# Characterizations of multiparameter Cox and Poisson processes by the renewal property

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**Abstract**

In the first part of this paper, we give two characterizations of the multiparameter Poisson process: one characterization of poisson process using the renewal property in the space $\mathbf{R}_+^d$, and the other one using the exponential distributions of random areas of rectangles. Finally, we generalize partly a characterization of the random measure associated with a renewal Cox process in $\mathbf{R}_+^d$.

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# 1 Introduction

In recent years, there have been many new results on the dynamical properties of random processes indexed by a multidimensional time parameter or by a class of sets. In particular, Ivanoff and Merzbach [5] provide a definition of the renewal property for general point processes on $\mathbf{R}_+^d$ in a manner that includes the Poisson process.

In the first part of this paper, we give two characterizations of the spatial Poisson process: one characterization using the renewal property in $\mathbf{R}_+^d$, and the other one using the exponential distributions of random areas of rectangles. In the second part, we generalize partly a characterization of the random measure associated with a renewal Cox process in $\mathbf{R}_+^d$.

# 2 Basic notation and definitions

The definitions and results in this section are taken from [5]. Points in $T = \mathbf{R}_+^d$ will be denoted by lower case letters such as $s$ or $t$, and sets in $T$ will be denoted by upper case letters. Families of sets in $T$ will be denoted by script letters.

$\mathcal{A}$ is the collection of rectangles $A_t := [0, t]$. For $t \in T$, $E_t$ denotes the "future" of $t$: $E_t = \{s \in T : t \leq s\} = \{s \in T : A_t \subseteq A_s\}$. $\mathcal{A}(u)$ is the collection of finite union of sets from $\mathcal{A}$. More generally, for any subset $B$ of $T$, its past is defined to be $A(B) = \bigcup_{t \in B} A_t$. Finally, let $\mathcal{C}$ be the class of sets of the form $\mathcal{C} = A \setminus \bigcup_{i=1}^k, A_i$ where $A, A_i \in \mathcal{A}$ and $k$ is finite.

We assume the existence of a sufficiently rich probability space $(\Omega, \mathcal{F}, P)$ on which we define our processes (i.e., the probability space is assumed to be large enough so that each of the random elements defined subsequently is measurable). Our processes will be indexed by $\mathcal{A}$, and more generally, when an $\mathcal{A}$-indexed process induces a random measure on $T$, it may be parameterized by the collection $\mathcal{B}$ of Borel sets of $T$. Moreover, since a set in $\mathcal{A}$ is characterized by its upper right corner, we can identify any $\mathcal{A}$-indexed process $X$ with its $T$-indexed counterpart $X_t = X_{[0,t]} = X_{A_t}$. For notational convenience, occasionally we shall use $X(A)$ or $X(t)$ instead of $X_A$, respectively $X_t$.

Let $N = \{N_{A_t} = N_t; t \in T\}$ be a point process (i.e., an integer-valued random measure; cf. [2]). We will always assume that $N$ is locally finite (i.e., $N_B < \infty \ \forall B, B \in \mathcal{B}$ and $B$ compact) and that $N_t = 0$ if one or more of the coordinates of $t$ is 0.

**Definition 2.1** *Let $N = \{N_{A_t} = N_t; t \in T\}$ be a point process on $T = \mathbf{R}_+^d$.*

- *$N$ is simple if each realization of $N$ satisfies $N_{\{t\}} = 0$ or 1 $\forall t \in T$. (Note the distinction between $N_t = N_{A_t}$ and $N_{\{t\}}$, the mass of $N$ on the singleton $\{t\}$.) If $N_{\{t\}} = 1$, then $t$ is a* jump point *of $N$.*

- *$N$ is strictly simple if whenever $t$ is a jump point of a realization of $N$ ($N_{\{t\}} = 1$), then $N(\partial A_t) = 1$ (i.e., there are no other jump points on $\partial A_t$).*

**Definition 2.2** *For an arbitrary set $B \in \mathcal{B}$,*

$$\min(B) = \{t \in B : s \not\leq t,\ \forall s \in B \text{ such that } s \neq t\}.$$

**Definition 2.3** *For $B \in \mathcal{B}$, we say that $t$ is an* exposed point *of $B$ if*

- $t \in B$,
- $(E_t)^\circ \cap B = \emptyset$, *and*
- *There exists $\epsilon > 0$ such that for each coordinate $t^{(i)}$ of $t = (t^{(1)}, \ldots, t^{(d)})$, $A_{(t^{(1)},\ldots t^{(i-1)},t^{(i)}+\delta,t^{(i+1)},\ldots,t^{(d)})} \subseteq B\ \forall \delta \leq \epsilon$.*

*The set of exposed points of $B$ is denoted by $\varepsilon(B)$.*

**Definition 2.4** *Let $N$ be a point process.*

- $\xi_n = \{t \in T : N_{t-} = N_{(0,t)} < n\}, n = 1, 2, \ldots$ .
- $\xi_n^+ = \overline{(\bigcup_{k\neq j}(E_{\tau_k^{(n)}} \cap E_{\tau_j^{(n)}}))^c}$. *($\overline{(\cdot)}$ denote the closure) If $\xi_n$ has only one exposed point, then $\xi_n^+ = T$.*
- $\Delta_N := \{\tau : N_{\{\tau\}} = 1\}$ *is the set of jump points of the process $N$.*

**Definition 2.5** *$\xi : \Omega \longrightarrow \mathcal{A}(u)$ is called a random set if for every $t \in T$, $\{\omega : t \in \xi(\omega)\} \in \mathcal{F}$; i.e., $\xi$ is a measurable mapping from $(\Omega, \mathcal{F})$ into $(\mathcal{A}(u), \mathcal{F}_{\mathcal{A}(u)})$, where $\mathcal{F}_{\mathcal{A}(u)} = \sigma\{\{D \in \mathcal{A}(u) : t \in D\}, t \in T\}$.*

The class of sets of the form $\{D \in \mathcal{A}(u) : t_1, t_2, \ldots, t_n \in D\}$ for $t_1, t_2, \ldots, t_n \in T$ is a $\pi$-system generating $\mathcal{F}_{\mathcal{A}(u)}$, so that $P\xi^{-1}$ is determined by

$$P\xi^{-1}\{D : t_1, \ldots t_n \in D\} = P\{\omega : t_1, \ldots, t_n \in \xi(\omega)\}.$$

In other words, the distribution of a random set is characterized by these probabilities. We can define independence between random sets as usual (cf. [9],[10]): two random sets $\xi$ and $\xi'$ are said to be independent if

$$P\{t_1, \ldots, t_n \in \xi \cap \xi'\} = P\{t_1, \ldots, t_n \in \xi\}P\{t_1, \ldots, t_n \in \xi'\}$$

for all $t_1, \ldots, t_n \in T$.

We restrict our attention to bounded sets in order to ensure that we are able to number the jump points of a point process lexicographically.

We now come to the definition of a multi-parameter renewal process, as introduced in Ivanof and Merzbach [5].

We introduce the following notation: for an arbitrary Borel set $B$ and $t \in T$, $B \ominus t = \{x - t : x \in B\}$.

**Definition 2.6** *Let $N$ be a strictly simple point process on $\mathbf{R}_+^d$ with associated $\xi_i$. $N$ is a renewal point process if for every $i \geq 1$ and every $A \in \mathcal{A}$,*

1. *For $i \geq 1$, given $\xi_i$ the random set $\left(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A\right) \ominus \tau_j^{(i)}$ are independent, $\forall j$, where $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ are the exposed points of $\xi_i \cap A$, numbered lexicographically.*

2. *For $i, j \geq 1$, given $\xi_i$*

$$\left(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A\right) \ominus \tau_j^{(i)} =_{\mathcal{D}} \xi \cap \left[\left(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\right) \ominus \tau_j^{(i)}\right] \tag{1}$$

*where "$=_{\mathcal{D}}$" denotes equality in distribution, $\xi$ is a copy of $\xi_1$ independent of $\xi_i$, and $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ are the exposed points of $\xi_i \cap A$, numbered lexicographically.*

**Theorem 2.7** *A Poisson process $N$ on $T$ is renewal if and only if it is homogeneous.*

# 3 Characterizations of the Poisson process in $\mathbf{R}_+^d$

In this section, we give some new characterizations of the spatial Poisson process by the renewal property. These characterizations are very similar to classical characterizations of the one parameter Poisson process.

## 3.1 First Characterization

Let $A \in \mathcal{A}$ be given. The stochastic processes that we consider in this section are indexed by sets included in $A$.

**Definition 3.1** *Let $\xi$ be a random set, $F_\xi(B) = P(B \subseteq \xi)$ for $B \in \mathcal{A}(u)$ is called the distribution function of $\xi$.*

**Proposition 3.2** *Let $D = \bigcap_{i=1}^{n} \overline{E_{z_i}^c}$, $\quad D' = \bigcap_{i=1}^{m} \overline{E_{z'_i}^c}$, where the points $z_1, \ldots, z_n$ are incomparable and the points $z'_1, \ldots, z'_n$ are incomparable in $\mathbf{R}_+^d$. Then, the probability $P(\xi \in [D, D'])$ is determined by $F_\xi$.*

**Proof:** Notice first that

- $P(\xi \subset D') = 1 - P(\xi \cap \{z'_1, \ldots, z'_m\} \neq \phi) = 1 - \left(\sum_{i=1}^{m} F_\xi(A(\{z'_i\})\right) - \sum_{i \neq j} F_\xi(A(\{z'_i, z'_j\})) + \cdots + (-1)^{n+1} F_\xi(A(\{z'_1, \ldots, z'_m\}))).$

- $P(D' \subseteq \xi | D \subseteq \xi) = \frac{F_\xi(D')}{F_\xi(D)}$

- $P(\xi \in [D, D']) = P(\xi \subseteq D' | D \subseteq \xi) F_\xi(D) = \left[1 - \sum_{i=1}^{m} \frac{F_\xi(A(\{z'_i\})}{F_\xi(A(\{z'_i\}) \cap D)} + \sum_{i \neq j} \frac{F_\xi(A(\{z'_i, z'_j\})}{F_\xi(A(\{z'_i, z'_j\}) \cap D)} + \cdots + (-1)^n \frac{F_\xi(A(\{z'_1, \ldots, z'_m\})}{F_\xi(A(\{z'_1, \ldots, z'_m\}) \cap D)}\right] F_\xi(D).$

Hence $P(\xi \in [D, D'])$ is determined by $F_\xi$. $\square$

**Corollary 3.3** *The distribution of $\xi$ is entirely determined by $F_\xi$.*

**Example 3.4** *If $F_\xi(D) = e^{-\lambda\mu(D)}$, where $\mu$ is a measure on $(\mathbf{R}^d_+, \mathcal{B})$, then:*

$$P(\xi \in [D, D']) = e^{-\lambda\mu(D)} \prod_{i=1}^{m} \Big(1 - e^{-\lambda\mu(A_{z'_i} \setminus D)}\Big).$$

**Comment 3.5** Let us note that $F_{\xi_{i+1}|\xi_i}(D'|D) = P(D' \subseteq \xi_{i+1}|\xi_i = D)$, $D, D'$ defined as in Proposition 3.2.

**Proposition 3.6** *Let $N$ be a renewal process, with associated $\xi_i$. Then $\forall D, D'$ as above:*

$$F_{\xi_{i+1}|\xi_i}(D'|D) = \begin{cases} \prod_j F_{\xi_1}(D'_j \ominus z_j) & D \subseteq D' \subseteq D^+ \\ 0 & otherwise \end{cases}$$

*where $z_j$ are the exposed points of $D$ and $D'_j = D' \cap E_{z_j}$ and $D^+$ is defined as in Definition 2.4*

**Proof:** From the renewal property, we obtain that $\xi_{i+1}$ is identical distributed to $\xi_1$ on the random set $(E_{z_j} \cap D^+) \ominus z_j$.

**Example 3.7** *Let $N$ be a homogeneous Poisson process, with parameter $\lambda$ and with associated random sets $\xi_i$. then:*

$$F_{\xi_{i+1}|\xi_i}(D'|D) = \begin{cases} e^{-\lambda\mu(D' \setminus D)} & D \subseteq D' \subseteq D^+ \\ 0 & otherwise \end{cases}$$

where $\mu$ is the Lebesgue measure.

**Theorem 3.8** *Let $F_\xi$ be the distribution function of $\xi$, then there exists one and only one renewal point process $N$ with $\xi_1 =_{\mathcal{D}} \xi$.*

**Proof:** (see [5]) Let $F_{\xi_i}$ be defined by $\xi$ as in Proposition 3.6. Denote the set $\varepsilon(\xi_1) = \min(\Delta_N)$ by $\epsilon_1$, given $\xi_{i-1}$, let $\epsilon_i = \min\{\Delta_N \cap (\xi^+_{i-1} \setminus \xi_{i-1})\}$. We observe that $\bigcup_n \epsilon_n = \Delta_N$.

Hence, if $\forall A \in \mathcal{B}$,

$$N_A = \sum_{\tau \in \Delta_N} I(\tau \in A) = \sum_{n=1}^{\infty} \sum_{\tau \in \epsilon_n} I(\tau \in A).$$

The process $N$ is renewal and is entirely determined by the distribution function of $\xi$ $\square$

**Theorem 3.9** *The following statements are equivalent:*

*(a) $N$ is a homogeneous Poisson process with parameter $\lambda$.*

*(b) $N$ is a renewal process and $\xi_1$ is exponentially distributed.*

**Proof:** (a) $\Rightarrow$ (b): By Theorem 2.7, the Poisson process is a renewal process. Since $F_{\xi_1}(B) = P(N(B) = 0) = e^{-\lambda\mu(B)}$, $\xi_1$ is exponential distributed.

(b) $\Rightarrow$ (a) Immediate by theorem 3.8. $\square$

## 3.2 Second Characterization

Another characterization of the Poisson process can be presented by the distributions of its inter arrival areas.

Let $t = (t_1, \ldots, t_d) \in R^d_+$. For any $a \geq 0$ and $i = 1, \ldots, d$, denote $t(a^{(i)})$ to be the point $(t_1, \ldots t_{i-1}, a, t_{i+1}, \ldots, t_d)$. Also, the product of the components of a point t will be denoted by $|t| = \prod_{j=1}^{d} t_j$, therefore $|t(1^{(i)})| = \prod_{j=1, j\neq i}^{d} t_j$.

**Definition 3.10** *Let $N$ be a homogenous Poisson process on $\mathbf{R}^d_+$ with associated sets $\xi_i$. A level of $\xi_n(\omega)$ from $t(0^{(i)})$ is*

$$Y^{(n)}_{t(0^{(i)})}(\omega) = \sup\{a | A_{t(a^{(i)})} \subseteq \xi_n\},$$

*where $t = t(t^{(i)}_i)$.*

**Proposition 3.11** $Y^{(n)}_{t(0^{(i)})} \sim \Gamma(n, \lambda |t(1^{(i)})|)$ $\forall n \geq 1$.

**Proof:** Let $0 < x$, $P(Y^{(n)}_{t(0^{(i)})} \leq x) = P(N_{t(x^{(i)})} \geq n) = \Gamma(n, \lambda |t(1^{(i)})|)$.

**Corollary 3.12** *If $N$ is a homogenous Poisson process, then the random area of the rectangle $|t(1^{(i)})| Y^{(n)}_{t(0^{(i)})} \sim \Gamma(n, \lambda)$, $\forall n \geq 1$.*

**Proof:** Calculation immediate from the distribution.

**Proposition 3.13** $Y^{(n)}_{t(0^{(i)})} - Y^{(n-1)}_{t(0^{(i)})} \sim \exp(\lambda |t(1^{(i)})|)$, $\forall n \geq 1$ *and the family* $\{Y^{(n)}_{t(0^{(i)})} - Y^{(n-1)}_{t(0^{(i)})}\}^{\infty}_{n=1}$ *is independent.* $(Y^{(0)}_{t(0^{(i)})} = 0)$.

**Proof:** Let $I$ be the segment obtained by the intersection between the point $t(0^{(i)})$ and the point $t(t^{(i)}_i)$. Let $N_{t(0^{(i)})}(t_i) = N_{t(t^{(i)}_i)}$ where $t(0^{(i)})$ is fixed. Hence $N_{t(0^{(i)})}(t_i) \sim poi(\lambda |t(1^{(i)})|)$ on $I$ where $Y^{(i)}_{t(0^{(i)})}$ are jump points. □

**Corollary 3.14** *Let $M^i_n(t) = |t(1^{(i)})| (Y^n_{t(0^{(i)})} - Y^{n-1}_{t(0^{(i)})})$ be called the inter arrival rectangles of the process, then $\forall n, \forall i, \forall t, M^i_n(t) \sim \exp(\lambda)$ and the family $\{M^i_n(t)\}^{\infty}_{n=1}$ is independent.*

**Definition 3.15** [4] *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_A, A \in \mathcal{A})$ be a family of sub $\sigma$ fields of $\mathcal{F}$. We say that this family satisfies (CI) if $\mathcal{F}_A$ and $\mathcal{F}_B$ are conditionally independent given $\mathcal{F}_{A\cap B}$, for all $A, B \in \mathcal{A}$.*

**Comment 3.16** [4] *(CI) corresponds to the (F4) condition introduced in* [1].

**Theorem 3.17** *Let $N$ be a simple point process in $\mathbf{R}^d_+$, $(\mathcal{F}_t, t \in \mathbf{R}^d_+)$ be the minimal filtration generated by $N$, and $M^i_n(t)$ its associated inter arrival rectangles. Then $N$ is a homogeneous Poisson process with parameter $\lambda$ if and only if (CI) holds, $\forall n, \forall i, \forall t, M^i_n(t) \sim \exp(\lambda)$ and the family $\{M^i_n(t)\}^{\infty}_{n=1}$ is independent.*

**Proof:** $\Rightarrow$ Proved above.
$\Leftarrow$ Let $i, i = 1, \ldots, d$. Since the process $N_{t(0^{(i)})}(t_i)$ is generated by $\{M_n^i(t)\}$, then it is a one parameter Poisson process and therefore $N_{t(0^{(i)})}(t_i) - \lambda t_i$ is martingale $\forall i$, where $t(0^{(i)})$ is fixed. By hypothesis $(CI)$ (see [1]), $N_t - \lambda t_1 \cdots t_d$ is a multiparameter martingale.

Hence $N$ is a homogeneous Poisson process with parameter $\lambda$ (see [4]). $\square$

# 4 The random measure associated with a renewal Cox process in $\mathbf{R}_+^d$

In the one parameter classical case, Kingman (see [6], [7], [8]) proved that a Cox process with rate $\Lambda$ is a renewal process if and only if there exists $\lambda > 0$ so that $\Lambda$ takes two values $\lambda$ or 0 only, and the event $\Lambda(t) = \lambda$ is an equilibrium regenerative event. Here we extend a part of this result to the multiparameter case answering an open question from [5].

**Definition 4.1** *Let $\Lambda = \{\Lambda(t), t \in T\}$ be a nonnegative integrable stochastic process. We say that the point process $N = \{N_A, A \in \mathcal{A}\}$ is a Cox process with intensity $\Lambda$ if $N$ conditionally on $\Lambda$ is a non-homogeneous Poisson process with mean measure $\Lambda_*(A) = \int\limits_A \Lambda(t)dt$.*

**Comment 4.2** *(see [3])Let $N$ be a Cox process then*

$$P(N(B_1) = k_1, ..., N(B_n) = k_n) = E_{\Lambda_*}(\prod_{i=1}^n e^{-\Lambda_*(B_i)} \frac{(\Lambda_*(B_i))^{k_i}}{k_i!})$$

$$= \int\limits_M (\prod_{i=1}^n e^{-\nu(B_i)} \frac{(\nu(B_i))^{k_i}}{k_i!}) d\Pi$$

*where $B_1, ..., B_n$ are disjoint Borel sets, $\nu$ is a measure on $\mathbf{R}_+^d$, $M$ is the collection of all the absolutely continue measures on $\mathbf{R}_+^d$, $\Pi$ is a probability measure on $M$.*

Any map $f : R_+ \rightarrow \mathcal{A}(u)$ which is increasing with respect to the partial order induced by the set-inclusion is called a **flow**.

**Definition 4.3** *A flow $f : R_+ \rightarrow \mathcal{A}(u)$ is* **stationary** *if for any $\boldsymbol{t}, \boldsymbol{s} \in R_+$, $\boldsymbol{s} \leq \boldsymbol{t}$, we have $\mu(f(\boldsymbol{t}) \backslash f(\boldsymbol{s})) = \boldsymbol{t} - \boldsymbol{s}$, where $\mu$ is Lebesgue measure.*

**Example 4.4** *Let $f$ be a flow such that $f : \boldsymbol{t} \rightarrow A_{(\sqrt[d]{t}, ..., \sqrt[d]{t})}$ then $f$ is a stationary flow because $\mu(f(\boldsymbol{t}) \setminus f(\boldsymbol{s})) = \boldsymbol{t} - \boldsymbol{s}$.*

**Example 4.5** *Given* $t \in R_+^d, t = (t_1, ..., t_d)$. *We denote* $\forall t \in R_+^d$, $B_t = A_{2t} - (t, 2t]$, *therefore* $|B_t| = (2^d - 1)|t|$.
*Let* $f$ *be a flow such that:*

$$x \mapsto I_{\{x \leq (2^d-1)|t|\}} \cdot B_{\sqrt[d]{\frac{x}{(2^d-1)|t|}}t} + I_{\{(2^d-1)|t| \leq x \leq |2t|\}} \cdot B_t \cup A_{(1+\sqrt[d]{\frac{x-(2^d-1)|t|}{|t|}})t}$$
$$+ I_{\{|2t| \leq x\}} \cdot A_{\sqrt[d]{\frac{x}{|t|}}t}$$

*then* $f$ *is a stationary flow.*

**Theorem 4.6** *Let* $N$ *be a renewal Cox process on* $\mathbf{R}_+^d$ *with intensity* $\Lambda^{(d)}$, *then* $\Lambda^{(d)}(t) = \lambda, 0$ *where* $\lambda$ *is a positive constant.*

**Proof:** Let $f : R_+ \to \mathcal{A}(u)$ be a stationary flow as example 4.5. Define $g(N)$ to be the point process on $\mathbf{R}_+$ with jump points $(\tau_1, \tau_2, \dots)$ by the following definition: $P(x \leq \tau_1) = F_{\xi_1}(D)$ where $D = f(x)$. The distribution of the other jump points are defined similarly: $P(x + s \leq \tau_{i+1} | \tau_i = s) = F_{\xi_{i+1}|\xi_i}(D'|D)$ for any $D', D$ such that $D \subseteq D' \subseteq D^+$ and $D' \backslash D = z_k + f(x)$ and $z_k$ is an exposed point of $D$. Then we obtain (see Proposition 3.6) $F_{\xi_{i+1}|\xi_i}(D'|D) = \prod_j F_{\xi_1}(D'_j \ominus z_j)$ where $D'_j \ominus z_j = \phi, \forall j \neq k$ and $D'_k \ominus z_k = f(x)$, so that $F_{\xi_{i+1}|\xi_i}(D'|D) = F_{\xi_1}(f(x))$. $F_{\xi_{i+1}|\xi_i}$ doesn't depend of $i$ and therefore $g(N)$ is a renewal process on $\mathbf{R}_+$.

Let $M^{(1)}$ be a collection of measures on $\mathbf{R}_+$ such that $M^{(1)} = \{\nu^{(1)} | \nu^{(1)}([0, x]) = \nu^{(d)}(f(x)), \nu^{(d)} \in M\}$ and $\forall k$, denote $F(B) = P(g(N)(B) = k)$. We compute the distribution of $g(N)$ in $B$ by its interarrival times:

$$F(B) = \int\limits_{M^{(1)} \times B} dF(\nu, \mathbf{t}) = \int\limits_{M^{(1)}} \int\limits_{B} dF(\nu, \mathbf{t})$$

(see [7], $\mu$ is the Lebesgue measure, denote $\mathbf{s} = \nu(\mathbf{t})$ )

$$= \int\limits_{M^{(1)}} \int\limits_{\nu(B)} dF(\nu, \nu^{-1}(\mathbf{s})) = \int\limits_{M^{(1)}} \int\limits_{\nu(B)} dF(\mu, \mathbf{s}) = \int\limits_{M^{(1)}} e^{-\nu(B)} \frac{(\nu(B))^k}{k!} d\Pi$$

and hence we obtain that $g(N)$ is a Cox process in $\mathbf{R}_+$, too.

From Kingman (see [7]) we obtain that $\Lambda^{(1)}(\mathbf{t}) = \lambda, 0$ ($\Lambda^{(1)}(\mathbf{t})$ is the intensity of $g(N)$). From the definition of $g(N)$ we obtain $\Lambda_*^{(d)}((t, (1+\sqrt[d]{\frac{h^d}{|t|}})t]) = \Lambda_*^{(1)}(((2^d-1)|t|, (2^d-1)|t| + h^d])$, hence

$$\Lambda^{(d)}(t) = \lim_{h \to 0} \frac{1}{h^d} \int\limits_{(t, (1+\sqrt[d]{\frac{h^d}{|t|}})t]} \Lambda^{(d)}(s) ds = \lim_{h \to 0} \frac{1}{h^d} \int\limits_{((2^d-1)|t|, (2^d-1)|t|+h^d]} \Lambda^{(1)}(\mathbf{s}) d\mathbf{s}$$
$$= \Lambda^{(1)}((2^d-1)|t|) = \lambda, 0.$$

$\square$

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