\}$ is locally uniformly bounded does not imply quasi-normality of the family $\mathcal{F}$.

It doesn't even imply Q_α-normality of $\mathcal{F}$ for any ordinal number α since the constructed sequence $\{f_n\}_{n=1}^\infty$ and all of its subsequences are not normal at any point of the continuum $\partial\Delta$. (For the exact definition of Q_α-normality we refer to [5].)

Remark 3. In [4] we gave for the first time a differential inequality that implies quasi-normality and does not imply normality.

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