CHAPTER II: FINITE DIMENSIONAL ALGEBRAS

1. Introduction

We wish now to consider in general the structure of finite dimensional algebras A, first over a field k and then over an integral domain R. Some of what we describe will be true of wider classes of rings and algebras.

The general setting in which we will be operating for the remainder of the book will be the following. There will be a commutative ring R which will, in fact, be an integral domain, and a distinguished maximal ideal $\mathcal{M}$, with a fixed isomorphism

$$R/\mathcal{M} \tilde{\to} k.$$

We have a second field, K, which is the quotient field of R, and we will work with this triple (k, R, K), where the ring R provides a bridge between the two fields k and K. We will have an R-algebra A which is free as an R-module, and this will give us a corresponding triple of algebras $(\overline{A}, A, \hat{A})$, where

$$\overline{A} = A \otimes k$$

and

$$\hat{A} = A \otimes K.$$

The algebra A will be a deformation over R, with $\overline{A}$ called the *special fiber* and $\hat{A}$ the *generic fiber*.

Example 1.1. Let R be $k[t]$, let $\mathcal{M}$ be the ideal (t), so that there is a natural homomorphism $f : R \to k$ with $\operatorname{Ker} f = \mathcal{M}$, given by substituting zero for t. The quotient field K of R is just $k(t)$, a transcendental extension of degree 1. Let

$$A = k[t][x]/(x^3 - t^2 x).$$

Then

$$\overline{A} \tilde{\to} k[x]/x^3,$$

while $x^3 - t^2 x$ has three distinct roots over K and thus we can apply the Chinese Remainder theorem, so

$$\hat{A} \tilde{\to} K \oplus K \oplus K.$$

Our first concern will be to describe the structure of an algebra over a field.

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2. Finite dimensional algebras over a field

Let A be an finite dimensional algebra over a field k. Let us assume that k is sufficiently large that $A/\operatorname{Rad}(A)\tilde{\to}\bigoplus_i M_{n_i}(k)$. The diagonal idempotents of the matrix blocks provide a decomposition of the identity 1_A into a sum of primitive orthogonal idempotents, modulo $Rad(A)$. We now lift these to a complete set of primitive idempotents of A by using the following theorem from ring theory:

Lemma 2.1. *Let A be a k-algebra, I an ideal such that $\cap_{i=1}^{\infty} I^i = 0$, and suppose that $A/I\tilde{\to}S$, where S is a direct sum of matrix blocks over a field k. Then f has a splitting, i.e., there is an embedding of $g : S \to A$ such the $f \circ g = id_S$. The primitive idempotents in two different liftings are conjugate.*

Corollary 2.1.1. *If $I = \operatorname{Rad}(A)$, then two different complete sets of orthogonal primitive idempotents for a finite dimensional algebra are conjugate. Any two dempotents whose reductions modulo I lie in the same matrix block are conjugate.*

Proof. If we have two different complete sets of primitive orthogonal idempotents, e_i and e_i', then their images in S are conjugate by the Skolem-Noether theorem. Since $\operatorname{Ker}(f) = \operatorname{Rad}(A)$, the invertible element $\overline{u}$ of S by which they are conjugated lifts to a unit u of A. Conjugation of e_i' by u provides a set e_i'' of idempotents such that $f(e_i) = f(e_i'')$ for each i. Then, by the lemma, the two sets are conjugate.

If we have two individual idempotents $\{e_1, e_2\}$ in S, lying in the same matrix blocks, then complete each e_i to a set of orthogonal primitive idempotents, using the fact that $1_A - e_i$ is an idempotent orthogonal to e_i, and apply the theorem. □

Example 2.1. Consider the upper triangular 2×2 matrices over k, which we will designate by T^2. The radical is the one-dimensional set of off-diagonal elements generated by E_{12}. The semisimple quotient is $S\tilde{\to}k \oplus k$. The different possible pairs of orthogonal primitive idempotents are $E_{11} + aE_{12}, E_{22} - aE_{12}$. If $u = E_{11} + aE_{12} + E_{22}$, then

$$u^{-1}E_{11}u = E_{11} + aE_{12},$$

and

$$u^{-1}E_{22}u = E_{22} - aE_{12}.$$

Thus every possible pair of orthogonal primitive idempotents is conjugate to the standard pair E_{11}, E_{22}

Definition 2.1. The *degree* $m(e_i)$ of a primitive idempotent e_i is the degree of the matrix block to which it belongs. It is also called the *multiplicity* of the idempotent, because it coincides with the number of conjugate idempotents in a decomposition of the identity I_A of A as a sum of orthogonal primitive idempotents.

Example 2.2. In a group algebra over a field whose characteristic does not divide the order of the group, the degree of a primitive idempotent e_i is the degree of the irreducible representation corresponding to the matrix block in which the e_i lies.

Definition 2.2. Let A be an algebra over a field k, and let e_i be a complete set of primitive idempotents. The *projective decomposition* of A with respect to this set is the module direct sum decomposition

$$A = \bigoplus Ae_j.$$

The *Peirce decomposition* of A is a vector space direct sum decomposition

$$A = \bigoplus e_i A e_j.$$

Note. With regard to the dimensions of the summands in the decomposition, there is always complete symmetry with regard to idempotents in the same matrix block. If we hold e_j fixed and allow the e_i to range within a given matrix block then the dimensions will be constant, because the matrix units provide isomorphisms between the summands corresponding to different summands.

Definition 2.3. The *basis graph* of a finite dimensional algebra A over a field k is a graph with one vertex for each primitive idempotent, and

(1) n_{ij} arrows from e_j to e_i, for $i \neq j$,
(2) $n_{ii} - 1$ loops at e_i,

where

$$n_{ij} = \dim_k(e_i A e_j).$$

Remark 2.1. This definition is independent of the particular choice of complete primitive idempotent set, since any other set is conjugate, and conjugation, being an automorphism of the algebra, will preserve the dimensions of the summands of the Peirce decomposition.

In order that the basis graph should convey more information about the structure of the algebra, we will consider the Peirce decomposition of the powers of the radical, which gives a filtration of the Peirce components. Define

$$n_{ij}^t = \dim_k e_i(J^t/J^{t+1})e_j, \text{for t} = 0, 1, ...$$

Definition 2.4. The *weighted basis graph* is obtained from the basis graph of an algebra A by assigning weight t to n_{ij}^t loops or arrows for $t \geq 0$ and weight 0 to one arrow from e_i to e_j if i and j are different but lie in the same matrix block in $A/\operatorname{Rad}(A)$. We represent non-idempotent matrix units by arrows with a solid head, and all other weighted arrows by using a number of arrowheads equal to the weight of the arrow.

Example 2.3. (Four-dimensional algebras) We draw the basis graphs of several different algebras of dimension four. Notice that the blocks are simply the connected components:

(1) $k \oplus k \oplus k \oplus k$,
(2) $M_2(k)$,
(3) $k \oplus T^2$,
(4) $k < x, y > /(x^2, y^2, xy + yx)$,
(5) An algebra with two idempotents e_1 and e_2, and two radical elements x and y satisfying
$$x = e_1 x e_2,$$
$$y = e_2 y e_1,$$
and
$$xy = yx = 0.$$
(6) $k[x]/(x^4)$,

In all but (4), the algebra is uniquely determined by the weighted basis graph. The weighted basis graph in (4) corresponds to an entire family of algebras of the form
$$k < x, y > /(x^2, y^2, xy - syx),$$
for s an element of k.

Example 2.4. We now give weighted basis graphs for the group algebra of S_3 for characteristic 2,3,and 5. Let
$$S_3 =< a, b | a^2 = b^3 = e, aba = b^2 > .$$
The character table, as was given in the exercises at the end of the previous chapter, is
$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}, N = diag(1, 1, 2), \text{and } M = diag(1, 2, 3).$$
Thus
$$W = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}.$$
Now,
$$W^{-1} = 1/6 \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}.$$
Since W represents the coefficients of the e_i in the basis of the $\hat{C}_j$, this means that the coefficients of the $\hat{C}_j$ in the block idempotents e_i are given by the columns of W^{-1}.

Case 1: $char(k) = 2$. In this case the first two blocks coalesce, giving a block defined by the fact that the third idempotent, which, mod 2,

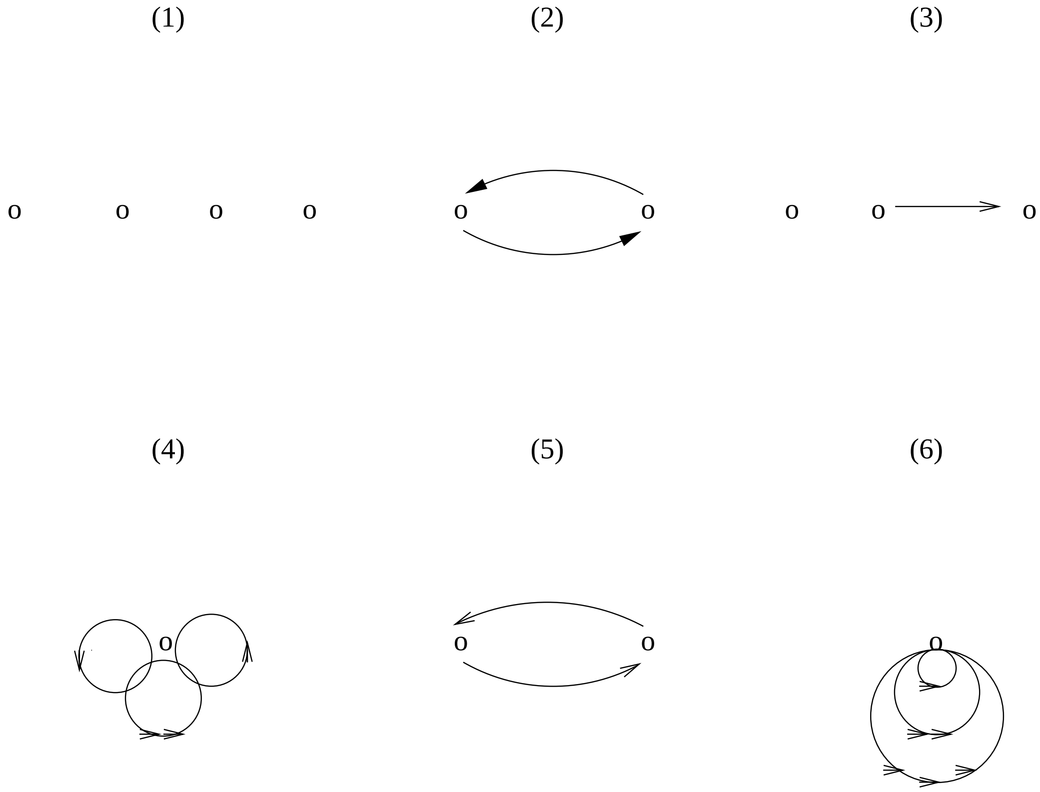


FIGURE 1. Basis graphs for some four dimensional algebras

equals $b + b^2$, is zero. Since b is invertible, this means that $b = 1$, so we have the group algebra of C_2 over a field of characteristic 2, which is isomorphic to $k[x]/(x^2)$, with $x = a - e$.

Case 2: $char(k) = 3$. Modulo 3, all the rows of W are identical, so there is only one block. The ideal J defined by $x = (b - b^2)$ is nilpotent, and its quotient is isomorphic to the semisimple algebra kC_2, so is the radical. The idempotents of kC_2 lift to idempotents $e_1 = (e + a)/2$ and $e_2 = (e - a)/2$. Thus we have the following nonzero Peirce components: $e_1(A/J)e_1 = e_1 + J$, $e_2(A/J)e_2 = e_2 + J$, $e_1(J/J^2)e_2 = e_1xe_2 + J^2$, $e_2(J/J^2)e_1 = e_2xe_1 + J^2$, $e_1(J^2)e_1 = e_1x^2e_1$, and $e_2(J^2)e_2 = e_2x^2e_2$. Thus the weighted basis graph has two vertices, two arrows of weight 1, and two loops of weight 2.

Case 3: $char(k) = 5$. This is the same as the characteristic zero case, so there are two blocks of dimension 1 and one matrix block of degree 2 and dimension 4.

Definition 2.5. The *points* of a finite dimensional algebra A are the conjugacy classes of primitive idempotents, under conjugacy by the set of units A^* of A. The set of points is denoted by $\mathcal{P}(A)$.

Lemma 2.2. *An element u of A is a unit if and only if it is a unit modulo* $\mathrm{Rad}(A)$.

Proof. One direction is trivial, for if v is the inverse of u in A, its image under the natural epimorphism is inverse to $\overline{u}$ in $A/\operatorname{Rad}(A)$.

In the other direction, we use induction on the powers of the radical. Let A_i denote $A/\operatorname{Rad}^i(A)$. Suppose u_{i+1} is an element of A_{i+1} and u_i is its image under the natural epimorphism from A_{i+1} onto A_i. If u_i is invertible with inverse v_i, then let v be an arbitrary lifting of v_i to A_{i+1}. Clearly, $u_{i+1}v = 1 + d$, for $d \in \operatorname{Rad}^i(A)$. Since $\operatorname{Rad}^i(A)/\operatorname{Rad}^{i+1}$ is an A_1 module, we now replace v by $v' = v(1-d)$, giving $u_{i+1}v' = 1$ Now suppose that

$$v'u_{i+1} = 1 + d'.$$

Multiplying on the left by v' and using fact that it is a left inverse, we get $d'v' = 0$. Multiplying the expression for $v'u_{i+1}$ by $(1 + d')$ on the right, we conclude that $d' = 0$, and thus v' is the desired inverse. □

Example 2.5. Let A be the fifth algebra in Example 2.3 with $k = \mathbb{Q}$. Thus $1_A = e_1 + e_2$, $x = e_1xe_2$ and $y = e_2ye_1$. Let $u_1 = (1/2)e_1 + (1/3)e_2 + 5x + 7y$. Set $v = 2e_1 + 3e_2$. Then

$$u_1v = e_1 + e_2 + 15x + 14y.$$

We thus define $v' = 2e_1 + 3e_2 - 30x - 42y$. Then $u_1v' = 1_A$, and

$$v'u_1 = e_1 + e_2 - 10x - 21y + 10x + 21y = 1_A$$

In Section 4 of Chapter 1, we stressed the importance of writing a module as a direct sum of indecomposable submodules. One of the most important modules over the k-algebra A is A as a left module over itself. In this case, the decomposition of 1_A as a sum of primitive orthogonal idempotents induces a decomposition of A as a direct sum of modules Ae_i.

Lemma 2.3. *If e_i is a primitive idempotent, then Ae_i is an indecomposable module.*

Proof. If we have a nontrivial direct sum decomposition

$$Ae_i = P' + P'',$$

then the idempotent decomposes as $e' + e''$. If one of these is zero, then one component P' or P'' contains a generator and thus is the entire module. If both are non-trivial, then both are idempotents by the rules of multiplication in a direct product, and they are orthogonal. This is a contradiction to the assumption that e_i is primitive. □

What made this proof work was the fact that if a module contained the generating element, it contained the entire module. A corresponding fact is true for module homomorphisms from Ae_i: the entire homomorphism is determined by the image of e_i. Suppose that $f : M \to N$ is an epimorphism of modules. Any map $g : Ae_i \to N$ then can be lifted to a map from Ae_i to M by taking and arbitrary element in the preimage of $g(e_i)$. This is equally true of any direct sum of modules generated by primitive idempotents, and the property is so important that it is generally taken as the defining property of this class of modules.

Definition 2.6. A module P over a ring A is called *projective* if for every homomorphism of A-modules $g : P \to N$ and every epimorphism of A-modules $f : M \to N$, there exists a homomorphism $h : P \to M$ such that $g = f \circ h$.

$$\begin{array}{ccc} & & P \\ & {}^{h}\swarrow & \downarrow {}^{g} \\ M & \xrightarrow{f} & N \end{array}$$

We have asserted that this is the defining property of modules which are direct sums of modules Ae_i, or, since each Ae_i is a direct summand of a copy of A, direct summands of free modules A^r. The equivalence of these properties, and of another which is also useful, is the subject of the following lemma.

Lemma 2.4. *Let A be a unitary Noetherian ring. For a finitely generated A-module P, The following properties are equivalent:*

(1) *P is a direct sum of modules isomorphic to Ae_i for e_i a primitive idempotent,*
(2) *P is projective,*
(3) *If $f : M \to P$ is an epimorphism, then there is a monomorphism $h : P \to M$ such that $id_P = f \circ h$.*

Proof. We show the equivalence of the given properties:

- **(1)** $\Rightarrow$ **(2)**: Since
$$Hom_A(\bigoplus P_i, N) \tilde{\to} \bigoplus Hom_A(P_i, N),$$
we check property (2) when $P = Ae_i$. Letting m be any preimage of $g(e_i)$, we define h by $h(a \cdot e_i) = a \cdot m$.
- **(2)** $\Rightarrow$ **(3)**: This is just a specialization to the case $g = id_P$.
- **(3)** $\Rightarrow$ **(1)**:The module P, like any other finitely generated module, is isomorphic to a quotient of a free module. But then, but property (3), it is a direct summand of a free module. Since A is Noetherian, 1_A has a decomposition as a sum of a finite number of primitive idempotents, for otherwise we could find an ascending sequence of left ideals generated by idempotents. Thus any free module A^r has a decomposition into factors of the form Ae_i. By the Krull-Schmidt Theorem, every direct summand has this form, so we get (1).

□

Lemma 2.5. *For a left Artinian ring A,the following sets are all in one-to-one correspondence:*

(1) $\mathcal{P}(A)$
(2) $\mathrm{Max}(A) = \{I \subseteq A | I maximal\}$
(3) $\mathrm{Irr}(A) = \{simple\ modules\ of A\}$

Proof. We have assumed that k is sufficiently large that

$$\phi : A/\operatorname{Rad}(A) \tilde{\to} \bigoplus_{i+1}^{t} M_{n_i}(k).$$

For each i, we define a maximal ideal as the preimage of the sum of the other blocks, and the maximal ideals are all of this form. The block corresponds to a well-defined simple module, an element of $\mathrm{Irr}(A)$, and finally, by Lemma 2.1, two primitive idempotents whose image under ϕ lie in the same block are conjugate, and the condition is also necessary, since conjugation induces an automporphism on each block. □

Dual to projective modules are injective modules. Let us first state what we mean by "dual". Let A be a finite dimensional algebra over a field. We surely expect A and its opposite algebra A^{op} to have essentially the same representation theory, since they differ only in a convention of left and right. If, for example, A had only a finite number of indecomposable modules, we would expect A^{op} to have the same number of indecomposable modules of each dimension.

The function which allows us to pass from modules over A to modules over A^{op} is called the *dualizing functor* or transpose. Let $A - mod$ be the collection of all finitely generated A-modules, and define

$$tr : A - mod \to A^{op} - mod$$

by

$$tr(M) = Hom_k(M, k).$$

Since M is finite dimensional as a k-vector space, then by standard properties of dual vector spaces, $tr(M)$ is a vector space of the same dimension as M. The action of an element a of A^{op}on a linear transformation $f : M \to k$ is given by

$$a \cdot f(m) = f(am),$$

which satisfies

$$(ab) \cdot f(m) = f(ab \cdot m) = a \cdot f(bm) = b \cdot (a \cdot f)(m).$$

Now we can take an injective module for A to be the dual of a projective module for A^{op}. The vector space dual interchanges submodules with quotients, epimorphisms with monomorphism, and for modules, the top with the socle. Each primitive idempotent e_i for A is also a primitive idempotent for A^{op}. If P_i is the projective module Ae_i and P_i' is the projective module $A^{op}e_i$, then $I_i = \mathrm{Hom}_k(P_i', k)$ is the corresponding injective module for A. Since e_i is primitive, it lies in unique matrix block of $A/\operatorname{Rad}(A)$. Let S_i be the simple module corresponding to that matrix block. Then $\mathrm{Top}(P_i)$ is isomorphic to the column of the matrix block generated by e_i, which is a simple module S_i. The top of P'_i is the simple S'_i corresponding to the row generated by e_i. Finally, I_i is a module whose *socle* is isomorphic to S_i.

In terms of the basis graph, P_i has a basis consisting of all arrows leaving e_i. The basis graph of A^{op} is just the basis graph of A with all arrows reversed, so in this dual graph, P'_i has a basis consisting of all arrows leaving e_i. Now we dualize, which reverse the arrows again, and thus I_1 has a basis corresponding to all the arrows *entering* e_i. It must be stressed that while P_i is always a left ideal of A, I_i is generally not, except for cases like group algebras where $A = A^{op}$ (and thus all injectives are projectives).

Lemma 2.6. *Let A be a finite dimensional algebra over a field k. For a finitely generated A-module I, The following properties are equivalent:*

(1) *I is a direct sum of modules isomorphic to* $\mathrm{Hom}(A^{op}e_i, k)$ *for e_i a primitive idempotent,*
(2) *If $g : N \to I$ is a homomorphism, and $f : N \to M$ is a monomorphism, then there is a homomorphism $h : M \to I$ such that $g = f \circ h$.*
(3) *If $f : I \to M$ is a monomorphism, then there is an epimorphism $h : M \to I$ such that $id_I = f \circ h$.*

Definition 2.7. The *skeleton* of a finite dimensional algebra A over a suficiently large field k is the algebra

$$A_0 = e_0 A e_0,$$

where e_0 is an idempotent which is the sum of one idempotent from each point, taken from a complete orthogonal decomposition of the identity 1_A.

The skeleton is a *basic* algebra, that it,

$$A_0 / \operatorname{Rad}(A_0) \tilde{\to} k^t.$$

The vertices in the basis graph of A_0 are in one-to-one correspondence with the points of A and of A_0.

One of the most important examples of an algebra useful in representation theory is the algebra $\mathrm{End}_A(M)$, where M is a module over a left Artinian ring A. A homomorphism ϕ of modules over a ring is idempotent if and only if it is a projection map. Since the image of a homomorphism of modules is itself a module, then the idempotent is a projection onto a submodule N of M. It is primitive if and only if it cannot be written as the sum of two orthogonal projections onto submodules of N. If such orthogonal projections exist, then N is a direct sum of their images, and conversely, if N decomposes as a direct sum, then the projections onto the summands give a decomposition of N, which is therefore decomposable. Thus the primitive idempotents of $\mathrm{End}_A(M)$ correspond to the indecomposable summands of M.

Now let us ask what are the points of $\mathrm{End}_A(M)$? These are conjugacy classes of primitive idempotents, under conjugacy by invertible elements of $\mathrm{End}_A(M)$. However, the invertible elements of $\mathrm{End}_A(M)$ are precisely the A-automorphisms of M. If π is a projection on an indecomposable submodule N, and $\alpha : M \tilde{\rightarrow} M$ is an automorphism, the homomorphism $\alpha \circ \pi \circ \alpha^{-1}$ is the projection on the isomorphic module $\alpha(N)$. Thus the points of $\mathrm{End}_A(M)$ are in one-to-one correspondence with isomorphism classes of indecomposable direct summands of M.

Example 2.6. Let A be the ring $k[x]/(x^3)$, for $k = \mathbb{Q}$. There are three nonzero indecomposable modules for A, these being isomorphic to the left ideals $N_3 = A$, $N_2 = (x)$, and $N_1 = (x^2)$. Each N_i is of dimension i. By a general theorem in ring theory,

$$\mathrm{End}_A(A) \tilde{\rightarrow} A^{op},$$

and since A in this case is commutative, $A^{op} \tilde{\rightarrow} A$. For the regular representation, there is a single point, corresponding to the conjugacy class of automorphisms of A.

Let us now take a second module N_1' isomorphic to N_1 and construct a module $M = N_1 \oplus N_1'$. Since x acts as 0 on N_1 and N_1', the A-module endomorphisms are just the k-linear transfomations of a vector space of dimension 2, to

$$\mathrm{End}_A(M) \tilde{\rightarrow} M_2(k).$$

A primitive idempotent is just a projection of a two dimensional vector space on any one dimensional vector space. By the Skolem-Noether theorem these are all conjugate, and thus there is a single point.

If we let $M = N_1 \oplus N_1' \oplus N_2 \oplus N_3$, then $\mathrm{End}_A(M)$ has three points, corresponding to the three isomorphism classes of indecomposables. The basis graph is given in Figure 2. The skeleton is the endomorphism algebra of $N_1 \oplus N_2 \oplus N_3$, which is a basic algebra because each indecomplable summand appears only once. Note that the natural endomorphism from N_2 onto its radical factors either through an epimorphism onto N_1 or a monomorphism into N_3. As we shall see later, the only endomporphisms between indecomposables which do not factor through an indecomposables are the epimorphisms and the monomorphisms.

Exercises

(1) Find the skeleton of each of the algebras in Example 2.3.
(2) If $A = A_1 \otimes_k A_2$ is the tensor product of two k-algebras, prove that $\mathrm{Rad}(A) = \mathrm{Rad}(A_1) \otimes A_2 + A_1 \otimes \mathrm{Rad}(A_2)$. Conclude that

$$\mathcal{P}(A) = \mathcal{P}(A_1) \times \mathcal{P}(A_2).$$

(3) Find the indecomposable injective and projective modules for T^3.

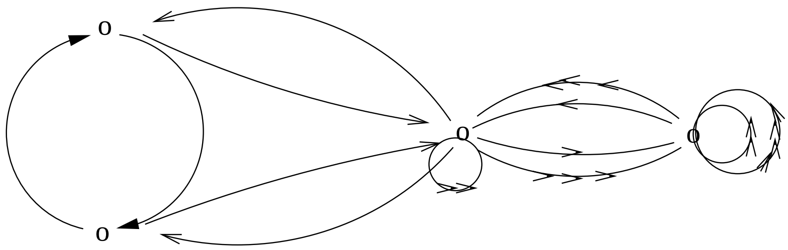


FIGURE 2. Weighted basis graph for an endomorphism algebra

(4) Calculate the basis graph of the algebra $\mathrm{End}_A(M)$, when $A = k[x]/(x^2)$ and M is the direct sum of the two indecomposable modules of dimensions 1 and 2.

3. Morita equivalence and the quiver

Two algebras A and B with the same skeleton have equivalent representation theories, in the sense that there are functors

$$\mathcal{F} : A - mod \to B - mod,$$

and

$$\mathcal{G} : B - mod \to A - mod,$$

such that the composition in either order maps a module to an isomporphic module, i.e., the two module categories are equivalent. Furthermore, the two functors are given by tensoring with bimodules, that is, there are bimodules ${}_AX_B$ and ${}_BY_A$,such that, for an A-module M and a B-module N, we have

$$\mathcal{F}(M) = Y \otimes_A M,$$

and

$$\mathcal{G}(N) = X \otimes_B N.$$

We will give the formal definition in terms of these bimodules:

Definition 3.1. Let A and B be finite dimensional algebras over a sufficiently large field k. The A and B are *Morita equivalent* if there exists bimodules ${}_AX_B$ and ${}_BY_A$, each projective as left and as right modules, such that

$$X \otimes_B Y \xrightarrow{\sim} A,$$

and

$$Y \otimes_Z X \xrightarrow{\sim} B.$$

The isomorphism is is an isomorphism of bimodules, but since the algebra structure of A is completely determined by its $A - A$ bimodule structure, and similarly for B, we have full information about A and B from X and Y.

Example 3.1. Let A be a finite dimensional algebra with

$$A/\operatorname{Rad}(A) \xrightarrow{\sim} S = \bigoplus_{i=1}^{t} M_{n_i},$$

and let $A_0 = e_0 A e_0$ be its skeleton, where e_0 is the idempotent formed by taking one primitive idempotent from each matrix block. Let X be Ae_0, and let Y be $e_0 A$. Then X is a direct sum of one copy of each projective indecomposable A-module and is thus projective on the left. The module X is also a right A_0 module, since $ae_0 \cdot e_0 a' e_0 \in X$ for any $a, a' \in A$.

Claim 1. *X is projective as a right A_0-module.*

Proof. Choose an embedding of S into A, which exists by Lemma 2.1. For each $i = 1, \ldots, t$, let e^i be the idempotent of the matrix block of S, let $\{E^i_{jk}\}$ be the matrix units associated with e^i,and let $E^i_{j_i j_i} = e^i e_0 = e_0 e^i$ be the primitive idempotent representing the block i in e_0. Then $X = \bigoplus_i e^i A e_0$, and each primitive idempotent E^i_{jj} in block i can be written as $E_{jj_i} e_0 E_{j_i j}$. Each summand is a sum of n_i copies of $E^i_{j_i j_i}(e_0 A e_0)$, and is thus projective.

□

The proof that Y is a left projective A_0 module is dual. That $Y \otimes_A X$ is isomorphic to A_0 is clear, since $A \otimes_A A \xrightarrow{\sim} A$. To show that $X \otimes_{A_0} Y$ is isomporphic to A, we use the method in the claim on both sides.

Example 3.2. The matrix algebras $A = M_s(k)$ and $B = M_t(k)$ have the same skeleton, which is just k. To see directly that they are Morita equivalent, we set

$$X = M_{st},$$

and

$$Y = M_{ts}.$$

Then

$$X \otimes_B Y \xrightarrow{\sim} A,$$

with the isomorphism given by matrix multiplication.

Definition 3.2. The *quiver* Q of a finite dimensional algebra A is the directed graph consisting of the vertices and weight one arrows of the skeleton A_0 of A in the weighted basis graph of A_0.

Definition 3.3. The *path algebra* over a field k of a directed graph Q is the algebra k[Q] which has as k-basis all the paths in Q, i.e., finite strings of arrows which are compatible in that one starts where the previous one ends. The product of two paths is zero if they are not compatible and is the composition if they are compatible.

The radical of the path algebra is generated by the arrows (including the loops).

(1) (2) (3)

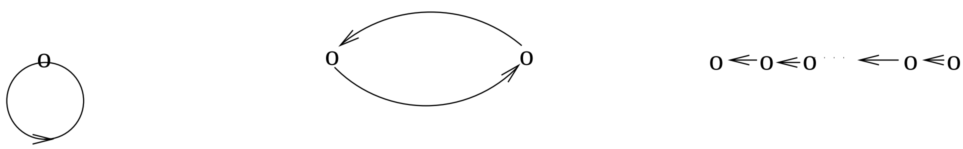


FIGURE 3. Quivers of some algebras

Example 3.3. (1) The quiver of $k[x]/(x^7)$ consists of a point and a loop α. The path algebra is infinite dimensional, with a basis consisting of all the powers of α.

(2) Over a field of characteristic 5, the quiver of D_5 consists of two points and one arrow in each direction. The path algebra is again infinite dimensional.

(3) If the quiver has no oriented cycles, i.e., no paths which return to the starting vertex, then the path algebra will be finite dimensional. The quiver of the algebra of upper triangular matrices T^n consists of a linear string of arrows on n vertices. In this case, the path algebra of the quiver has the same dimension as the algebra. See Figure 3.

One of the most important quivers in representation theory is the Auslander-Reiten quiver. A general definition is outside the material of this book, but in the particular case when the algebra A is of *finite representation type*, i.e., has a finite number of isomorphism classes of indecomposable modules, then, if we let M be the direct sum of one copy of each indecomposable, the Auslander-Reiten quiver is just the quiver of the algebra $\mathrm{End}_A(M)$.

The vertices of the quiver correspond to the indecomposable modules. We generally assume that A is basic, with primitive idempotents e_i, and, when this suffices to distinguish the modules, we label the vertex corresponding to an indecomposable N by the dimension vector d, an integral vector whose components are

$$d_i = \dim_k(e_i N).$$

The arrows correspond to the *irreducible maps*, i.e., maps which do not factor through any other indecomposable. In the finite representation case, every other homomorphism between indecomposables is a linear combination of the compositions of irreducible maps.

The following very basic results are true for algebras other than those of finite representation type:

Lemma 3.1. *Every irreducible map is either a monomorphism or an epimorphism.*

$$
\begin{tikzcd}[column sep=small]
\overline{1}\,0\,0 \arrow[dr] & & 0\,1\,0 \arrow[dr] & & 0\,0\,\underline{1} \\
& 1\,\overline{1}\,0 \arrow[ur] \arrow[dr] & & 0\,\underline{1}\,1 \arrow[ur] & \\
& & \underline{1}\,1\,\overline{1} \arrow[ur] & &
\end{tikzcd}
$$

FIGURE 4. Auslander-Reiten quiver for T^3

Lemma 3.2. *The irreducible maps to a projective indecomposable P are the embeddings of the indecomposable summands of* $\mathrm{Rad}(P)$ *into P.*

Lemma 3.3. *The irreducible maps from an injective indecomposable I are the natural epimorphisms of I onto the indecomposable summands of the socle quotient* $I/\operatorname{Soc}(I)$.

Example 3.4. (Upper triangular matrices) Let our algebra A be T^n, the upper triangular 3×3 matrices. The dimension vectors of the indecomposables are all 3-vectors with entries 0 or 1 such that the nonzero entries form a connected interval. The irreducible maps are precisely the maps into the radical or onto the socle quotient. In this case only indecomposable which is not either projective or injective is the module with dimension vector (0,1,0) but that is rare. Usually most modules are neither. We indicate the top of a projective module by a bar, and the socle of an injective module by an underline. The Auslander-Reiten quiver is given in Figure 4. The mesh-like appearance of the quiver in not accidental. Each diamond or triangle of the mesh corresponds to a short exact sequence which is called, in Auslander-Reiten theory, an *almost split sequence.*

Since A is Morita equivalent to its skeleton, and Morita equivalence preserves the representation theory, the quiver is the central object in algebra representation theory. It is not generally used in group representation theory because it does not preserve the action of the group. The algebra corresponding to the skeleton in group respresentation theory is the source algebra, which will decribed later.

Because the skeleton is a basic algebra, one can make use in algebra representation theory of the following useful result:

Theorem 3.1. *If A is a basic algebra of finite dimension over a field k, then A is isomorphic to a quotient of the path algebra $k[Q*]$ of the dual of the quiver of A, by an ideal contained in $Rad^2(k[Q^*])$ and containing a finite power $Rad^n(k[Q^*])$ of the radical.*

Definition 3.4. The *Cartan matrix* of a basic algebra A_0 with r idempotents $\{e_i\}_{i=1}^r$ is the matrix $r \times r$ matrix $[c_{ij}]$, where

$$c_{ij} = \dim_k(\mathrm{Hom}_A(Ae_j, Ae_i)).$$

The Cartan matrix of a general algebra is the Cartan matrix of its skeleton.

Lemma 3.4. *The Cartan matrix represents:*

(1) *The numbers $n_{ji} = e_j A e_i$, which, except in the loop case, represents the number of arrows from i to j in the basis graph of the skeleton.*
(2) *The multiplicity of $S_j \xrightarrow{\sim} Ae_j/Rad(Ae_j)$ as a composition factor of Ae_i.*

Proof. (1)We have already said that a homomporphism f from the indecomposable projective module Ae_j is completely determined by the image of e_j. However, since f is an A homomorphism, we have $f(e_j) = f(e_j e_j) = e_j f(e_j)$, so $f(e_j)$ lies in the right ideal $e_j A$. Therefore, the vector space of possible images of e_j in the left ideal Ae_j is the vector space $e_j A e_i$, and the dimension of $Hom(Ae_j, Ae_i)$ is the dimension of the space of possible images.

(2) We can form a composition series of the module Ae_i by taking a refinement of the radical series, writing each quotient $\mathrm{Rad}^k(A)/\mathrm{Rad}^{k+1}(A)$ as a direct sum of its Pierce components, and then choosing a basis for each summand. Thus the number of composition factors isomorphic to S_j is

$$\sum_k n_{ji}^k = n_{ji} = c_{ij},$$

by part (1). □

Corollary 3.4.1. *The dimension of the projective module Ae_i is the sum of the numbers in the i-column of the Cartan matrix.*

From the point of view of algebra representation theory, the simplest cases are finite dimensional algebras which are path algebras over a very select set of quivers, quivers whose underlying directed graph is one of the graphs known as the Dynkin diagrams. Dynkin diagrams include four infinite families, A_n, B_n, C_n, and D_n, together with a few exceptional graphs denoted by E_n, n = 6,7,8 and F_4, G_2. The subscript always refers to the number of simple modules. Some of these types depend on field extensions, so the only ones which occur for a sufficiently large field are the A_n, D_n and E_n, which are given in Figure **??**. Note that they are all trees. They have the useful but unusual property that every submodule of a projective module is itself projective.

The path algebra of any quiver with these underlying graphs is of finite representation type. The Auslander-Reiten quiver is the quiver of the endomorphism algebra $\mathrm{End}_A(M)$, where M is the sum of one copy of each indecomposable. It is a strip, with the projective module on the left, and the injective modules on the right. The dimension of the algebra and the Cartan matrix depend on the orientation of the arrow. By a fundamental theorem of Gabriel, the number of indecomposable modules for the algebras based on the Dynkin diagrams does not depend on the orientation.

Example 3.5. Consider an algebra whose quiver is the Dynkin diagram D_4 with all arrows oriented toward the point with valency 3, which we will number as 1. Total algebra has dimension 7, and there are three projectives of dimension 2. If we now reverse the direction of the arrow from 2 to 1, the total dimension will now be 9. The projective at 3 and 4 will have dimension 3, the projective at 1 will have dimension 2, and the projective at 2 will have dimension 1 The Cartan matrices will be

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The dimension of the projective Ae_i is read off the Cartan matrix by summing the numbers in the ith column, as stated in Corollary 3.4.1. The number of indecomposable modules in each case is 12. The dimension vectors of the indecomposable consist of all connected sets of 1's, as well as one "heightest weight" indecomposable which has dimension 2 at the vertex of valency 3.

Exercises

(1) Calculate the quiver of T^n.
(2) Prove that $M_{st}(k) \otimes M_t(k) M_{ts}(k) \xrightarrow{\sim} M_s(k)$, where the map is given by matrix multiplication.
(3) Given natural numbers r and n and a partition
$$\nu = (n_1, \ldots, n_r)$$
of n into r parts, we defined, in Chapter 1, the algebra R_ν of upper triangular block matrices. If m is another natural number, and
$$\mu = (m_1, \ldots, m_r)$$
is a partition of m into r parts, then we can define a $R_\nu - R_\mu$ bimodule $M_{\nu\mu}$ of upper triangular block matrices where the ij block has dimension $n_i \times m_j$. Prove that $M_{\nu\mu}$ and $M_{\mu\nu}$ determine a Morita equivalence between the algebras R_ν and R_μ.
(4) If $A = A_1 \otimes A_2$ is the tensor product of two algebras with quivers Q_1 and Q_2, and points sets $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$, show

that the quiver Q of A, which was shown before to have vertex set $\mathcal{P}(A_1) \times \mathcal{P}(A_2)$, is the join of the two graphs $Q_1 \times \mathcal{P}(A_2)$, with the graph $\mathcal{P}(A_1) \times Q_2$.

(5) Over a sufficiently large field of characteristic 2, show that the group algebra of $C_6 \times C_2$ consists of three points, each with two loops.

(6) Find the kernal of the map from the path algebra of the dual quiver onto the algebras in Example 3.3.

4. Finite rank lattices over an integral domain

As we indicated in the introduction to this chapter, we will generally be considering not just finite dimensional algebras over a field, but also algebras over an integral domain R. We will always require that our algebra A be free of finite rank over the commutative ring R. The fact that A is free as an R-algebra, i.e., that $A \xrightarrow{\sim} R^n$, will be denoted by saying that A is an *R-lattice.* We will designate a particular maximal ideal $\mathcal{M}$, and let $k = R/\mathcal{M}$, $K = QuotField(R)$. This then induces two algebras, the special fiber

$$\overline{A} = A \otimes_R k,$$

and the generic fiber

$$\hat{A} = A \otimes_R K.$$

Example 4.1. Let R be $k[t]$, let $\mathcal{M} = (t)$, and let $K = k(t)$. Let Q be the quiver of (5) in Example 2.3. Let A be the quotient of the path algebra $R[Q]$ by the ideal I generated by $(xyx - t^2x,\ yxy - t^2y)$, where we recall that $e_1xe_2 = x$ and $e_2ye_2 = y$. When $t = 0$ this is an indecomposable algebra, and if $char(k) = 3$, it is isomorphic to the group algebra of S_3. When $t \neq 0$, multiplying the relation $xyx - t^2x$ on the left by y gives a new primitive idempotents

$$e_1' = (1/t^2)xy,$$

and similarly the other relation gives an idempotent

$$e_2' = (1/t^2)yx.$$

We get a full set of matrix units by setting

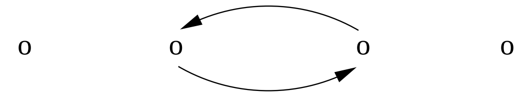
$$E_{12} = (1/t)x, \text{and} E_{21} = (1/t)y.$$

Since $e_1'e_1 = e_1' = e_1e_1'$ we have another idempotent $e_1'' = e_1 - e_1'$ and similarly $e_2'' = e_2 - e_2'$. Thus the algebra when we substitute any value of t except 0 is isomorphic to

$$k \oplus k \oplus M_2(k),$$

which we will call the *general fiber*, and the generic fiber is

$$K \oplus K \oplus M_2(K),$$



$(\overline{A})$

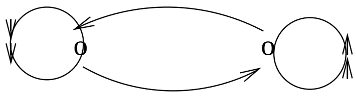


FIGURE 5. Deformations of kS_3

which corresponds to the group algebra of S_3 over a field of characteristic other than 2 or 3.

One of the most important things to note is that the primitive idempotents of the special fiber $\overline{A}$ are the residues mod $\mathcal{M}$ of idempotents e_1 and e_2 of A, but in A these "lifted" idempotents are no longer primitive. Rather,

$$e_1 = e_1' + e_1'',$$

and

$$e_2 = e_2' + e_2''.$$

The basis graphs for $\overline{A}$ and $\hat{A}$ are given in Figure 5. Both have dimension 6. When the four idempotents coallesce into two, we get two new radical elements, xy and yx, both of which appear as loops.

If we can lift idempotents from $\overline{A}$ to A, then it turns out that we can lift the maximal semisimple algebra guaranteed by Lemma 2.1. That the maximal semisimple algebra in the special fiber can be lifted to the entire deformation is a property known as "straightening out" in deformation theory. We will show in this chapter that it is true whenever R is a complete local ring, e.g., $k[[t]]$. Deformations over nonlocal rings like $k[t]$ or $\mathbb{Z}$ can often be "straightened out" if we make a change in the parameter ring, in order to add the roots to certain equations or "throw away" bad maximal ideals. We will not get involved, here, in issues of changing the parameter ring. We will continue to give examples of deformations over $k[t]$, because the possibility of substituting various values of t makes the deformation easy to visualize, but they will all be examples for which "straightening out" holds without any change in the parameter ring.

Definition 4.1. An R-algebra S which is an R-lattice is *R-simple* if it is isomorphic to $M_n(R)$ for some n. It is *R-semisimple* if it is the direct sum of R-simple algebras.

The process of "straightening out" is the lifting of the maximal k-semisimple subalgebra $\overline{S}$ of $\overline{A}$ to an R-semisimple subalgebra of A. A complete orthogonal primitive set of idempotents $\overline{e}_i$ lifts to a complete orthogonal set of idempotents e_i of A. These induced a complete orthogonal set of idempotents $\hat{e}_i$ of $\hat{A}$, with $\hat{e}_i = e_i \otimes K$. The idempotents $\hat{e}_i$ are generally *not* primitive, but they still induce a Peirce decomposition.

Lemma 4.1. *If A is a deformation of $\overline{A}$ over R such the the maximal semisimple algebra of $\overline{A}$ lifts to an R-semisimple algebra, then the basis graph of $\overline{A}$ is obtained from the basis graph of $\hat{A}$ by coallescing vertices, and adding as many loops at a coallesced vertex as is necessary to make up for the lost idempotents and ensure that*

$$dim_k \overline{e}_i \overline{A} \overline{e}_i = dim_K \hat{e}_i \hat{A} \hat{e}_i.$$

Proof. Let e_i be the lifting of a complete orthogonal idempotent set $\overline{e}_i$ from $\overline{A}$ to A. Since A is an R-lattice of rank n, and the elements of R lie in the center of A, the Peirce components with respect to the idempotents e_i are also R-lattices. Tensoring with k and with K then ensures that

$$\dim_k \overline{e}_i \overline{A} \overline{e}_j = rank_R e_i A e_j = \dim_K \hat{e}_i \hat{A} \hat{e}_j.$$

When $i \neq j$, the number of arrows from j to i is the sum of all the number of arrows from the components of $\hat{e}_j$ to the components of $\hat{e}_i$. When $j = i$, the total dimension of $\overline{e}_i \overline{A} \overline{e}_i$ is the total number of arrows between the primitive constituents, all of which become loops after collapsing, plus the number m of idempotents collapsing to e_i. By the definition of the basis graph, we must add an extra $m - 1$ loops to fill out the basis. □

4.1. **Complete Local Rings and Idempotent Lifting.** A ring R is *local* if it has a unique maximal ideal $\mathcal{M}$ and $R/\mathcal{M}$ is a field k. It is *complete* if

$$\cap_{i=1}^{\infty} \mathcal{M}^i = 0.$$

If the ideal $\mathcal{M}$ is principal, generated by and element π, then we say that R is a complete discrete valuation ring, and we can define a valuation $v : R \to \mathbb{Z}R$ by sending each element r to the lowest integer i such that $r \in (\pi^i)$. The k-algebra $k[[t]]$ is one example of a complete discrete valuation ring. Another example, the p-adic integers for p a prime integer, will be discussed later.

When the ring R is local, we have $\mathcal{M} = \mathrm{Rad}(R)$. Furthermore, any simple A-module V is also an R-module, so by Nakayama's lemma applied to R, $\mathcal{M}V \neq V$. However, since $\mathcal{M}A$ is an ideal of A, the $\mathcal{M}AV$ is a proper submodule of the simple module V, and is thus 0. Since the ideal $\mathcal{M}A$ annihilates all simple A-modules, it is contained in $\mathrm{Rad}(A)$.

Theorem 4.1. *Let R be a complete Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field $k \xrightarrow{\sim} R/\mathcal{M}$. Let the R-algebra A be an R-lattice of finite rank, and set*

$$\overline{A} = A \otimes_R K.$$

Then a complete orthogonal set of idempotents $\overline{e}_i$ of $\overline{A}$ lifts to a complete orthogonal set of idempotent e_i of A.

Proof. Because R is Noetherian, $\cap_{i=1}^{i} nfty\mathcal{M}^i = 0$. This follows from Nakayama's lemma because $\mathcal{M} = \mathrm{Rad}(R)$, and the intersection is stable under multiplication by $\mathcal{M}$. If we fix a finite basis for A as an R-module, then in order to specify the idempotents e_i, it suffices to find the coefficients as elements of R. Because R is complete, it actually suffices to find a sequence of elements e_{ij} in

$$A_j = A \otimes_R R/\mathcal{M}^j,$$

which give complete orthogonal sets of idempotents for each j. Since we have not required that the idempotents be primitive, it actually suffices to prove the theorem for a pair of idempotents, and then proceed by induction on the localizations of A at these two idempotents. But then we can further reduce to the problem of lifting the idempotent e_{1j-1} of A_{j-1} to e_{1j} to A_j, since once we have done that we can set

$$e_{2j} = 1_{A_j} - e_{1j}.$$

Let e' be an arbitrary lifting to A_j of e_{1j-1}. Set

$$f = e'^2 - e' \in \mathcal{M}^{j-1}A_j.$$

The element f lies in $\mathcal{M}^{j-1}A_j$, which is an A_1-module. Since $f \in e'A_je'$, and e' act as an idempotent in A_1, we get that

$$e'f = fe' = e'fe'.$$

and define

$$e'' = e' + f - e'f - fe'.$$

Then e'' is the desired idempotents, because, using the fact that $f^2 = 0$,

$$e''^2 = e'^2 + (e' - e'^2)f - e'fe' + f(e' - e'^2) - e'fe' \tag{1}$$
$$= e'^2 - 2f^2 - 2e'fe' \tag{2}$$
$$= e' + e'^2 - e' - e'f' - fe' \tag{3}$$
$$= e' + f - e'f - fe'. \tag{4}$$

□

Corollary 4.1.1. *If $\overline{S} = \overline{A}/\operatorname{Rad}(\overline{A}) \xrightarrow{\sim} \bigoplus_{i=1}^{\ell} M_{q_i}(k)$, then A contains a subalgebra $S \xrightarrow{\sim} \bigoplus_{i=1}^{\ell} M_{n_i'}(R)$, and S is the maximal R-semisimple subalgebra of A.*

Proof. By Theorem 4.1, the idempotents on the diagonal of the matrix blocks in $\overline{S}$ lift to idempotents in A, and they are still orthogonal after the lifting. For each block M_{q_i}, let $\{E^i_{jk}\}$ be the matrix units. Conjugation by a cyclic permutation matrix permutes the primitive idempotents in the block. Let $W_i = w_1, w_2, \ldots, w_{q_i}$, with $w_1 = 1_{\overline{A}}$ be the set of permutation matrices such that $w_j E^i_{jj} w_j^{-1} = E^i_{11}$. By

Lemma 2.2, these can be lifted to units in A. For any $j = 2,3,\ldots,q_i$, let f_j be the result of conjugating the lifting of E^i_{jj} by the unit lifting w_j. Let e be the lifting of E^i_{11}. Then

$$\overline{e} = \overline{f}_j = E^i_{11}.$$

Consider the element

$$u = 1_A - e - f + 2ef.$$

This is a unit, since it reduces to $1_{\overline{A}}$ mod(Rad(A)). Furthermore,

$$eu = ef = uf,$$

so $f = u^{-1}eu$. Multiplying w_j by u, and doing this for each j, we get a new set W'_i of elements of A which conjugate one idempotent e_i to an orthogonal set of idempotents which reduce to the diagonal elements of the i-th matrix block M_{q_i}. If $W'_i = \{w'_1, \ldots, w'_{q_i}\}$, then the jk-matrix unit for block i will be given by the product

$$w'^{-1}_j e_i w'_k.$$

$\square$

4.2. **Rosenberg's lemma.** We quote without proof the following useful result from Thevenaz, Thm 3.1(h).

Lemma 4.2. *Let A be an R-algebra of finite rank, let $\mathcal{B}$ be an ideal, and let e be an idempotent of $A - \mathcal{B}$. Let $\overline{A} = A/\mathcal{B}$. For any ideal $\mathcal{I}$ of A, let $\overline{\mathcal{I}} = \mathcal{B} + \mathcal{I}/\mathcal{B}$, and let $\overline{e}$ be its image in $\overline{A}$. Then $e \in A$ if and only if $\overline{e} \in \overline{\mathcal{I}}$.*

We now come to a result which will be crucial in the sequel:

Lemma 4.3. *(Rosenberg's lemma)A primitive idempotent e lies in the sum of a family of ideals if an only if it lies in one of the ideals.*

Proof. By the previously quoted result Lemma 4.2, it suffices to prove the result for the quotient of A by its radical, which is a semisimple algebra. If the idempotent was primitive in A, it is still primitive modulo the radical, for if it had a decomposition, the two constituents could be lifted to A by using Theorem 4.1. A primitive idempotent in a semisimple algebra must belong to a unique matrix blocks. The ideals in a semisimple algebra are sums of the matrix blocks, so if the idempotent lies in the sum of a family of ideals, one of those ideals $\mathcal{I}$ contains the matrix block containing the primitive idempotent. $\square$

4.3. **Algebras with simple top.** One of the most important cases in group representation theory is the case where $\overline{S} \xrightarrow{\sim} M_n(k)$. In this case, if we let f be a lifting of the first primitive idempotent in this matrix algebra, we get that in fact

$$A \xrightarrow{\sim} M_n(fAf).$$

The algebra A is Morita equivalent to the algebra fAf, which is, in fact, its skeleton.

4.4. **The p-adic numbers.** The ring R which is most often used in group representation theory is based on the p-adic numbers. Let I be the ideal in $\mathbb{Z}$ generated by the integral prime p. Consider the direct product of the rings $\mathbb{Z}/I^n$ for natural numbers n. Let $\mathbb{Z}$ be the subset of this product consisting of all sequences $(a_1,\ a_2,\ \dots)$ for which we have, if $n < m$, that $a_n \equiv a_m mod I^n$. This is a complete local ring with unique maximal ideal consisting of all elements with $a_1 = 0$. The elements are usually written as infinite series in powers of p, given by

$$a_1 + (a_2 - a_1) + (a_3 - a_2) + \dots.$$

For example, the number -1 is given by a sequence with $a_n \equiv p^n - 1 \mod I^n$. As a series this is written

$$(p-1) + (p-1)p + (p-1)p^2 + \dots.$$

The residue field k is $\mathbb{Z}/p\mathbb{Z}$.

To calculate the characteristic of the quotient field, note that for any integer s which is relatively prime to p, if we let d be the order of p in the finite group $\mathbb{Z}/s\mathbb{Z}$, we have

$$s | p^d - 1.$$

This means that any rational number with denominator prime to p can be written in the form $r/p^d - 1$. Writing r as a number base p, and using the fact that, in $\mathbb{Z}_p$,

$$1/(p^d - 1) = 1 + p^d + p^{2d} + p^{3d} + \dots,$$

we find that $\mathbb{Z}$ contains all the rational numbers with denominator prime to p. The quotient field then contains all rational numbers, and is thus of characteristic 0. Usually we need k to contain more roots of unity, but this can be done by taking a finite integral extension of $\mathbb{Z}$.

Exercises.

(1) Let R be the integral extension of $\mathbb{Z}_2$ given by $\mathbb{Z}_2[x]/(x^2+x+1)$. Show that $\overline{x}$ is a cube root of unity in R. Show that $1+2+4+8\dots$ is a square root of unity.
(2) Show that the residue field k of the ring R given in the previous exercise is a finite field of four elements.
(3) Consider the group algebra of $D_6 = \langle a, b | a^2 = b^6 = e, aba = b^5 >$. Show that $1 + b^2 + b^4$ and $b^2 + b^4$ are central idempotents in kD_6.
(4) * Lift the idempotents in the previous exercise to idempotents for RD_6.
(5) * Show that the second idempotent in the previous exercise corresponds to a block with simple top.

5. Self-injective and symmetric algebras

The algebras which occur in modular group are diametrically opposed to the Dynkin quiver algebras. The quiver always has cycles, so one much give relations to describe the algebra. Except for one very special case, there are an infinite number of indecomposable modules. Furthmore, every projective module is also injective.

Definition 5.1. An algebra is called *self-injective* if the projective and injective modules coincide.

The algebra (f) in Example 2.3 is a self-injective algebra. The indecomposable projective-injective modules are the arrows going from one idempotent to the other.

The top of an indecomposable projective is always one dimensional, but its socle can have larger dimension. Similarly the socle of an indecomposable injective is one-dimensional, but the top can be laarger. Note that in a self-injective algebra, the top and the socle of an indecomposable projective-injective must both be one dimensional, but they can correspond to different simple modules.

All group algebras, and all blocks of group algebras, are self-injective, but they satisfy an additional property which forces the top and socle to correspond to the same simple module when we consider the algebra over a field. This means that for each vertex in the weighted basis graph, there is a unique loop of maximal weight.

Definition 5.2. Let R be a complete local ring (and we allow the case $R = k$, a field) and let A be a finite rank R-algebra which is an R-lattice. The algebra A is called *symmetric* if

$$\phi :_A A \xrightarrow{\sim} Hom_R(A_A, R),$$

and this map is symmetric in the sense that

$$\phi(a)(b) = \phi(b)(a).$$

This immediately shows that A is self-injective, since the indecomposable projective modules are the direct summands of ${}_AA$, whereas the indecomposable injective modules are the direct summands of $Hom_R(A_A, R)$.

Lemma 5.1. *If R is a complete local ring (including the case of a field) and A is an R-algebra which is a finite rank R-lattice, then the following conditions are equivalent:*

(1) *A is symmetric,*
(2) *There is a symmetric R-bilinear form*

$$\phi : A \times A \to R,$$

such that for all a, b, $c \in A$, $\phi(ab, c) = \phi(a, bc)$, and also ϕ induces an isomorphism of R-modules

$$\phi : A \to Hom_R(A, R).$$

The form ϕ is called an associative, symmetric, unimodular form for A.

(3) *There is an R-linear map $\lambda : A \to R$ with the following three properties:*
  - *For all a, $b \in A$, $\lambda(ab) = \lambda(ba)$;*
  - $\mathrm{Ker}(\lambda)$ *does not contain any nonzero left ideal of A.*
  - *For any R-linear map $h : A \to R$, there exists $a \in A$ such that $h(b) = \lambda(ab)$ for all $b \in A$.*

The function λ is called a symmetrizing function for A.

Proof. (1) $\Rightarrow$ (2):If θ is the isomorphism in (1), define $\phi(a, b) = \theta(a)(b)$. It is R-bilinear because θ is R-linear and maps into a vector space of R-linear maps. It is symmetric because of the symmetry property of θ.

(2) $\Rightarrow$ (3): Given ϕ, we define λ by

$$\lambda(b) = \phi(1_A, b).$$

The associativity and symmetry then imply the $\lambda(ab) = \phi(1_A, ab) = \phi(a, b) = \phi(b, a) = \lambda(ba)$. If $\mathrm{Ker}(\lambda)$ contains a principal left ideal Aa, then $\phi(a)(-)$ is zero, so ϕ does not induce a one-to-one map, and if the third condition does not hold, then ϕ does not induce an onto map.

(3) $\Rightarrow$ (1): Define $\theta(a)(b) = \lambda(ab)$. The symmetry of λ ensures that θ is symmetric, and the remaining two properties ensure that θ is on-to-on and onto. □

To get the desired result about the isomorphism of the top and the socle we must restrict to the field case, because the top is usually not an $R - lattice$.

Corollary 5.1.1. *If R is a field k, and A is a symmetric k-algebra, then for any indecomposable projective module P, we have* $\mathrm{Soc}(P) \xrightarrow{\sim} Top(P)$.

Proof. Let e be a primitive idempotent such that P is isomorphic to Ae. There is some element a in $\mathrm{Soc}(ae)$ such that $\lambda(a) \neq 0$. Since $Soc(Ae) \subset Ae$, we get $ae = a$. Then, by the symmetry of λ, $\lambda(ea) \neq 0$, so $ea \neq 0$. Define a homomorphism $g : Ae \to Soc(Ae)$ by sending be to bea. This is a nonzero homomorphism. since A is self-injective, $\mathrm{Soc}(Ae)$ is simple, so g is onto. Since the image is simple, $Ker(g)$ contains $\mathrm{Rad}(Ae)$. Thus by the first isomorphism theorem, $\mathrm{Top}(Ae) \xrightarrow{\sim} \mathrm{Soc}(Ae)$. □

Lemma 5.2. *A group algebra $A = RG$ over a complete local ring is a symmetric algebra.*

Proof. The symmetrizing function $\lambda : A \to R$ is given by $\lambda(\sum_{g\in G} a_g g) = a_e$. To show that λ is symmetric, we set

$$a = \sum_{g\in G} a_g g, b = \sum_{g\in G} b_g g.$$

Then, using the fact that $gg^{-1} = g^{-1}g$ for all g in G, we calculate that

$$\lambda(ab) = \sum g \in Ga_g b_g^{-1} gg^{-1}.$$

In order to show that λ does not vanish on any left ideal, it suffices to show this for any ideal of the form Ab for $b \neq 0$ in A, since any left ideal containing a nonzero element b contains the entire principal ideal Ab. One of the coefficients b_g is nonzero, and then $\lambda(g^{-1}b) = b_g$, which is nonzero as required.

Finally, suppose that $h : A \to R$ is an R-linear functional. Then h is an R-linear combination of the functionals

$$h_g : A \to R$$

given by $h(g) = 1$, and $h(g') = 0$ for $g' \neq g$, i.e., the dual basis to the standard basis of group elements. However,

$$h_g(b) = b_g = \lambda(g^{-1}b).$$

If $h = \sum_{g\in G} c_g h_g$, then letting

$$a = \sum_{g\in G} c_g g^{-1},$$

we get $h(b) = \lambda(ab)$, as desired. □

Example 5.1. Let A be an algebra over a field k. We do not require A to be basic. Let S be the maximal semisimple algebra. Let E^i_{jk} be a set of matrix units for one of the blocks of S, a $q_i \times q_i$ matrix, and denote E^i_{11} by e_1. Consider the weighted basis graph of A. At the primitive e_1, there is one loop of maximal weight. Let s_1 be a basis element coresponding to that loop, a generator of the one dimensional socle of e_1Ae_1. The left ideal generated by s_1 is the vector space with basis $E^i_{j1}s_1$, where of course $E^i_{11}s_1 = s_1$ If $j \neq 1$, then

$$\lambda(E_{j1}s_1) = \lambda(s_1E_{j1}) = \lambda(0),$$

so since λ does not vanish on the entire left ideal, we must have $\lambda(s_1) \neq 0$. Multiplying by an appropriate elements of k, we may assume that $\lambda(s_1) = 1$. For every Peirce component fAe_i of A we get that $\dim_k(fAe_i) = \dim_k(eAf_i)$, and the bases can be paired so that the product of paired elements is s_1. This means that the weighted basis graph is extremely symmetric.

Exercises:

(1) Let $A = M_n(R)$. Show that the function $\lambda : A \to R$ given by $\lambda(N) = tr(N)$ is a symmetrizing function for A. (Hint: To show that λ does not vanish on a nonzero principal left ideal, multiply the generator of the ideal by an appropriately chosen matrix unit.)

(2) If f is any idempotent of A, define a symmetric, bilinear form $\phi : fAf \times fAf \to R$ by $\phi(a, b) = \lambda(ab)$. Use this form to prove that fAf is also a symmetric algebra. Thus, in particular, every block of a symmetric algebra is symmetric.
(3) Calculate the weighted basis graph for C_6 over a field of characteristic 2 and find the paired arrows.
(4) * Calculate the pairing for the group D_6 over a field of characteristic 2.