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THE UNIFORM DISTRIBUTION OF FRACTIONAL PARTS OF RECURRENT SEQUENCES

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Let $f(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - \ldots - a_0$ be a polynomial with integer coefficients of degree $n \geqslant 1$ and rreducible over $\mathbf{Z}$; let $\lambda_1, \ldots, \lambda_n$ be the roots of $f(\lambda)$ and D the discriminant.

We consider the recurrence equation with the characteristic polynomial $f(\lambda)$:

$$\Psi(x+n) = a_{n-1}\Psi(x+n-1) + \ldots + a_0\Psi(x) \qquad (x = 1, 2, \ldots). \tag{1}$$

Let $(\omega(x))$ (for $x = 1, 2, \ldots$) be some sequence of real numbers satisfying (1), with the initial conditions $\omega(\nu) = \omega_\nu$ for $\nu = 1, \ldots, n$.

If $0 < \sigma \leqslant 1$ and $p \geqslant 1$ is an integer, let $N_\sigma(P)$ be the number of solutions of the inequality

$$\{\omega(x)\} < \sigma, \text{ for } x = 1, \ldots, P.$$

The quantity $N_\sigma(P)$ has been most extensively studied in the case $n = 1$, that is, when $\omega(x) = \omega_1\lambda_1^{x-1}$ is an exponential function with an integer base. For example, Korobov [1] has constructed, for prime λ, a ω_1 such that

$$N_\sigma(P) = \sigma P + O(P^{1/3}(\log P)^{4/3}). \tag{2}$$

In [2] this result was extended to any integer $\lambda_1 > 1$. For $n \geqslant 1$ examples of uniformly distributed sequences were given in [3], [4], and [5]. It follows from [6] that for almost all initial conditions

$$\overline{\lim_{P\to\infty}} \sup_\sigma |N_\sigma(P) - \sigma P| P^{-1/2} > 0.$$

A more detailed survey can be found in [7].

In this note we construct, for any $n \geqslant 1$ and $|\lambda_\nu| \neq 1$ ($\nu = 1, \ldots, n$), initial conditions $\omega_1, \ldots, \omega_n$ such that the sequence $(\omega(x))$ ($x = 1, 2, \ldots$) satisfies (2).

It is easily shown that there is an integer $q_1 > 1$ and a prime $p \geqslant 3$ such that $p \mid f(q_1), p \nmid a_0 D$. Hence, $p \nmid q_1 f'(q_1)$ and we can choose a sequence (q_k) ($k = 1, 2, \ldots$) such that

$$f(q_k) \equiv 0 \pmod{p^k}, \quad q_k \equiv q_{k-1} \pmod{p^{k-1}}. \tag{3}$$

We denote by τ_k the exponent of q_k modulo p^k (for $k = 1, 2, \ldots$). Because the polynomials $f(\lambda)$ and $\lambda^{\tau_1} - 1$ are relatively prime, it follows that there is a β such that $\tau_1 = \ldots = \tau_\beta \neq \tau_{\beta+1}$.

LEMMA 1. *For an integer $k \geqslant \beta$, $\tau_k = \tau_1 p^{k-\beta}$.*

The proof follows from Lemma 1 of [8].

Let $n_1 = 0, n_{k+1} = n_k + [p^{k/2}k^2]\,\tau_k$ (for $k = 1, 2, \ldots$), and suppose that only the first m of the roots $\lambda_1, \ldots, \lambda_n$ are of modulus greater than 1.

For integers $a_{i,r} \in [0, p^r)$ ($i = 1, \ldots, n; r = 1, 2, \ldots$) we set

$$A_{\nu,k} = \sum_{r=1}^{k} \lambda_\nu^{n_k - n_r} p^{k-r} \sum_{i=1}^{n} a_{i,r}\lambda_\nu^{i-1} \qquad (\nu = 1, \ldots, m; \quad k = 1, 2, \ldots).$$

LEMMA 2. *There exist integers $a_{i,r} \in [0, p^r)$ such that*

$$A_{1,r}\lambda_1^i + \ldots + A_{n,r}\lambda_n^i \equiv q_r^i \pmod{p^r} \qquad (i = 1, \ldots, n; \quad r = 1, 2, \ldots).$$

If the integers $a_{i,j}$ (for $i = 1, \ldots, n$ and $j = 1, \ldots, r-1$) are defined, then for the choice of $a_{i,r}$ ($i = 1, \ldots, n$) we obtain a system of n linear congruences with the determinant $a_0 D \not\equiv 0 \pmod p$.

Suppose that the $a_{i,r}$ are chosen as indicated in Lemma 2. We set

$$\begin{cases} \theta_\nu = \sum_{r=1}^{\infty} \lambda_\nu^{-n_r} p^{-r} \sum_{i=1}^{n} a_{i,r}\lambda_\nu^{i-1} & (\nu = 1, \ldots, m), \\ \omega_\nu = \theta_1\lambda_1^\nu + \ldots + \theta_m\lambda_m^\nu & (\nu = 1, \ldots, n). \end{cases} \tag{4}$$

LEMMA 3. *Suppose that $|\lambda_\nu| \neq 1$ for $\nu = 1, \ldots, n$; let $\omega(x)$ (for $x = 1, 2, \ldots$) be a sequence satisfying* (1) *with the initial conditions* (4). *Then*

$$\{\omega(n_r+x)\}=\{q_r^x p^{-r}+o(p^{-r})\},$$
$$x=r^2,\ \ldots,\ [p^{r/2}r^2]\,\tau_r-r^2 \qquad (r=1,\ 2,\ \ldots).$$

The lemma follows from Lemma 2, from (4), and the relation

$$\theta_\nu=A_{\nu,k}\lambda_\nu^{-n_k}p^{-k}+O(\lambda_\nu^{-n_{k+1}}) \qquad (\nu=1,\ \ldots,\ m).$$

THEOREM. *Suppose that* $|\lambda_\nu|\neq 1$ *for* $\nu=1,\ldots,n$; *let* $\omega(x)$ *for* $x=1,2,\ldots$ *be a sequence satisfying* (1) *with initial conditions determined by* (4). *Then the following asymptotic formula holds*:

$$N_\sigma(P)=\sigma P+O(P^{1/3}(\log P)^{4/3}).$$

The proof follows immediately from Lemma 3 and the estimates obtained in [2].

REMARK 1. If $m=1$, that is, if λ_1 is a Pisot number, then the uniform distribution remainder of the sequence $(\theta_1\lambda_1^x)$ (for $x=1,\ldots,P$) is $O(P^{1/3}(\log P)^{4/3})$.

REMARK 2. It is easy to see that the condition on $f(\lambda)$ to be irreducible involves no loss of generality.

REMARK 3. A similar method allows us to obtain a generalization of this result to the case of a recurrent sequence of s-dimensional vectors.

References

[1] N. M. Korobov, The distribution of fractional parts of exponential functions, Vestnik Moskov. Univ. Ser. I. Mat. Mekh. **21**:4 (1966), 42–46. MR **33** # 5600.

[2] M. B. Levin, The distribution of fractional parts of an exponential function, Izv. Vyssh. Ucheb. Zaved. Matematika **21**:11 (1977), 50–57.
= Soviet Math. (Izv. Vuz.) **21**:11 (1977), 41–47.

[3] N. M. Korobov, Multidimensional problems of the distribution of functional parts, Izv. Akad. Nauk SSSR Ser. Mat. **17** (1953), 389–400. MR **15**–511.

[4] A. M. Polosuev, On the uniformity of the distribution of a system of functions forming the solution of a system of linear finite-difference equations of the first order. II, Vestnik Moskov. Univ. Ser. I. Mat. Mekh. **15**:6 (1960), 20–25. MR **30** # 1201.

[5] M. B. Levin, The uniform distribution of the sequence $\{\alpha\lambda^x\}$, Mat. Sb. **98** (1975), 207–222. MR **53** # 10732.
= Math.-USSR-Sb. **27** (1975), 183–197.

[6] V. P. Leonov, *Nekotorye primeneniya starshikh semiinvariantov k teorii statsionarnykh sluchainykh protsessov* (Some applications of higher semi-invariants to the theory of stationary random processes) Nauka, Moscow 1964. MR **30** # 5371.

[7] A. G. Postnikov, Ergodic problems in the theory of congruences and of Diophantine approximation, Proc. Steklov Math. Inst. **82** (1966). MR **35** # 5409.
= Proc. Steklov Inst. Mat. **82** (1966).

[8] N. M. Korobov, Trigonometric sums with exponential functions, and the distribution of the symbols in periodic fractions, Mat. Zametki **8** (1970), 641–652. MR **43** # 6165.
= Math. Notes **8** (1970), 831–837.

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