Fields Institute Communications
Volume **00**, 0000

Deformations, Tiltings, and Decomposition Matrices

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Dedicated to Professor Vlastimil Dlab

Abstract. Let $\tilde{A}$ be a deformation of a k-algebra A_0 over an integral domain R, such that $\tilde{A} \otimes_R K$ is a separable algebra for K = Quotient Field(R). We show that any tilting complex T for A_0, lifts to a tilting complex $\tilde{T}$ for $\tilde{A}$ and defines a deformation $\tilde{B}$ of $\tilde{B} = \mathrm{End}_{D^b(\tilde{A})}(\tilde{T})$. Then the decomposition matrix $D_{\tilde{B}}$ of $\tilde{B}$ is obtained from the decomposition matrix $D_{\tilde{A}}$ of $\tilde{A}$ by operations on the columns determined by the summands of $\tilde{T}$ and multiplication of certain rows by (-1). We use this method on the Broué conjecture for the non-principal block of $\tilde{A}_6$, giving an explicit one-step tilting complex of length 2.

1 Introduction

Let A be a k algebra, where k is a field sufficiently large that $A/\operatorname{Rad} A$ is separable. For simplicity one might assume k algebraically closed and A basic, but we do not so in order to allow applications to the theory of representations of groups.

Let R be an integral domain with a distinguished maximal ideal m such that $R/m \xrightarrow{\sim} k$, and a quotient field K.

Definition A *deformation* $\tilde{A}$ of A over (R, m) is a flat R-module $\tilde{A}$ together with an isomorphism

$$A \xrightarrow{\sim} \tilde{A} \otimes_R R/m.$$

The deformation is called *separable* if

$$\bar{A} = \tilde{A} \otimes_R K$$

is a separable algebra. We note that $\tilde{A}$ itself will generally not be separable.

2000 *Mathematics Subject Classification.* 16G10, 20C05, 16S80.

Work is partially supported by a grant from the Bar-Ilan University Research Authority.

There are various examples in the literature of k-algebras with a separable deformation. Brauer tree algebras have a separable deformation over $k[t]$, as to various algebras for which the Donald-Flanigan problem has been solved. Group blocks have a separable deformation over a complete discrete valuation ring with residue field of characteristic p and quotient field of characteristic 0.

The examples given so far have been symmetric, but this is not necessary. The algebra A of dimension four whose quiver consists of two vertices and two arrows in opposite directions deforms to a matrix algebra. The algebra A is self injective but not symmetric. The join of two copies of A by identifying one vertex of each and adding a loop is not self injective but deforms to two matrix algebras.

2 Decomposition Matrices

In the sequel we consider identity idempotents e' of matrix algebras $M_d(k)$, and define $\deg e' = d$.

Assume henceforward that R is either a ring with the idempotent lifting property, e.g.,a complete discrete valuation ring, or is the coordinate ring of an affine curve, e.g., $k[t]$. Let $S_1, \dots, S_\ell$ be the simple modules in $A/\operatorname{Rad}(A)$, and choose a complete orthogonal set of idempotents for A, $e_1, \dots, e_\ell$, such that e_i is the identity for S_i, embedded as a subalgebra of a. Then, taking an etale cover of R if necessary when A is the coordinate ring of a curve, we may assume that the idempotents lift to $\tilde{e}_1, \dots, \tilde{e}_n$ in $\tilde{A}$, a complete orthogonal set of idempotents (generally not primitive).

We note that by various "straightening out theorems", an algebra basis well behaved w.r.t. $\{e_i\}$ lifts to a basis of $\tilde{A}$ well behaved w.r.t. $\{\tilde{e}_i\}$. This leads to the main theorem of [11] that a degeneration of algebras corresponds to a degeneration of basis graphs.

Definition If A is an algebra with separable deformation $\tilde{A}$ over (R, m), $\tilde{e}_1, \dots, \tilde{e}_\ell$ are liftings of the identities of subalgebras S_i, and $f_1, \dots, f_m$ are the block idempotents of the blocks $\tilde{A} \otimes_R K$, then we define the decomposition matrix

$$D_{\tilde{A}} = [d_{ij}]^{n \times r}$$

where

$$d_{ij} = \frac{\deg f_i \tilde{e}_j}{\deg \tilde{e}_j}.$$

Note The f_i are canonical. Since the choice of a complete orthogonal set of idempotents is determined up to conjugation by a unit in the algebra, this definition is independent of the choice of the e_i and of the choice of the lifting $\tilde{e}_i$. It is not difficult to see that for group blocks this coincide with standard definitions of the decomposition matrix.

3 Tilting and Decomposition Matrices

We now assume our algebra A can be tilted to an algebra B. We recall the relationship between tilting and deformation for any deformation, not only for a separable deformation. First assume that $T$is a classical tilting module, and $\tilde{A}$ a deformation of k-algebras. It was proven in [2] that there is a deformed tilting module $\tilde{T}$ of $\tilde{A}$, lifting T, such that if we define $\tilde{B} = \operatorname{End}_{\tilde{A}}(\tilde{T})$, then $\tilde{B}$ is a flat deformation of $B = \operatorname{End}_A(T)$.

Example [2] The diagrams which follow are given in terms of the "basis graph", a directed graph, with a vertex for every idempotent and arrows for every basis element in $\mathrm{Rad}(A)$, according to the Peirce decomposition. The number of arrowheads indicate the depth in the radical series. The basis graph behaves well under flat deformation (which preserves dimension) whereas the quiver behaves badly. The algebra $\tilde{A}$ is a copy of the Dynkin quiver algebra A_3 with linear orientation. In $\tilde{B}$ the arrows with solid arrowheads represent matrix units, and thus $\tilde{B}$ is Morita equivalent to the Dynkin quiver algebra A_3 with the middle vertex a source.

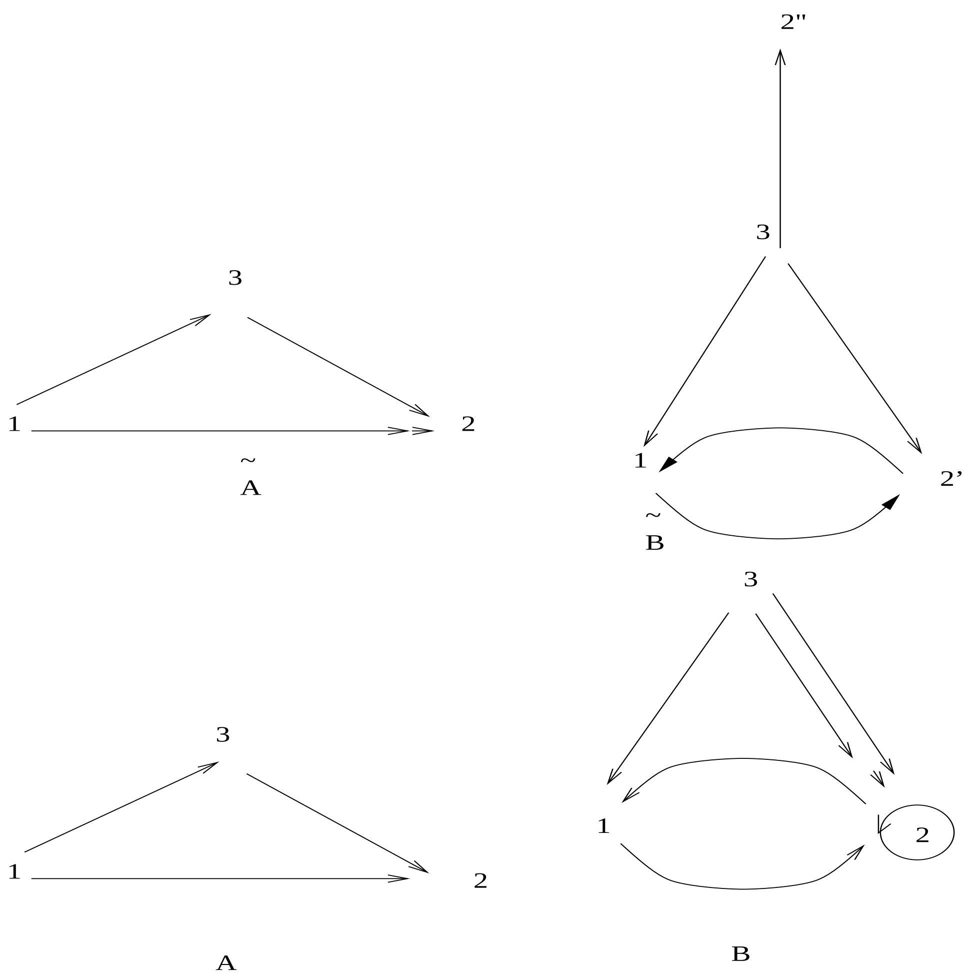

Figure 1

Jeremy Rickard [6] generalized the theorem in [2] in two directions: he allowed a tilting complex instead of just a tilting module, and he also allowed a wider range of parameter rings. We assume henceforward that T is a tilting complex of projective modules as defined in [5] and that $\tilde{A}$ is a separable deformation.

Theorem *Let A be an algebra with separable deformation $\tilde{A}$ and projective indecomposable modules $P_1, \ldots, P_\ell$. Let $T = \oplus T_j^*$, with*

$$T_j^{*d} = \bigoplus_{t=1}^{t_d} Q_{t,j}^d,$$

where each $Q_{t,j}^d$ is isomorphic to an indecomposable projective module $P_{\sigma(j,t,d)}$. Let $\tilde{B} = \mathrm{End}_{D^b(\tilde{A})}(\tilde{T})$. Then for a suitable renumbering of the simples of B and of $\tilde{B}$,

$$D_{\tilde{B}} = S \cdot D_{\tilde{A}} M,$$

where S is a diagonal matrix with entries ± 1, and M is an invertible square matrix of integers with entries

$$m_{ij} = \bigoplus_d (-1)^d \bigoplus_t^{t_d} \delta_{i\sigma(j,t,d)}.$$

Proof By Rickard's theorem, $\overline{T} = \tilde{T} \otimes_R K$ is a tilting complex for $\tilde{A} \otimes_R K$. A tilting complex for a separable algebra must be homotopic to a complex $\hat{T}$ in which each simple module appears in a unique degree, since otherwise the condition

$$\mathrm{Hom}(\overline{T}, \overline{T}[v]) = 0, \quad v \neq 0$$

would not hold. Let d_i be the degree in which N_i occurs in the complex $\hat{T}$ homotopic to $\overline{T}$. The degree of the corresponding simple module in $\tilde{B} \otimes K$ will be the total number of copies of N_i in degree d_i, after we have cancelled out the copies in other degrees via homotopy. We have taken only one copy of each summand in forming T, so this total degree is the sum of the degree for each summand T_j^*, i.e.,

$$d_i' = \bigoplus d_{ij}'.$$

Since maps in a complex of modules for a separable algebra are the sum of zero maps and isomorphisms and the isomorphisms cancel out under homotopy, the maps are irrelevant.

Let us fix N_i and T_j^*, and try to calculate d_{ij}'. Let d be a degree, and let $Q_{t,j}^d$ be one of the terms in T_j^*, in degree d, isomorphic to $P_{\sigma(j,t,d)}$ for some index $s = \sigma(j,t,d)$ in $\{1, \ldots, \ell\}$. The number of copies of N_i in P_s is d_{is}, where $D_{\tilde{A}} = [d_{is}]$. Thus we get

$${d_{ij}'}^d = \bigoplus_{t=1}^{t_d} \bigoplus_{s=1}^{\ell} (-1)^{d-d_i} d_{is} \delta_{s\sigma(j,t,d)}, \tag{1}$$

$$d_{ij}' = (-1)^{d_i} \bigoplus_d \bigoplus_{t=1}^{t_d} \bigoplus_{s=1}^{\ell} (-1)^d d_{is} \delta_{s\sigma(j,t,d)}. \tag{2}$$

We now claim that d_{ij}' is in fact the (ij)-th entry in $D_{\tilde{B}}$. The idempotents corresponding to the simple modules are $\Pi_{T_j^*}$, the projections onto the j-th summand. Because we have taken a single copy of each summand, the degree of this idempotent is one. We choose a complex homotopic to $\overline{T}$ in which all maps are zero. The endomorphism of $\overline{T}$ corresponding to N_i' is the projection on the sum of copies of N_i, which we denote $\overline{\Pi}_i^*$. The composition of the two idempotents $\overline{\Pi}_i \circ \Pi_{T_j^*}$ is the projection on the sum of the copies of N_i in $T_j^{*\prime}$. The degree is the number of copies of N_i in this complex, i.e., d'_{ij}. This is then the (ij)-th entry in $D_{\tilde{B}}$. Let $s_i = (-1)^{d'_i}$ and let S be the diagonal matrix with entries s_i. Formula (2) for the entry d'_{ij} of $D_{\tilde{B}}$ shows that

$$D_{\tilde{B}} = S \cdot D_{\tilde{A}} \cdot M$$

where the (u, j) entry of M is $\bigoplus_d \bigoplus_t (-1)^d \delta_{u\sigma(j,t,d)}$. □

4 Factorizations into Elementary Tiltings for Symmetric Algebras

The matrix M does not actually determine the tilting complex. There are different possible "foldings" of the tilting complex, and different foldings correspond to different images of the simple modules under the induced stable equivalence. We will, in particular, be interested in factoring M into steps corresponding to tilting complexes of the following type, based on [8] and [4], where the "folding" is indicated by the partition $I' \cup I''$ of $I - I_0$, [9], [12], [13].

Definition Let I be the set of simples and consider a partition $\lambda(I_0, I'I'')$, $I = I_0 \cup I' \cup I''$. If P_i, $i \in I$, are the indecomposable projectives, we define

$$\begin{array}{lll} P_i^* : & 0 \to P_i \to 0 & i \in I_0 \\ P_i^* : & 0 \to P_i \to Q_i \to 0 & i \in I' \\ P_i^* & 0 \to R_i \to P_i \to 0 & i \in I'' \end{array}$$

where Q_i is the injective hull of the largest quotient of P_i by a submodule not containing elements from I_0, and R_i is the projective cover of the largest submodule of P_i whose cokernel does not contain simples from I_0.

Definition A map is *tight* if it does not factor through any P_t with $t \in I_0$. Note that the map in the P_i^* are tight.

If both I' and I' are nonempty, we require the following two conditions:
Condition 1: If $i \in I'$, $j \in I''$, and the composition $P_i \to Q_i \to R_j \to P_j$ is zero, then one of the two compositions of length two is zero.
Condition 2: If $t, s \in I_0$, $i \in I'$, $i \in I''$, and if $m_j : P_t \to P_j$ and $m_i : P_i \to P_j$ are tight maps with a commutative square

$$\begin{array}{ccc} P_t & \stackrel{m_i}{\to} & P_j \\ \downarrow & & \downarrow \\ P_i & \stackrel{m_i}{\to} & P_j \end{array}$$

then it is homotopic to zero, i.e., the diagonal map factors through a map from P_j to P_i.

Proposition 1 *If the partition λ satisfies Condition 1 and Condition 2, then $T(\lambda) = \oplus P_i^*$ is a tilting complex.*

Proof Since Q_i and R_i are sums of copies of projectives P_i with $i \in I_0$, we easily construct mapping cones for the other P_j, i.e., $j \in I' \cup I''$, which will generate these P_j.

Thus we need only to show that $\mathrm{Hom}(T(\lambda), T(\lambda)[m]) = 0$ for $m \neq 0$.

We begin with summands

$$\begin{array}{c} P_j \\ \downarrow \\ R_i \xrightarrow{f_i} P_i \end{array}$$

Since P_j is projective and there is no map from P_j into coker f_i then any map must factor through R_i and thus is homotopic to zero. Dually, if there is a map

$$\begin{array}{lc} P_j & \\ \downarrow & \\ P_i \xrightarrow{g} Q_i & \end{array}$$

it can only be well defined if the composition is zero but then P_i maps into the kernel of g_i, and the map must be zero. Since each projective is also injective, we have no maps in the opposite direction either, by a dual argument.

The same arguments used above show that

$$\begin{array}{ccc} & R_i & \to P_i \\ & \downarrow & \\ R_j \xrightarrow{f_i} P_j & & \end{array}$$

is homotopic to zero, because the fact that $\operatorname{coker}(f_j)$ contains no simples from I_0 means that we can factor on the left. In the case of the opposite side

$$\begin{array}{ccc} R_i \to & P_i & \\ & \downarrow \ell & \\ & R_j & \to P_j \end{array}$$

if it is well defined, both compositions with ℓ are zero. However, we have already shown above that if $\ell \circ f_i = 0$, then $\ell = 0$. The case on this opposite side is dual.

We are left with the "mixed" conditions. We begin with a well-defined chain map

$$\begin{array}{ccc} P_i & \xrightarrow{g_i} & Q_i \\ \downarrow & & \downarrow \\ R_j & \xrightarrow{f_j} & P_j \end{array}$$

We need to prove that there is a homotopy $h : Q_i \to R_j$ which gives $\ell_1 = h \circ g_i$ and $\ell_2 = f_j \circ h$. We first note that since coker f_i contains no copies of simples from I_0, there is an h_2 such that $\ell_2 = f_j \circ h_2$. If $\ell_1 = h_2 \circ g_i$, then we are finished. If not, we can replace ℓ_2 by 0 and ℓ_1 by $\ell_1' = \ell_1 - h_2 \circ g_i$. Since $\ker g_i$ contains no simples from I_0, there exists a factorization $\ell_1' = h_1 \circ g_i$. Since $f_j \circ h_1 \circ g_i = 0$ and $h_1 \circ g_i \neq 0$, we conclude by Condition 1 that $f_j \circ h_1 = 0$, and thus h_1 is a homotopy for the new chain map ℓ_1'. Thus $h = h_1 + h_2$ is a homotopy for the original chain map.

We must now consider the opposite direction

$$\begin{array}{ccc} R_j & \to & P_j \\ \downarrow & & \downarrow \\ P_i & \to & Q_i \end{array}$$

It suffices to check this map on each summand of R_j and Q_i, so we may assume that we have

$$\begin{array}{ccc} P_t & \xrightarrow{m_j} & P_j \\ \downarrow & & \downarrow \\ P_i & \xrightarrow{m_i} & P_j \end{array}$$

where m_i and m_j are tight maps. Then by Condition 2, the composition factors through a homomorphism $P_j \to P_i$ and thus the map is homotopic to zero, as desired. $\square$

Example If λ is a partition for the indices of a Brauer star algebra with indices $\{1, \ldots, e\}$, then a partition will be one satisfying the two conditions in which intervals of I', I_0, I'' alternate cyclically. Suppose $e = 12$, $I_0 = \{2, 4, 5, 11\}$, $I' = \{1, 9, 10\}$, $I'' = \{3, 6, 7, 8, 12\}$. The resulting Brauer tree algebra has the form

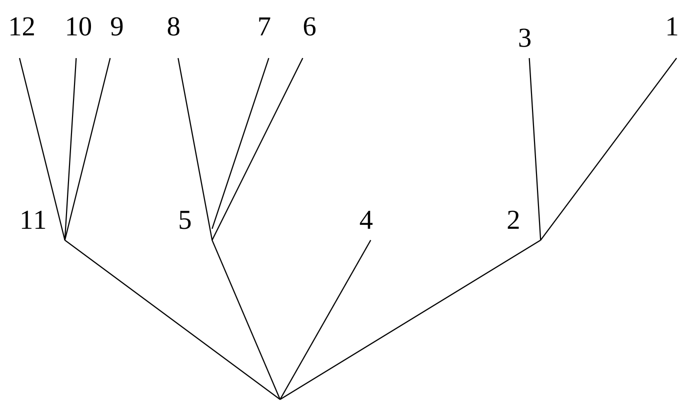

Figure 2

where the edges at the exceptional vertex correspond to elements of I_0, the edges on the clockwise side of the point in the non exceptional vertices correspond to elements of I', and those on the counter clockwise side to I''. In [10], we show that the Green correspondents of the simple modules are determined by the I' and I''.

5 The Broué Conjecture

Let us consider the application of our main theorem to an example for the Broué conjecture. In the case of blocks of group algebras, our main theorem is a consequence of Theorem 3.1 in [1]. The principal blocks for a number of simple groups have been handled by Okuyama, so we consider one of them, namely that of A_6, for $p = 3$. The conjecture has already been settled for the block by Okuyama [4] using different methods, but here we will give the tilting complex explicitly. The normalizer of the defect group is C_4 acting on $C_3 \times C_3$. The quiver of the principal block is

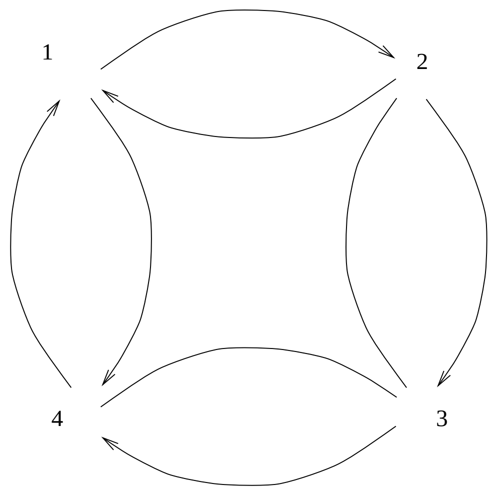

Figure 3

Let α_i be the clockwise arrow starting at i, and β_i the counterclockwise arrow starting at i. We have relations $\alpha^3 = 0$, $\beta^3 = 0$, $\alpha\beta = \beta\alpha$, for

$$\alpha = \sum \alpha_i$$

$$\beta = \sum \beta_i.$$

The decomposition matrix is

$$D_{\tilde{b}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The decomposition matrix of the block of A_6 is given, after a permutation of rows to bring the exceptional character to the bottom, by

$$D_{\tilde{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

There is a stable equivalence (induced by Green correspondence), given by

$$X_1 = 1 \qquad X_2 = \begin{matrix} & & 2 \\ & 3 & \\ 4 & & \end{matrix} \qquad X_3 = \begin{matrix} & 3 & \\ 4 & & 2 \\ & 3 & \end{matrix} \qquad X_4 = \begin{matrix} 4 & & \\ & 3 & \\ & & 2 \end{matrix}$$

Using Okayama's strategy of choosing I_0 with large Green correspondent, we take $I_0 = \{3\}$, $I' = \emptyset$, $I'' = \{1, 2\}$ and get the tilting complex

$$\begin{array}{llcl} P_1^*: & 0 \to & P_3 \oplus P_3 & \to P_1 \\ P_2^*: & 0 \to & P_3 & \to P_2 \\ P_3^*: & 0 \to & P_3 & \to 0 \\ P_4^*: & 0 \to & P_3 & \to P_4 \end{array}$$

The effect on the decomposition matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We now choose $I_0 = \{2, 4\}$, $I' = \emptyset$, $I'' = \{1, 3\}$. The map from P_2^* and P_4^* to P_1^* are isomorphisms on P_3, and similarly with the maps from P_2^* and P_4^* to P_3^*.

So when we take mapping cones we get=

$$\begin{aligned} P_1'^*: &\quad\quad\quad P_2 \oplus P_4 \to P_1 \\ P_2'^*: &\quad P_3 \to P_2 \\ P_3'^*: &\quad P_3 \to P_2 \oplus P_4 \\ P_4'^*: &\quad P_3 \to P_4 \end{aligned}$$

The new matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

It differs from $D_{\tilde{B}}$ only by a permutation of the columns.

Okuyama [4] proves, using his Lemma 2.1, that this sequence of elementary tiltings produces the desired tilting. We comment that, given a candidate tilting complex such as P'^*, one can check directly whether or not it sends each of the X_i to the corresponding simple S_i. Let X_i^* be a projective resolution of X_i. For each X_i, there is a unique degree n_i such that $\mathrm{Hom}(P'^*, X_i^*[n_i]) \neq 0$, modulo homotopy. In this example, the degrees are $n_1 = 0$, $n_2 = 1$, $n_3 = 2$, $n_4 = 1$. by Okuyama's Lemma 1.3 (2),then suffices to check that $\Omega^{n_i}(\mathrm{Hom}(P'^*, X_i^*[n_i])$ is the simple module corresponding to $P_i'^*$, plus projectives.

We hope to turn this into a general method.

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