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What is a multi-parameter renewal process?

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The concept of the renewal property is extended to processes indexed by a multidimensional time parameter. The definition given includes not only partial sum processes, but also Poisson processes and many other point processes whose jump points are not totally ordered. A new version of the waiting time paradox is proven for multidimensional Poisson processes, and is shown to imply the renewal property. Finally, martingale properties of renewal processes are studied.

Keywords: Point process; Renewal process; Stopping set; Partial sum process; Poisson process; Multiparameter martingale

AMS 2000 Subject Classifications: 60K05; 60G55; 60G48

1. Introduction

In recent years, there have been many new results on the dynamical properties of random processes indexed by a multidimensional time parameter or by a class of sets. In particular, the concepts of both martingales and Markov processes have been extended with some success to processes with more general index sets (cf. [13,14,20,28,33]). The major challenge in extending dynamical concepts which are characterized in terms of the future behaviour of the process given the past is finding the appropriate analogues of "past" and "future" when the parameter set is only partially ordered.

Perhaps the simplest dynamical structure on $\mathbf{R}_+$ is that of the *renewal process*. An ordinary renewal point process N on $\mathbf{R}_+$ is determined by the partial sums of a sequence of i.i.d. non-negative random variables $(X_1, X_2, \dots) : N_t = \sum_{i=1}^{\infty} I(\tau^{(i)} \leq t)$, where $\tau^{(i)} = \sum_{j=1}^{i} X_j$. A similar definition may be given on $\mathbf{R}_+^d$: if $(X_1, X_2, \dots)$ is a sequence of i.i.d. $\mathbf{R}_+^d$-valued random variables and $\tau^{(i)} = \sum_{j=1}^{i} X_j$, then for $z \in \mathbf{R}_+^d$ the renewal point process N is defined by $N_z = \sum_{i=1}^{\infty} I(\tau^{(i)} \leq z)$ (here, "$\leq$" denotes the usual partial order on $\mathbf{R}_+^d$). This approach

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has been taken by numerous authors and analogues of several classical theorems have been proven (cf. [1,4,8–10,19,26,29]). However, the shortcoming of this approach is that it forces the jump points $\tau^{(1)}, \tau^{(2)}, \ldots$ of the multidimensional renewal point process to be totally ordered. Thus, according to this definition the most fundamental renewal process on $\mathbf{R}_+$, the Poisson process, is *not* renewal on $\mathbf{R}_+^d$.

The purpose of this paper is to provide a satisfactory definition of the renewal property for general point processes on $\mathbf{R}_+^d$ in a manner that includes both the partial sum process defined above and the Poisson process. Our original (and more ambitious) goal was to extend the concept of the renewal property to processes indexed by a class of sets from a general topological space (such as in the framework of Refs. [13] and [14]), but it became apparent that a concept of "positive translation" would be necessary. Although, this extension is feasible, we have chosen for the moment to restrict our attention to Euclidean space, given the complexity of the definitions and notation required. In fact, in our experience this apparently simple notion was the most challenging dynamical structure to extend to a more general framework. Nevertheless, as observed in Ref. [5], the Poisson and renewal processes are the "archetypal point processes" on $\mathbf{R}_+$, and so it is important to understand how the renewal mechanism can be generalized.

To describe the basic idea behind the approach taken here, we return to the classical renewal process on $\mathbf{R}_+$. The renewal process describes a recurring phenomenon. Each time there is an occurrence of the phenomenon, the process begins anew until the time of the next occurrence. The successive recurrence times are the jump points of the renewal point process. The main idea is that at each jump point of the renewal process, the pattern of the process after the point does not depend on the pattern before the point. As well, the conditional laws of the jump points given their predecessors are identical and each depends only on the location of the preceding jump. The idea of *successive* jump points is crucial, and we must find a way of extending this to time points which are only partially ordered.

We begin by observing that the jump points of any simple point process N on $\mathbf{R}_+$ form an increasing sequence $(\tau^{(1)}, \tau^{(2)}, \ldots)$ of stopping times with respect to the minimal filtration generated by N: $\mathcal{F}_t = \sigma\{N_s : s \leq t\}$. As is well known, this is no longer the case in $\mathbf{R}_+^d$; in fact it has been observed repeatedly in the literature on multi-parameter and set-indexed processes that the appropriate analogue of the stopping time is a *stopping set* (or equivalently its boundary, which is called a stopping line). A formal definition will be given in Section 3, but we note here that the concept of stopping is fundamental for any sort of dynamical structure, and finding the appropriate definition was the key that permitted the development of a useful set-indexed martingale theory (cf. [16]). Stopping sets are also the starting point for renewal theory: the idea is to define the multidimensional renewal property in terms of the increasing sequence of stopping sets $\xi_1, \xi_2, \ldots$, where $\xi_i = \{t \in \mathbf{R}_+^d : N((0,t)) \leq i-1\}$, and $N((0,t))$ denotes the number of jump points in the open set $(0,t)$ (We observe that on $\mathbf{R}_+$, $\xi_i = [0, \tau^{(i)}]$). The set of points of discontinuity of N is equal to $\cup_i \partial\xi_i$ ($\partial\xi_i$ denotes the boundary of ξ_i). In a manner that will be made precise, we shall define the renewal property in such a way that the conditional law of ξ_i given $\xi_1, \ldots, \xi_{i-1}$ depends only on the configuration of ξ_{i-1}, and this law is independent of i.

This kind of phenomenon can occur in numerous real-life situations. First, we consider the partial sum process described earlier. Although, the structure of the partial sum process is very simple, its mathematical properties are profound and its applications are many: it can be considered as the product of d independent renewal processes on the line or it can model d

(possibly dependent) queues where the components of the vector $(\tau_1^{(i)}, \ldots, \tau_d^{(i)})$ denote the i-th time that each of the queues is empty. It can be used to model an item with several components that break down at random times and are repaired separately. More generally, renewal processes whose jump points are not totally ordered include the Poisson process and certain generalizations. This type of process has important applications in the environmental sciences. For example, it can be used to model the spread of a disease in a forest subject to prevailing winds, or the migration and capture of a fish population moving in ocean currents. In each case, the point of view adopted is essentially dynamical: the process evolves in a general (northeasterly) direction and renews itself at each stopping line. More precisely, each diseased tree will infect others northeast of it because of prevailing winds from the southwest; likewise, a captured fish swimming in ocean currents indicates the presence of other fish swimming in the same direction from the point of capture. Even as an approximate model, our approach has the advantage of being easy to simulate. As will be seen, our renewal process consists of the superposition of conditionally independent "single line" processes as defined in Section 2.

In Section 2, we present generalities on point processes and introduce a set-indexed framework, followed in Section 3, by a discussion of the special dynamical structure of a point process in $\mathbf{R}_+^d$. In Section 4, we define the concept of a multi-parameter renewal process. In Section 5, we study the partial sum process in the context of our definition of the renewal property. We give a renewal formula and show that some simple extensions cannot be renewal. Section 6 is devoted to Poisson processes. We prove an extension of the waiting time paradox (WTP) for the homogeneous Poisson process on $\mathbf{R}_+^d$ and show that it is in fact equivalent to the renewal property. Moreover, among the class of Poisson processes, the renewal property is characterized by the homogeneity of the process. Section 7 is devoted to the martingale properties of renewal processes. The power of martingale methods in the statistical analysis of point processes on $\mathbf{R}_+$ is well-established, and these techniques have recently been extended to point processes on more general spaces in Ref. [15]. We construct compensators for renewal processes with respect to two different filtrations, giving rise to two different types of martingales. Examples are given, including a brief discussion of how the compensator can be used to analyze the asymptotic behaviour of superpositions of renewal processes. In Section 8, we conclude with general comments and directions for further research.

2. Generalities on point processes

In this section, we examine the general structure of a point process on a partially ordered space T. In particular, we develop a decomposition of any point process on T into the sum of point processes each consisting only of non-comparable jump points. This will be the key result required for the renewal property. The framework will be essentially the same as in Ref. [13], and generally we follow the same notation. However, since we need translations, our parameter set T will be the positive quadrant of the Euclidean space $\mathbf{R}^d$ equipped with the usual partial order.

Points in $T = \mathbf{R}_+^d$ will be denoted by lower case letters such as s or t, and sets in T will be denoted by upper case letters. Families of sets in T will be denoted by script letters and we will use Greek letters for random sets. We denote the interior, closure and boundary of sets by "$(\cdot)^\circ$", "$\overline{(\cdot)}$" and "$\partial(\cdot)$", respectively.

$\mathcal{A}$ is the collection of rectangles $A_t := [0, t]$. For $t \in T$, E_t denotes the "future" of t: $E_t = \{s \in T : t \le s\} = \{s \in T : A_t \subseteq A_s\}$. Next, let $D_t = \overline{E_t^c}$. A_t can be interpreted as the strict past of t, and D_t its wide past. Finally, let $\mathcal{C}$ be the class of sets of the form $C = A \backslash \cup_{i=1}^k, A_i$ where $A, A_i \in \mathcal{A}$ and k is finite ($\mathcal{C}$ can be thought of as the class of "generalized rectangles").

More generally, for any subset B of T, its past is defined to be $A(B) := \cup_{t \in B} A_t$, its future is $E(B) := \cup_{t \in B} E_t$, and $D(B) := \overline{E(B)^c}$.

We assume the existence of a sufficiently rich probability space $(\Omega, \mathcal{F}, P)$ on which we define our processes (i.e. the probability space is assumed to be large enough that each of the random elements defined subsequently is measurable). Our processes will be indexed by $\mathcal{A}$, and more generally, when an $\mathcal{A}$-indexed process induces a random measure on T, it may be parameterized by the collection $\mathcal{B}$ of Borel sets of T. Moreover, since a set in $\mathcal{A}$ is characterized by its upper right corner, we can identify any $\mathcal{A}$-indexed process X with its T-indexed counterpart $X_t = X_{[0,t]} = X_{A_t}$. For notational convenience, occasionally we shall use $X(A)$ or $X(t)$ instead of X_A, respectively, X_t.

Let $N = \{N_{A_t} = N_t; t \in T\}$ be a point process (i.e. an integer-valued random measure; cf. [5]). We will always assume that N is locally finite (i.e. $N_B < \infty \, \forall B$, $B \in \mathcal{B}$ and B compact) and that $N_t = 0$ if one or more of the coordinates of t is 0.

Definition 2.1. Let $N = \{N_{A_t} = N_t; t \in T\}$ be a point process on $T = \mathbf{R}_+^d$.

- N is simple if each realization of N satisfies $N_{\{t\}} = 0$ or $1 \; \forall t \in T$ (Note the distinction between $N_t = N_{A_t}$ and $N_{\{t\}}$, the mass of N on the singleton $\{t\}$). If $N_{\{t\}} = 1$, then t is a jump point of N.
- N is strictly simple if whenever t is a jump point of a realization of N ($N_{\{t\}} = 1$), then $N(\partial A_t) = 1$ (i.e. there are no other jump points on ∂A_t).

In figure 1, we see an example of a realization of a strictly simple point process N on a set $A_t = [0, t], t \in \mathbf{R}_+^2$. The jump points of N are illustrated in figure 1(a), and the values of N_s for $s \in [0, t]$ are given in figure 1(b).

In Ref. [13], point processes are generally assumed to be simple but not necessarily strictly simple. *Here we shall assume without further comment that all point processes are strictly simple*, which is essential for the renewal property defined in the next section. Note that a point process N is characterized by the set of its jump points since for any Borel set B, N_B is the number of jump points contained in B. It follows trivially that N is an *increasing* process: $N_B \ge 0 \, \forall B \in \mathcal{B}$. We observe that a point process N may be regarded either as a random measure indexed by $\mathcal{B}$ or as a stochastic process indexed by T (or equivalently by $\mathcal{A}$). As a random measure, the distribution of N is characterized by the finite dimensional probabilities

$$P(N_{B_1} = k_1, \ldots, N_{B_n} = k_n), \quad B_1, \ldots, B_n \in \mathcal{B}, \quad 1 \le n < \infty.$$

However, by additivity, the finite dimensional distributions of the T- (equivalently, $\mathcal{A}$−) indexed stochastic process N

$$P(N_{t_1} = k_1, \ldots, N_{t_n} = k_n), \quad t_1, \ldots, t_n \in T, \quad 1 \le n < \infty$$

determine the law of N as a random measure (cf. [5], Proposition 6.2.III).

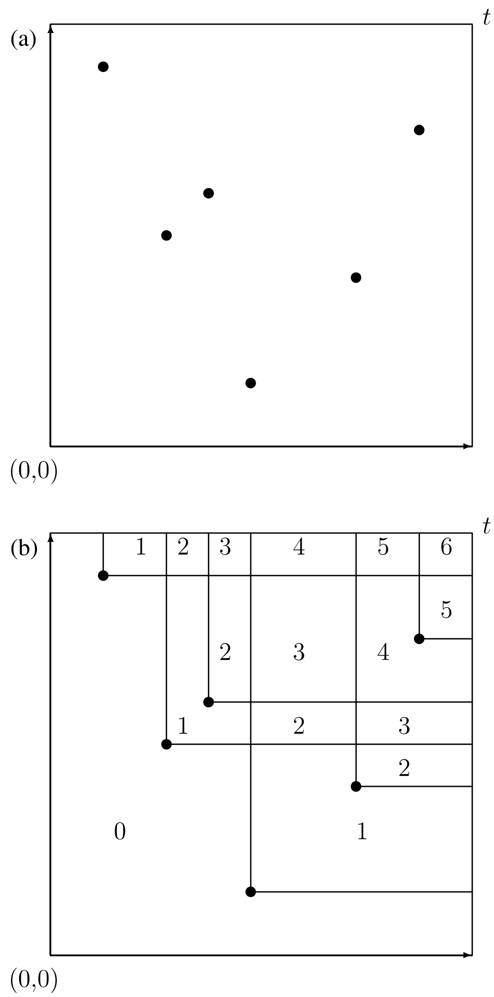

Figure 1. A realization of a strictly simple point process on A_t and its values: (a) jump points of N; (b) values of N_s for $s \in [0, t]$.

Given a point process N, we may define a number of associated random sets and points. Their measurability properties will be discussed later in this section. First, we need the following:

Definition 2.2.

1. For an arbitrary set $B \in \mathcal{B}$,

$$\min(B) = \{t \in B : s \not\leq t, \quad \forall s \in B \text{ such that } s \neq t\}.$$

   The set B is called incomparable if $B = \min(B)$.
2. For $B \in \mathcal{B}$, we say that t is an exposed point of B if

- $t \in B$,
- $(E_t)^\circ \cap B = \emptyset$, and

- There exists $\varepsilon > 0$ such that for each coordinate $t^{(i)}$ of $t = (t^{(1)}, \ldots, t^{(d)})$, $A_{(t^{(1)}+\delta, \ldots t^{(i-1)}+\delta, t^{(i)}, t^{(i+1)}+\delta, \ldots, t^{(d)}+\delta)} \subseteq B \ \forall \delta \leq \varepsilon$.

The set of exposed points of B is denoted by $\varepsilon(B)$. It is an incomparable set.

The sets for which we will be identifying exposed points will be of the form $B = \overline{(\cup_{t \in S} E_t)^c} = \cap_{t \in S} D_t$, where S is a finite or countable subset of T. In this case, the exposed points of B are precisely $\min(S)$.

DEFINITION 2.3. Let N be a point process. Denote:

- $\xi_n := \{t \in T : N_{t-} = N_{(0,t)} < n\}, \quad n = 1, 2, \ldots$
- $\varepsilon(\xi_n)$ is the countable incomparable set of exposed points of ξ_n. The points in $\varepsilon(\xi_n)$ will be denoted by $\{\tau_j^{(n)}, j = 1, \ldots, \}$; the numbering $\tau_1^{(n)}, \tau_2^{(n)} \ldots$ may be defined arbitrarily. The points $\tau_j^{(n)}, j = 1, \ldots,$ are called the exposed points of N on the random set ξ_n.
- $\xi_n^+ := \overline{\left(\cup_{k \neq j}\left(E_{\tau_k^{(n)}} \cap E_{\tau_j^{(n)}}\right)\right)^c} = \cap_{k \neq j}\left(D_{\tau_k^{(n)}} \cup D_{\tau_j^{(n)}}\right)$. If ξ_n has only one exposed point, then $\xi_n^+ = T$.
- $\varepsilon_N := \cup_{i=1}^{\infty} \varepsilon(\xi_i) = \{\tau_j^{(i)}, \quad i = 1, \ldots; j = 1, \ldots\}$. This is the set of all exposed points of N.
- $\Delta_N := \{\tau : N_\tau = 1\}$. This is the set of jump points of the process N.

Comments 2.4.

1. As illustrated in figure 2(a), we note that $\xi_n = \cap_i D_{\tau_i^{(n)}}$. Thus, the random sets ξ_n are determined by $\varepsilon(\xi_n)$ and vice versa. Likewise, ξ_n^+ is determined by $\varepsilon(\xi_n)$, and the set $\xi_n^+ \backslash \xi_n$ is the disjoint union of the sets $(E_{\tau_j^{(n)}})^\circ \cap \xi_n^+, \quad j = 1, 2, \ldots$ (figure 2(b)).
2. We have that $N_t = \sum_{\tau \in \Delta_N} I(\tau \in A_t) = \sum_{\tau \in \Delta_N} I(\tau \leq t)$. The process N is characterized not only by Δ_N, but also by the increasing sequence of random sets (ξ_n), since $N_t \geq n$ if and only if $t \in \overline{\xi_n^c}$. In fact, we have

$$
\begin{aligned}
\overline{\xi_1^c} &= \cup_{\tau \in \Delta_N} E_\tau \\
\overline{\xi_2^c} &= \cup_{\tau_1, \tau_2 \in \Delta_N, \tau_1 \neq \tau_2} (E_{\tau_1} \cap E_{\tau_2}) \\
&\vdots \\
\overline{\xi_n^c} &= \cup_{\tau_1, \ldots \tau_n \in \Delta_N, \tau_1 \neq \cdots \neq \tau_n} (E_{\tau_1} \cap \ldots \cap E_{\tau_n}) \\
&\vdots
\end{aligned}
$$

   It should also be observed that for any $t \in T$, $\xi_n \cap A_t$ is a finite union of sets in $\mathcal{A}$.
3. N is also characterized by the sequence $(\partial \xi_n)$. These are called "lines of separation" in Ref. [23]. See Ref. [23], for a detailed analysis of the different random lines associated with a general point process.
4. It can be observed that $\varepsilon(\xi_i) = \min(\partial \xi_i)$.
5. In general, $\Delta_N \subseteq \varepsilon_N$. If $\tau \in \Delta_N$ and $N_\tau = i$, then $\tau = \tau_j^{(i)}$ for some j. Conversely, while all of the exposed points of ξ_1 are in Δ_N (in fact, $\varepsilon(\xi_1) = \min(\Delta_N)$), if $i > 1$, $\tau_j^{(i)}$ is not necessarily a jump point of N (see figure 2(a)—the jump points are indicated with $\bullet$, and other exposed points with $\circ$). However, $\tau_j^{(i)}$ will always be the supremum of one or more jump points. Therefore, the number of exposed points of ξ_i in any compact set will always be finite.

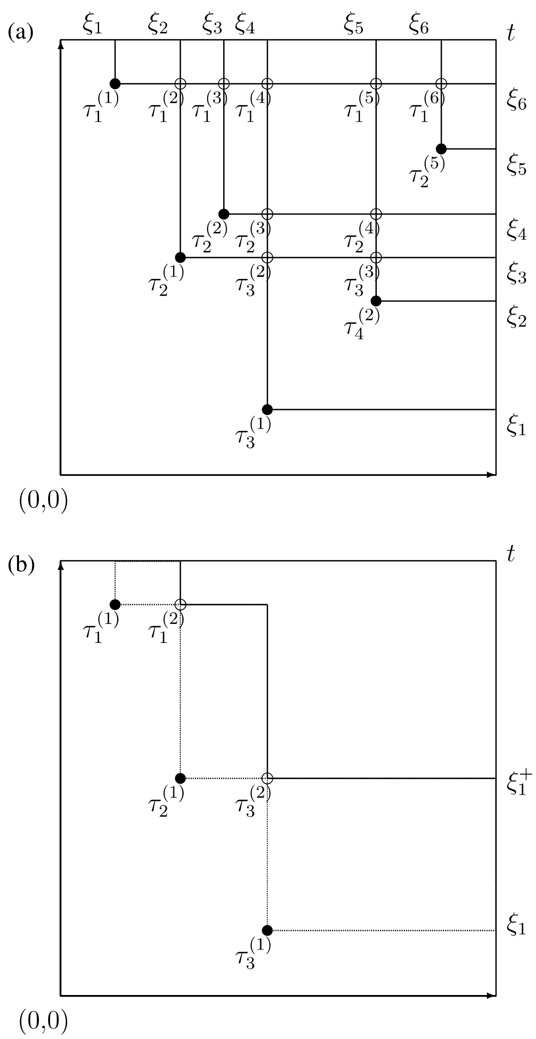

Figure 2. Random sets and points associated with a realization of *N*: (a) the random sets of *N* and their exposed points; (b) the random set $\xi_1^+ \backslash \xi_1$.

6. The exposed points of ξ_i are all incomparable, by definition. However, if $i < h$ then for every j there exists k such that $\tau_j^{(i)} \leq \tau_k^{(h)}$ and it is straightforward to see that $\xi_i \subseteq \xi_{i+1} \subseteq \xi_i^+, \ \forall i$. As well, the exposed points of ξ_{i+1} include the exposed points of ξ_i^+. This is illustrated in figure 2(b).

The simplest point process on $\mathbf{R}_+$ is the *single jump process*; that is, a process of the form $N_t = I(t \geq \tau)$ for some non-negative random variable τ. Clearly, one can define an analogue in $\mathbf{R}_+^d$ for τ a T-valued random variable. However, a similar but more general structure is that of the *single line process*:

Definition 2.5. A point process M is called a single line process if Δ_M is an incomparable set, which is the case if and only if $\Delta_M = \varepsilon(\xi_1)$. We say that M is the single line process associated with the random set ξ_1.

In figure 3 we illustrate the single line process M associated with ξ_1 (refer to figures 1 and 2). The jump points are given in figure 3(a). In figure 3(b) we observe that in general a single line process takes more values than 0 and 1: it takes all the values between 0 and $|\Delta_M|$ (the number of exposed points of ξ_1).

It is clear that the superposition of single line processes is not a single line process in general. However, we do have the following:

LEMMA 2.6. Let $M_1, M_2, \dots$ be single line processes associated with the random sets $\xi_1^{(1)}, \xi_1^{(2)}, \dots$, respectively. If the exposed points of $\xi_1^{(1)}, \xi_1^{(2)}, \dots$ (equivalently, the jump points of $M_1, M_2, \dots$) are almost surely incomparable, then $M = \sum_{i=1}^{\infty} M_i$ is a single line process associated with the random set $\xi_1 = \cap_{i=1}^{\infty} \xi_1^{(i)}$.

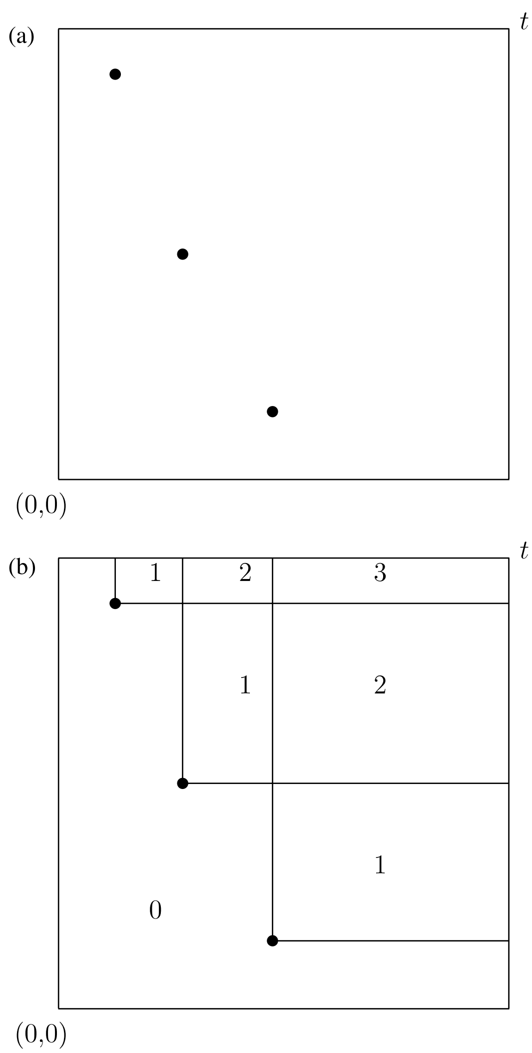

Figure 3. A single line process M and its values: (a) jumb points of M; (b) values of M_s for $s \in [0, t]$.

Proof. This is a straightforward consequence of the observation that the set of exposed points of ξ_1 satisfies $\varepsilon(\xi_1) = \cup_{i=1}^{\infty}\varepsilon(\xi_1^{(i)})$. □

The next proposition will be essential to understanding the definition of the structure of a renewal process. It is motivated by the trivial observation that any point process on the line can be expressed as a sum of single jump processes. More precisely, if a point process on $\mathbf{R}_+$ is generated by the jump times $(\tau_1, \tau_2, \dots)$, then it may be written as the sum of single jump processes of the form $M_n(t) = I(t \geq \tau_n) = 1 - I_{(0,\tau_n)}(t)$. The same idea will be used for point processes on $\mathbf{R}_+^d$, replacing the random interval $(0, \tau_n)$ by a random set.

PROPOSITION 2.7. Any (strictly simple) point process N can be decomposed into the sum of single line processes $M_1, M_2, \dots$:

$$N_t = \sum_{i=1}^{\infty} M_i(t).$$

Proof. Let N be a point process and $\{\xi_n\}$ its associated random sets as defined in Definition 2.3. Let M_1 be the point process consisting of the points of $\varepsilon(\xi_1) = \min(\Delta_N)$ (i.e. $\Delta_{M_1} = \varepsilon(\xi_1) = \min(\Delta_N)$), and notice that it is a single line process. Denote the incomparable set $\varepsilon(\xi_1)$ simply by ε_1.

Let $\varepsilon_2 = \min\{\Delta_N \cap (\xi_1^+ \backslash \xi_1)\}$ (the *minimum* points of the set of jump points of N contained in $(\xi_1^+ \backslash \xi_1)$). Let $\rho_2 := \cap_{\tau \in \varepsilon_2} D_\tau$, and $\rho_2 = R_+^d$ if $\varepsilon_2 = \emptyset$. Notice that ε_2 is an incomparable random set, and let M_2 be the single line point process associated with ρ_2 (i.e. $\Delta_{M_2} = \varepsilon_2$). Notice that $\xi_2 = \rho_2 \cap \xi_1^+$.

Continuing, given ξ_{i-1}, let $\varepsilon_i = \min\{\Delta_N \cap (\xi_{i-1}^+ \backslash \xi_{i-1})\}$. Then $\rho_i := \cap_{\tau \in \varepsilon_i} D_\tau$, and $\rho_i = R_+^d$ if $\varepsilon_i = \emptyset$. Let M_i be the single line process associated wih ρ_i ($\Delta_{M_i} = \varepsilon_i$), and it follows that $\xi_i = \rho_i \cap \xi_{i-1}^+$.

Finally, we observe that $\cup_n \varepsilon_n = \Delta_N$ and for any $t \in T$,

$$N_t = \sum_{\tau \in \Delta_N} I(\tau \leq t) = \sum_{n=1}^{\infty} \sum_{\tau \in \varepsilon_n} I(\tau \leq t) = \sum_{n=1}^{\infty} M_n(t).$$

□

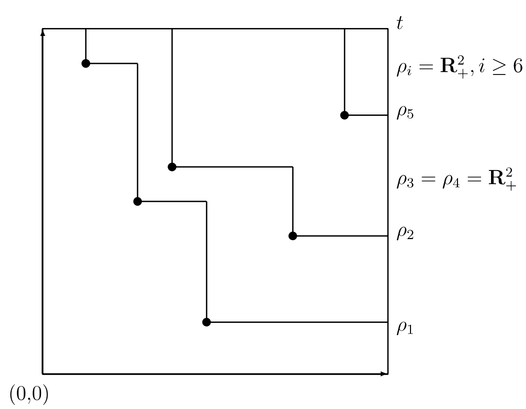

Figure 4. The decomposition of N into single line processes.

The random sets ρ_i defined in the proof above are shown in figure 4 for the realization of N illustrated in figures 1 and 2. In fact, this decomposition is not unique (another such decomposition is given in Ref. [23]), but it is this particular decomposition that is required for renewal processes.

DEFINITION 2.8. A closed set $B \subseteq T$ is called a lower layer if $t \in B \Rightarrow A_t \subseteq B$. The collection of lower layers is denoted by $\mathcal{L}$.

Clearly, for any Borel set B, the sets $A(B)$ and $D(B)$ are lower layers.

Now we introduce the measurability properties of random sets and point processes.

DEFINITION 2.9. $\xi : \Omega \to \mathcal{L}$ is called a random lower layer (rll) if for every $t \in T$, $\{\omega : t \in \xi(\omega)\} \in \mathcal{F}$; i.e. ξ is a measurable mapping from $(\Omega, \mathcal{F})$ into $(\mathcal{L}, \mathcal{F}_{\mathcal{L}})$, where $\mathcal{F}_{\mathcal{L}} = \sigma\{\{D \in \mathcal{L} : t \in D\}, t \in T\}$.

Comments 2.10.

1. This definition is classic (see for example, Refs. [22,25,33]). In fact, $\mathcal{F}_{\mathcal{L}} = \sigma\{\uparrow D : D \in \mathcal{L}\}$, where $\uparrow D = \{D' \in \mathcal{L} : D \subseteq D'\}$. Indeed, $\{D \in \mathcal{L} : t \in D\} = \uparrow A_t$, so $\mathcal{F}_{\mathcal{L}} \subseteq \sigma\{\uparrow D : D \in \mathcal{L}\}$. To show the reverse inclusion, let $T_D = \{t_1, t_2, \dots\}$ be a countable dense subset of D. Then it is easily seen that $\uparrow D = \cap_n \{D' \in \mathcal{L} : t_n \in D'\}$ and so $\sigma\{\uparrow D : D \in \mathcal{L}\} \subseteq \mathcal{F}_{\mathcal{L}}$.
2. For any point process N, the random sets ξ_n are rlls. Indeed,

$$\{t \in \xi_n\} = \cap_i \{N_{s_i} \leq n - 1\} \in \mathcal{F}, \tag{1}$$

where $s_1, s_2, \dots$ is a countable dense subset of $[0, t)$. We also observe that $\{t \in (\xi_n)^\circ\} \in \mathcal{F}$ since

$$\{t \in (\xi_n)^\circ\} = \cup_{r=1}^{\infty} \cap_{m \geq r} \left\{ t + \frac{1}{m} \in \xi_n \right\} \tag{2}$$

where $t + (1/m) = (t^{(1)} + (1/m), \dots, t^{(d)} + (1/m))$. Henceforth, we will refer to $\xi_1, \xi_2, \dots$ as the rll's associated with the point process N.

3. The class of sets of the form $\{D \in \mathcal{L} : t_1, t_2, \dots, t_n \in D\}$ for $t_1, t_2, \dots, t_n \in T$ is a π-system generating $\mathcal{F}_{\mathcal{L}}$, so $P\xi^{-1}$ is determined by

$$P\xi^{-1}\{D : t_1, \dots t_n \in D\} = P\{\omega : t_1, \dots, t_n \in \xi(\omega)\}.$$

In other words, the distribution of a rll is characterized by these probabilities. We can define independence between random sets as usual (cf. [22,32]): two rlls ξ and ξ' are said to be independent if

$$P\{t_1, \dots, t_n \in \xi \cap \xi'\} = P\{t_1, \dots, t_n \in \xi\} P\{t_1, \dots, t_n \in \xi'\}$$

for all $t_1, \dots, t_n \in T$. We observe that since ξ, ξ' are lower layers, it suffices to consider only incomparable points $t_1, \dots, t_n$.
4. The finite dimensional distributions of a point process N determine and are determined by the joint distributions of its associated lower layers (ξ_n). The first assertion follows

immediately from equation (1). On the other hand,

$$P\{N_{t_1} = k_1, \ldots, N_{t_n} = k_n\} = P\{t_1 \in (\xi_{k_1+1})^\circ \backslash (\xi_{k_1})^\circ, \ldots, t_n \in (\xi_{k_n+1})^\circ \backslash (\xi_{k_n})\}$$

and from equation (2) it is clear that $P\{t \in (\xi_{k+1})^\circ \backslash (\xi_k)^\circ\}$ is determined by the joint distribution of ξ_k and ξ_{k+1}.

3. Stopping sets and point processes

In order to develop the dynamical properties of a point process $N = \{N_A, A \in \mathcal{A}\}$, we require a filtration. In general, a filtration indexed by the class $\mathcal{A}$ is a class of σ-fields $\{\mathcal{F}_A : A \in \mathcal{A}\}$ satisfying the following conditions:

- $\mathcal{F}_A \subseteq \mathcal{F}$ and $\mathcal{F}_A$ contains the P-null sets $\forall A \in \mathcal{A}$.
- If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mathcal{F}_A \subseteq \mathcal{F}_B$.
- $\mathcal{F}_{\cap A_i} = \cap \mathcal{F}_{A_i}$ whenever (A_i) is a decreasing sequence in $\mathcal{A}$.

We will generally assume that the filtration is the minimal filtration generated by the process N:

$$\{\mathcal{F}_A, \ A \in \mathcal{A}\} = \left\{\mathcal{F}_A^{\mathrm{N}}, \ A \in \mathcal{A}\right\}$$

where $\mathcal{F}_A^N = \sigma\{N_{A'}, A' \in \mathcal{A}, A' \subseteq A\} \bigvee \{P\text{-null sets}\}$. For B a finite union of sets in $\mathcal{A}$, we can also define $\mathcal{F}_B^N = \sigma\{N_{A'}, A' \in \mathcal{A}, A' \subseteq B\} \bigvee \{P\text{-null sets}\}$. Outer continuity of the filtration follows from the fact that N is a point process (cf. [14]). When no ambiguity arises, the superscript N will be suppressed in the notation.

We now define a stopping set, which is a rll with a specific geometric structure and measurability properties. We refer the reader to Ref. [13] for more details on stopping sets.

DEFINITION 3.1. A rll ξ is a stopping set with respect to a filtration $\{\mathcal{F}_A : \mathcal{A} \in \mathcal{A}\}$ if

- For each $\omega \in \Omega$, $\xi(\omega)$ is a finite union of sets from $\mathcal{A}$.
- $\{\omega : t \in \xi(\omega)\} \in \mathcal{F}_{A_t}, \quad \forall t \in T$.

In [13], the second condition above was replaced by the equivalent statement that $\{\omega : A \subseteq \xi(\omega)\} \in \mathcal{F}_A, \ \forall A \in \mathcal{A}$ (as observed in [13], this definition is analogous to that of Rosanov [28], who called these sets "co-compatible", and is stronger than the definition introduced in Refs. [20] and [33] which requires measurability of the reverse inclusion).

We note here that the rll's ξ_i associated with a point process N are not necessarily finite unions of sets from $\mathcal{A}$ since any point process can have an infinite number of incomparable jump points. We do have the following:

LEMMA 3.2. Let N be a point process and $\{\xi_i\}$ the associated rll's. For every $s \in T$ and all i, $\xi_i \cap A_s$ is a ($\mathcal{F}^N$-) stopping set.

Proof. It is clear that $\xi_i(\omega) \cap A_s$ may be expressed as a finite union of sets from $\mathcal{A}$. Next, for $t \in A_s$, if $(t_1, t_2, \ldots)$ is a countable dense subset of $(0, t) = A_t^\circ$, then

$$\{\omega : t \in \xi_i(\omega) \cap A_s\} = \cap_j \{\omega : N_{t_j}(\omega) \leq i - 1\} \in \mathcal{F}_{A_t}.$$

Thus, $\xi_i \cap A_s$ is a stopping set. □

There are some immediate consequences of the preceding lemma. First, it is trivial that $\{t \in \xi_i\} \in \mathcal{F}_{A_t}$, $\forall t$. Next, referring to equation (2), by outer continuity of the filtration we have that $\{t \in (\xi_i)^\circ\} \in \mathcal{F}_{A_t}$, $\forall t$.

It is important to point out that if ξ is a $\mathcal{F}$-stopping set, then $\{\xi \subseteq B\} \in \mathcal{F}_B$, $\forall B \in \mathcal{A}(u)$ where $\mathcal{A}(u)$ denotes the class of finite unions of sets from $\mathcal{A}$ ([13], Lemma 1.5.8). The following definition now makes sense.

Definition 3.3. (cf. [13]) Let ξ be a stopping set with respect to the filtration $\mathcal{F}$.

- $\mathcal{F}_\xi$ denotes the σ-algebra generated by the stopping set ξ:

  $$\mathcal{F}_\xi = \{F \in \mathcal{F} : F \cap \{\xi \subseteq B\} \in \mathcal{F}_B, \quad \forall B \in \mathcal{A}(u)\}.$$

  If ξ takes on at most countably many values, then $\{\xi \subseteq B\}$ may be replaced by $\{\xi = B\}$ in the preceding statement.
- Let X be a measurable mapping from Ω to some measure space $(S, \mathcal{S})$. The random element X is said to be $\mathcal{F}_\xi$-measurable if

  $$\{X \in B\} \in \mathcal{F}_\xi, \ \forall B \in \mathcal{S}.$$

  The following proposition will be required in the sequel.

Proposition 3.4. Let ξ be a stopping set with respect to an arbitrary filtration $\mathcal{F}$ and let $\tau_1, \tau_2, \ldots$ be the exposed points of $\xi \cap A$, $A \in \mathcal{A}$, where $\tau_1, \tau_2, \ldots$ are numbered lexicographically (τ_m is defined to be ∞, $\forall m > n$ when $\tau_1, \ldots, \tau_n$ exhaust $\varepsilon(\xi \cap A)$). Then for every i, τ_i is $\mathcal{F}_{\xi \cap A}$-measurable.

Proof. We introduce the following notation: for any $s \in \mathbf{R}_+$, $g_n(s)$ is the smallest dyadic of order n strictly larger than s. For $t = (t^{(1)}, \ldots, t^{(d)}) \in \mathbf{R}_+^d$, $g_n(t) = (g_n(t^{(1)}), \ldots, g_n(t^{(d)}))$. For $A = A_t = [0, t] \in \mathcal{A}$, $g_n(A_t) = A_{g_n(t)} = [0, g_n(t)]$. For $B = \cup_{i=1}^k A_{t_i}$, $g_n(B) = \cup_{i=1}^k g_n(A_{t_i}) = \cup_{i=1}^k A_{g_n(t_i)}$.

Now consider the sequence $(g_n(\xi \cap A))$. This is a decreasing sequence of stopping sets ([13], Lemma 1.5.6), and $\xi \cap A = \cap_n g_n(\xi \cap A)$. We shall show that for any $t \in T$ and $B \in \mathcal{A}(u)$,

$$\{\tau_i \leq t\} \cap \{\xi \cap A \subseteq B\} \in \mathcal{F}_{g_r^2(B)}, \quad \forall r$$

where $g_r^2(B) = g_r(g_r(B))$. As noted previously, the filtration is outer-continuous since it is generated by a point process, and so it follows that

$$\{\tau_i \leq t\} \cap \{\xi \cap A \subseteq B\} \in \mathcal{F}_B,$$

which implies that τ_i is $\mathcal{F}_{\xi \cap A}$-measurable.

Denote the exposed points of $g_n(\xi \cap A)$ numbered lexicographically by $\tau_1^n, \tau_2^n, \ldots$. Given $n(\omega)$ large enough, there is a one-to-one correspondence between the exposed points of $\xi(\omega) \cap A$ and those of $g_n(\xi(\omega) \cap A)$, and so it is clear that $\tau_i = \lim_{n\to\infty} \tau_i^n$, $\forall i$. Therefore, for every r we have

$$\{\tau_i \leq t\} \cap \{\xi \cap A \subseteq B\} = \cup_{m\geq r} \cap_{n\geq m} \left\{\tau_i^n \leq g_n(t)\right\} \cap \{g_n(\xi \cap A) \subseteq g_n(B)\}. \tag{3}$$

We have now reduced the problem to the following: let $\xi \subseteq A$ be a stopping set which takes its values in $\mathcal{A}_n(u)$, the class of finite unions of sets of the form A_s, where $s = (s^{(1)}, \ldots, s^{(d)})$ and $s^{(i)}$ is a dyadic of order n, $\forall i$. Let $\tau_1, \tau_2, \ldots$ be the exposed points of ξ numbered lexicographically. It will be shown that for every i,

$$\{\tau_i = t\} \cap \{\xi \subseteq B\} \in \mathcal{F}_{g_n(B)}, \tag{4}$$

$\forall B \in \mathcal{A}_n(u)$ and $\forall t = (t^{(1)}, \ldots, t^{(d)})$ with $t^{(j)}$ a dyadic of order n, $j = 1, \ldots, d$. Substituting equation (4) in equation (3) (noting that B in equation (4) is replaced by $g_n(B)$), we see that the event in equation (3) is $\mathcal{F}_{g_r^2(B)}$-measurable.

It remains to prove equation (4). To avoid trivialities, assume $t \in B$ and that $B \subseteq A$. The points in B with all coordinates dyadics of order n will be denoted by B_n. Strict inequality in the lexicographic order will be denoted by "$\prec$". We begin with τ_1:

$$\{\tau_1 = t\} \cap \{\xi \subseteq B\} = \left(\cap_{s\in B_n, s\prec t} \mathcal{O}_s\right) \cap \mathcal{U}_t \cap \{\xi \subseteq B\},$$

where (letting $t + (1/2^{n+1}) = (t^{(1)} + (1/2^{n+1}), \ldots, t^{(d)} + (1/2^{n+1}))$,

$$\mathcal{O}_s = \left\{s + \frac{1}{2^{n+1}} \in \xi\right\} \cup \left(\{s \in \xi\} \cap \left[\cup_{i=1}^d \left\{\left(s^{(1)} + \frac{1}{2^{n+1}}, \ldots, s^{(i-1)} + \frac{1}{2^{n+1}}, s^{(i)}, s^{(i+1)} + \frac{1}{2^{n+1}}, \ldots, s^{(d)} + \frac{1}{2^{n+1}}\right) \notin \xi\right\}\right]\right)$$

and

$$\mathcal{U}_t = \{t \in \xi\} \cap \left\{t + \frac{1}{2^{n+1}} \notin \xi\right\} \cap \left[\cap_{i=1}^d \left\{\left(t^{(1)} + \frac{1}{2^{n+1}}, \ldots, t^{(i-1)} + \frac{1}{2^{n+1}}, t^{(i)}, t^{(i+1)} + \frac{1}{2^{n+1}}, \ldots, t^{(d)} + \frac{1}{2^{n+1}}\right) \in \xi\right\}\right].$$

Since both $s, t \in B$, we have $\mathcal{O}_s, \mathcal{U}_t \in \mathcal{F}_{g_n(B)}$ and equation (4) is satisfied for $i = 1$,

Now, we assume that equation (4) has been satisfied for $i - 1$. Defining $\mathcal{O}_s, \mathcal{U}_t$ in exactly the same way, we have

$$\{\tau_i = t\} \cap \{\xi \subseteq B\} = \cup_{v_1\prec\cdots\prec v_{i-1}\prec t\in B_n} \cap_{j=1}^{i=1} \{\tau_j = v_j\} \cap \left(\cap_{\substack{s\in B_n, v_{i-1}\prec s\prec t \\ s\in\cap_{j=1}^{i-1} E_{v_j}^c}} \mathcal{O}_s\right) \cap \mathcal{U}_t \cap \{\xi \subseteq B\},$$

and equation (4) is satisfied for i. This completes the proof. □

In the corollaries below, we restrict our attention to bounded sets in order to ensure that we are able to number the jump points of a point process lexicographically.

Corollary 3.5. Let ξ_i be a rll associated with the point process N. Then for any $A \in \mathcal{A}$, $\xi_i^+ \cap A$ is $\mathcal{F}_{\xi_i \cap A}$-measurable.

Proof. We must show that for any $t \in A$, $\{t \in \xi_i^+ \cap A\} \cap \{\xi_i \cap A \subseteq D\} \in \mathcal{F}_D\, \forall D \in \mathcal{A}(u)$. Trivially, $\{t \in \xi_i \cap A\} \in \mathcal{F}_{\xi_i \cap A}$, so it is enough to consider $\{t \in (\xi_i^+ \backslash \xi_i) \cap A\}$. Let $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ be the exposed points of $\xi_i \cap A$ numbered lexicographically and to avoid trivialities, assume that $t \in A$. We have

$$\left(\xi_i^+ \backslash \xi_i\right) \cap A = \left[\left(\overline{\left(\cup_{k \neq j}\left(E_{\tau_k^{(i)}} \cap E_{\tau_j^{(i)}}\right)\right)^c}\right) \Big\backslash \left(\cap_h D_{\tau_h^{(i)}}\right)\right] \cap A$$

and so $t \in (\xi_i^+ \backslash \xi_i) \cap A$ if and only if $\tau_h^{(i)} \in A_t^\circ$ for some h and $\tau_k^{(i)} \in (A_t^\circ)^c$, $\forall k \neq h$. Therefore, still assuming that $t \in A$,

$$\begin{aligned}\{t \in (\xi_i^+ \backslash \xi_i) \cap A\} \cap \{\xi_i \cap A \subseteq D\} = \cup_h \big(\big\{\tau_h^{(i)} \in A_t^\circ\big\} \cap \cap_{k \neq h} \big\{\tau_k^{(i)} \in (A_t^\circ)^c\big\}\big) \\ \cap \{\xi_i \cap A \subseteq D\} \in \mathcal{F}_D,\end{aligned}$$

by the preceding proposition. □

Corollary 3.6. Let $N = \sum_{j=1}^\infty M_j$ be the decomposition of the point process N into the sum of single line point processes, as defined in Proposition 2.7. Let (ξ_i) be the rll's associated with the point process N and for $j \geq 1$, let ρ_j be the random set defining the single line process M_j. Then for all $A \in \mathcal{A}$ and $i, j \geq 1$, $\xi_i^+ \cap A$ and $\rho_j \cap A$ are stopping sets.

Proof. Fix $A \in \mathcal{A}$ and to avoid trivialities, assume $t \in A$. By definition, $t \in \xi_i^+$ if and only if there is at most one exposed point of ξ_i in A_t. Let $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ be the exposed points of $\xi_i \cap A_t$ numbered lexicographically. It follows from Proposition 3.4 that

$$\left\{t \in \xi_i^+ \cap A\right\} = \cap_{h=2}^\infty \left\{\tau_h^{(i)} = \infty\right\} \in \mathcal{F}_{\xi_i \cap A_t} \subseteq \mathcal{F}_{A_t}, \tag{5}$$

and so $\xi_i^+ \cap A$ is a stopping set.

Next, we note that $\rho_1 = \xi_1$ and so we assume that $j > 1$. By definition, $t \in \rho_j$ if and only if $t \in \xi_{j-1}^+$ and $N(A_s \backslash \xi_{j-1}) = 0$, $\forall s \in A_t^\circ$. Therefore, if $(s_1, s_2, \ldots)$ is a dense subset of $[0, t)$,

$$\{t \in \rho_j \cap A\} = \left\{t \in \xi_{j-1}^+ \cap A\right\} \cap \cap_h \{N(A_{s_h} \backslash \xi_{j-1}) = 0\}.$$

By equation (5) the first term on the right is in $\mathcal{F}_{A_t}$. Since

$$N\left(A_{s_h} \backslash \xi_{j-1}\right) = N\left(A_{s_h}\right) - N\left(A_{s_h} \cap \xi_{j-1}\right)$$

and $N(A_{s_h} \cap \xi_{j-1})$ is $\mathcal{F}_{A_{s_h} \cap \xi_{j-1}}$-measurable (cf. [13], Lemma 1.5.9), the second term is in $\mathcal{F}_{A_t}$ as well. Thus, $\rho_j \cap A$ is a stopping set. □

4. Multi-parameter renewal processes

We now come to the definition of a multi-parameter renewal process. As alluded to in the introduction, our definition will ensure that the law of ξ_{i+1} depends only on the configuration (in particular, the exposed points) of ξ_i, and not on i.

What is the idea behind the renewal property? First, we introduce the following notation: for an arbitrary Borel set B and $t \in T$,

$$B \oplus t = \{x + t : x \in B\}, \quad B \ominus t = \{x - t : x \in B\}.$$

Consider a point process N and its associated rll's $\{\xi_i\}$. From Comment 2.4 we have that ξ_i^+ is determined by ξ_i, and that $\xi_i^+ \setminus \xi_i$ is the union of the disjoint incomparable sets $(E_{\tau_j^{(i)}})^\circ \cap \xi_i^+$, $j = 1, 2, \dots$ (and note that $E_{\tau_j^{(i)}} \ominus \tau_j^{(i)} = T$). Also, ξ_i^+ is an upper bound for ξ_{i+1}: $\xi_i \subseteq \xi_{i+1} \subseteq \xi_i^+ (\xi_i \subset \xi_{i+1}$ if $\xi_i \neq T)$. Next, we recall the single line processes M_i associated with the rlls ρ_i defined in the proof of Proposition 2.7: the jump points of M_1 are the minimal jump points of N, and the jump points of M_{i+1} are the minimal jump points of N in the set $\xi_i^+ \setminus \xi_i$. As noted previously,

$$\xi_{i+1} = \xi_i^+ \cap \rho_{i+1} = \xi_i \cup \left(\cup_j \left(E_{\tau_j^{(i)}} \right)^\circ \cap \xi_i^+ \cap \rho_{i+1} \right), \tag{6}$$

and the sets in the union on the right hand side of equation (6) are disjoint. In light of these observations, the renewal property will be consistent with certain principles:

- The "renewal times" generated by ξ_i will be the exposed points $\varepsilon(\xi_i) = \{\tau_j^{(i)}, j = 1, 2, \dots\}$.
- Given ξ_i, the process N should behave independently on each of the disjoint sets $(E_{\tau_j^{(i)}} \cap \xi_i^+)^\circ$, $j = 1, 2, \dots$.
- Given ξ_i, the law of the single line process generated by the minimal points of N on $(E_{\tau_j^{(i)}} \cap \xi_i^+)^\circ$ should be the same as the law generated by the points of the single line process M_1, each translated by $\tau_j^{(i)}$, which lie in $(\xi_i^+)^\circ$.

These properties are consistent with the renewal property on $\mathbf{R}_+$: after each renewal time, the next jump time is an independent copy of the first jump time, suitably translated. In terms of random sets, this property can be expressed as follows: given $\mathcal{F}_{\tau^{(i)}}$,

$$(\tau^{(i)}, \tau^{(i+1)}] =_{\mathcal{D}} (\tau^{(i)}, \tau^{(i)} + \tau],$$

where τ is independent of $\mathcal{F}_{\tau^{(i)}}$ and $\tau =_{\mathcal{D}} \tau^{(1)}$ ("$=_{\mathcal{D}}$" denotes equality in distribution). Equivalently, in terms of sets and remembering that on $\mathbf{R}_+$, $\xi_1 = [0, \tau^{(1)}]$, given $\mathcal{F}_{\xi_i} (= \mathcal{F}_{\tau^{(i)}})$

$$\left(E_{\tau^{(i)}} \cap \xi_{i+1} \right) \ominus \tau^{(i)} =_{\mathcal{D}} \xi,$$

where ξ is independent of $\mathcal{F}_{\xi_i}$ and $\xi =_{\mathcal{D}} \xi_1$.

In higher dimensions we replace the concept of the first jump time $\tau^{(1)}$ with the first rll ξ_1 of N. The renewal times associated with ξ_i are its exposed points and each of these renewals is on a disjoint set bounded above by ξ_i^+, since $\xi_{i+1} \subseteq \xi_i^+$, by definition. This bound is not an issue on the line since in the case that ξ_i has a single exposed point, $\xi_i^+ = T$.

We now formally define the renewal property. First, although the rll's ξ_i associated with a point process N are not necessarily stopping sets, we may define $\mathcal{F}_{\xi_i} = \vee_{t \in T} \mathcal{F}_{\xi_i \cap A_t}$.

Definition 4.1. Let N be a (strictly simple) point process on $\mathbf{R}_+^d$ with associated rll's $\{\xi_i . N\}$ is a renewal point process if for every $i \geq 1$ and every $A \in \mathcal{A}$ and $\tau_1^{(i)}, \tau_2^{(i)}, \dots$ the exposed points of $\xi_i \cap A$ numbered lexicographically,

1. given $\mathcal{F}_{\xi_i}$, the rll's $(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A) \ominus \tau_j^{(i)}$ are independent, $\forall j$, and
2. given $\mathcal{F}_{\xi_i}$,

$$\left(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A\right) \ominus \tau_j^{(i)} =_{\mathcal{D}} \xi \cap \left[\left(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\right) \ominus \tau_j^{(i)}\right], \quad \forall j \tag{7}$$

where "$=_{\mathcal{D}}$" denotes equality in distribution (in this instance, of rlls), and ξ is an independent copy of ξ_1 also independent of $\mathcal{F}_{\xi_i}$.

In order to clarify the above definition, we observe that by Proposition 3.4

$$\begin{aligned} &P\Big(t_1, \dots, t_n \in \Big(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A\Big) \ominus \tau_j^{(i)} | \mathcal{F}_{\xi_i}\Big) \\ &= P\Big(\Big\{t_1, \dots, t_n \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big\} \cap \Big\{N\Big(\cup_{h=1}^n \Big(\tau_j^{(i)}, t_h + \tau_j^{(i)}\Big)\Big) = 0\Big\} | \mathcal{F}_{\xi_i}\Big) \\ &= I\Big(t_1, \dots, t_n \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P\Big(N\Big(\cup_{h=1}^n \Big(\tau_j^{(i)}, t_h + \tau_j^{(i)}\Big)\Big) = 0 | \mathcal{F}_{\xi_i}\Big) \end{aligned} \tag{8}$$

and if ξ is independent of $\mathcal{F}_{\xi_i}$,

$$\begin{aligned} &P\Big(t_1, \dots, t_n \in \xi \cap \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big] | \mathcal{F}_{\xi_i}\Big) \\ &= I\Big(t_1, \dots, t_n \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P(t_1, \dots, t_n \in \xi) \\ &= I\Big(t_1, \dots, t_n \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P\big(N\big(\cup_{h=1}^n (0, t_h)\big) = 0\big). \end{aligned} \tag{9}$$

Therefore, condition 2 above may be restated equivalently as:

2′. For $i, j \geq 1$, and $t_1, \dots, t_n \in T$,

$$\begin{aligned} &I\Big(t_1, \dots, t_n \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P\Big(N\Big(\cup_{h=1}^n \Big(\tau_j^{(i)}, t_h + \tau_j^{(i)}\Big)\Big) = 0 | \mathcal{F}_{\xi_i}\Big) \\ &= I\Big(t_1, \dots, t_n \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P\Big(N\Big(\cup_{h=1}^n \Big(0, \tau_j^{(i)}\Big)\Big) = 0\Big) \end{aligned} \tag{10}$$

where $\tau_1^{(i)}, \tau_2^{(i)}, \dots$ are the exposed points of $\xi_i \cap A$, numbered lexicographically.

Trivially, a renewal process on $\mathbf{R}_+$ may be constructed given the law of $\tau^{(1)}$ (equivalently, ξ_1) by simply taking a sequence $\tau_1, \tau_2, \dots$ of i.i.d. copies of $\tau^{(1)}$. Let $\tau^{(1)} = \tau_1$ and for $i \geq 1$, given $\tau^{(i)}$ define $\tau^{(i+1)} = \tau^{(i)} + \tau_{i+1}$. To motivate the construction of a renewal process on $\mathbf{R}_+^d$, we express this in terms of sets. If $\xi^j = [0, \tau_j]$, then $\xi_1 = \xi^1$ and for $i \geq 1$,

$$\xi_{i+1} = (\xi^{i+1} \oplus \tau^{(i)}) \cup \xi_i = (\xi^{i+1} \oplus \tau^{(i)}) \cup D_{\tau^{(i)}}.$$

We can extend this construction to renewal processes on $\mathbf{R}_+^d$.

THEOREM 4.2. Given the law of any rll ξ associated with a single line process on $\mathbf{R}_+^d$, there exists a renewal point process N with $\xi_1 =_{\mathcal{D}} \xi$.

Proof. It is enough to define the single line processes $M_1, M_2, \dots$ associated with N on an arbitrary set $A = A_t \in \mathcal{A}$. Let $\xi_j^i, i, j \geq 1$ be i.i.d. copies of ξ and N_j^i the associated single line processes. Let $M_1 = N_1^1$ be the single line point process associated with $\rho_1 = \xi_1 := \xi_1^1$.

We proceed iteratively: given ξ_i, let $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ be the exposed points of $\xi_i \cap A$, numbered lexicographically. The jump points of M_{i+1} on $(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A)$ will be defined to be the jump points of N_j^{i+1} falling in the set $(E_{\tau_j^{(i)}} \cap \xi_i^+)^\circ \cap A$ after translation by $\tau_j^{(i)}$. Accordingly, we define M_j^{i+1} to be the single line process with jump points

$$\Delta_{M_j^{i+1}} = \left(\Delta_{N_j^{i+1}} \oplus \tau_j^{(i)}\right) \cap \left(E_{\tau_j^{(i)}} \cap \xi_i^+\right)^\circ \cap A,$$

and in terms of the associated rll's,

$$\rho_j^{i+1} \cap A = \cap_{\tau \in \Delta_{M_j^{i+1}}} D_\tau,$$

and $\rho_j^{i+1} = \mathbf{R}_+^d$ if $\Delta_{M_j^{i+1}} = \emptyset$. Since $\tau_j^{(i)}$ and $\xi_i^+ \cap A$ are both $\mathcal{F}_{\xi_i \cap A}$-measurable, the rll's $\rho_1^{i+1} \cap A, \rho_2^{i+1} \cap A, \ldots$ (and hence,the associated point processes $M_1^{i+1}, M_2^{i+1}, \ldots$) are conditionally independent given $\mathcal{F}_{\xi_i}$. By definition, the jump points of $M_1^{i+1}, M_2^{i+1}, \ldots$ fall on disjoint incomparable sets, and so the rll

$$\rho_{i+1} \cap A := \cap_j \rho_j^{i+1} \cap A$$

generates the single line process $M_{i+1} = \sum_j M_j^{i+1}$. Let $\xi_{i+1} \cap A = \xi_i^+ \cap \rho_{i+1} \cap A$. Now, by definition

$$\left(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A\right) \ominus \tau_j^{(i)} = \left(\rho_j^{i+1} \cap A \cap E_{\tau_j^{(i)}}\right) \ominus \tau_j^{(i)} \tag{11}$$

$$= \xi_j^{i+1} \cap \left[\left(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\right) \ominus \tau_j^{(i)}\right]. \tag{12}$$

Equations (11) and (12), respectively, show that conditions 1 and 2 of Definition 4.1 are satisfied. Therefore, the point process N with associated rll's $\xi_n := \cup_{A \in \mathcal{A}} (\xi_n \cap A)$ is a renewal point process with $\xi_1 =_{\mathcal{D}} \xi$. □

Comment. By restricting the jump points at each step in the construction above to the open sets $(E_{\tau_j^{(i)}} \cap \xi_i^+)^\circ$, we ensure that the resulting point process is strictly simple. For example, consider the deterministic single line process in $\mathbf{R}_+^2$ with jump points at $(1, 2)$ and $(2, 1)$. Under the construction above, the associated renewal process has jump points at $(1 + 2n, 2 + 2n)$ and $(2 + 2n, 1 + 2n)$, $n = 0, 1, 2, \ldots$. If instead we did the construction on the closed sets $E_{\tau_j^{(i)}} \cap \xi_i^+$, the resulting process would no longer be strictly simple, and, therefore, not renewal.

5. Partial sum point processes

As discussed in the introduction, the partial sum point process is the simplest example of a renewal process and there are many important applications of this structure. It is based on i.i.d. copies of a *single jump process*. We note here that single jump processes indexed by a general partially ordered set have already been studied extensively; see [13] and the references therein for a complete discussion, including a construction of the related compensator.

As mentioned previously, all preceding work on multi-dimensional renewal processes has been based on the partial sum process. Another kind of partial sum process is defined with

Minkowski sums of random sets as in [6], but as this is not a point process it will not be discussed further.

Let τ be a random variable taking its values in T, and let F be its distribution function:

$$F(t) = P(\tau \le t) = P(\tau \in A_t), \quad \forall t \in T.$$

We suppose that F is absolutely continuous and so with probability one, τ satisfies $0 \neq \tau^{(i)}, i = 1, \ldots, d$.

Let $\tau_1, \tau_2, \tau_3, \ldots$ be independent copies of τ and for $k \ge 1$, denote

$$\tau^{(k)} = \sum_{i=1}^{k} \tau_i, \quad D_k = D_{\tau^{(k)}}.$$

The random variables $(\tau^{(k)})$ are totally ordered and define the jump points of a strictly simple point process, the sequence of random sets (D_k) is strictly increasing, and the process defined by $N_t = \sum_{i=1}^{\infty} I_{\{\tau^{(i)} \in A_t\}} = \sum_{i=1}^{\infty} I_{\{t \notin D_i^\circ\}}$, $t \in T$, is clearly a renewal point process with associated rll's $\xi_k = D_k$. Note that this expression illustrates the decomposition of a point process into the sum of single line processes: in this case, the single line processes are defined by $M_i(t) := I_{\{\tau^{(i)} \in A_t\}}$, $i = 1, 2, \ldots$.

Denote by F^{*k} the k-fold convolution of $F : F^{*1} = F$ and

$$F^{*k}(t) = \int_T F^{*(k-1)}(t-u) F(du), \quad t, u \in T, \ \ k \ge 2.$$

F^{*k} is the distribution function of $\tau^{(k)}$ and it defines a measure on the Borel sets of T. For any integer n and $k \ge 1$, define:

$$F_k(t_1, \ldots, t_n) = P(t_1, \ldots, t_n \in \xi_k), \quad t_1, \ldots, t_n \in T.$$

Using the fact that F is absolutely continuous, we observe that

$$P(t_1, \ldots, t_n \in \xi_k) = P\Big(\tau^{(k)} \in \Big(\cup_{i=1}^n A_{t_i}^\circ\Big)^c\Big) = F^{*k}\Big(\cap_{i=1}^n \Big(A_{t_i}^\circ\Big)^c\Big) = F^{*k}\Big(\cap_{i=1}^n A_{t_i}^c\Big),$$

and obtain the renewal formula:

PROPOSITION 5.1. If F is absolutely continuous, then $\forall k \ge 1$, $\forall n \ge 1$, $\forall t_1, \ldots, t_n \in T$

$$F_k(t_1, \ldots, t_n) = F^{*k}\Big(\cap_{i=1}^n A_{t_i}^c\Big).$$

Thus, the law of the point process N is completely determined by the convolution distributions $F^{*k}, k = 1, 2, \ldots$.

We will revisit the partial sum renewal process in Section 7, where we will consider its martingale properties.

It is tempting to try to generalize the simple construction of the partial sum process; i.e. to construct a point process as a superposition of suitably translated (by $\tau_j^{(i)}$, $i, j \ge 1$) independent copies of the same single line process M. However, in general it is not possible to construct a renewal process in this way if M can have more than one jump point. To see this, note that if M has a fixed number ℓ of incomparable jump points $\tau_1^{(1)}, \ldots, \tau_\ell^{(1)}$, then if ξ_1 is the rll associated with M, the superposed version of M translated by $\tau_i^{(1)}$ cannot have any points outside of $(E_{\tau_i^{(1)}} \cap \xi_1^+) \ominus \tau_i^{(1)}$, if the resulting point process is to be renewal. However, if $\ell \geq 2$ then $(E_{\tau_i^{(1)}} \cap \xi_1^+) \ominus \tau_i^{(1)} \neq T$, and with positive probability one or more of the points of M will fall outside of some of these regions. Nevertheless, the law of M can be used to construct a renewal process N as in Theorem 4.2. In this case, the number of jump points of N on $\partial\xi_n$ can take on any value between 0 and ℓ^n.

6. Poisson processes

In this section, we consider the renewal property of the Poisson process. A multidimensional analogue of the Waiting Time Paradox (WTP) will be defined, and in analogy to the Poisson process on $\mathbf{R}_+$, we will show that the following conditions (which will be made precise subsequently) are equivalent for (strictly simple) point processes on $T = \mathbf{R}_+^d$:

1. N is a homogeneous Poisson process.
2. N is a stationary point process with independent increments.
3. N satisfies the WTP.
4. N is a Poisson renewal process.
5. N is a renewal point process with independent increments and no fixed atoms.

The definition of the Poisson process on T is classic:

Definition 6.1. A point process N on T is a Poisson process if there exists a boundedly finite nonatomic Borel measure Λ such that for every finite family of disjoint bounded Borel sets $B_1, \ldots, B_n \subset T$ and $k_1, \ldots, k_n \in \mathbf{N}$, we have

$$P(N_{B_i} = k_i, i = 1, \ldots, n) = \prod_{i=1}^{n} \frac{\Lambda(B_i)^{k_i}}{k_i!} e^{-\Lambda(B_i)}.$$

Λ is called the mean measure of N. If Λ is absolutely continuous with density (λ_t) with respect to Lebesgue measure μ, (λ_t) is called the intensity of N. If $\lambda_t \equiv \lambda$ μ-a.s., then N is a *homogeneous* Poisson process.

An arbitrary point process N is *stationary* if the finite dimensional distributions are invariant under translation: i.e. for all $B_1, \ldots, B_n \in \mathcal{B}, n \in \mathbf{N}$ and $t \in T$,

$$P(N_{B_1} = k_1, \ldots, N_{B_n} = k_n) = P(N_{B_1 \oplus t} = k_1, \ldots, N_{B_n \oplus t} = k_n).$$

N has *independent increments* if for all disjoint $B_1, \ldots, B_n \in \mathcal{B}, n \in N$, $N_{B_1}, \ldots, N_{B_n}$ are independent random variables. N has a *fixed atom* at $t \in T$ if $P(N(\{t\}) = 1) > 0$. Note that a point process with independent increments cannot have any fixed atoms if it is stationary.

It is well known that a simple point process without fixed atoms has independent increments if and only if it is a Poisson process and a Poisson process is stationary if and only if it is homogeneous. Thus, conditions 1 and 2 are equivalent.

Next, recall that a homogeneous Poisson process on $\mathbf{R}_+$ is characterized by the *WTP*: for any ($\mathcal{F}$-) stopping time τ and $t > 0$,

$$N(\tau, \tau + t] =_{\mathcal{D}} N(0, t].$$

The following is an analogue of the WTP for point processes on $T = \mathbf{R}_+^d$:

DEFINITION 6.2. A (strictly simple) point process N on $\mathbf{R}_+^d$ satisfies the WTP if

$$P(N_{B\oplus Z} = k|\mathcal{F}_\xi) = P(N_{B\oplus Z} = k) = P(N_B = k) \quad \forall k = 0, 1, 2, \ldots, \tag{13}$$

where ξ is any $\mathcal{F}$-stopping set, Z is any $\mathcal{F}_\xi$-measurable random element in T such that $Z \notin \xi^\circ$ a.s., and B is an arbitrary Borel set containing no points with a coordinate equal to 0 and such that $\mu(\partial B) = 0$ (μ denotes Lebesgue measure).

Heuristically, the WTP states that the law of N on a set B remains unchanged under a (random) translation of B into the "future" of an $\mathcal{F}_N$-stopping set (by assumption, $(B\oplus Z) \cap \xi = \emptyset$). In particular, Z can be one of the exposed points of ξ_n for some n. This can be regarded as a strong form of stationarity and is close to the strong Markov property studied in Refs. [14] and [24]. In fact, we shall see that it would be equivalent to state the WTP in terms of *deterministic* lower layers and points, which is ostensibly a weaker statement. This apparently weaker property suffices to prove that N is a homogeneous Poisson process, which in turn implies equation (13). This is proven in the following theorem, which is of independent interest:

THEOREM 6.3. A simple point process N on $T = \mathbf{R}_+^d$ is a homogeneous Poisson process if and only if it satisfies the WTP (for "if", it suffices that the WTP hold for deterministic sets and points).

Proof.
Only if: Assume that N is a homogeneous Poisson process with intensity $\lambda_t \equiv \lambda$. Using the notation introduced in the proof of Proposition 3.4, it is easily seen that

$$N_{B\oplus g_n(Z)} \longrightarrow_n N_{B\oplus Z} \text{ a.s.}$$

since $(B \oplus Z)^\circ \subseteq \liminf_n(B \oplus g_n(Z)(\subseteq \limsup_n(B \oplus g_n(Z)) \subseteq \overline{(B \oplus Z)}$, and $E[N(\partial(B \oplus Z))] = \lambda\mu(\partial(B \oplus Z)) = \lambda\mu(\partial B) = 0$. Next, note that $g_n(Z)$ is $\mathcal{F}_\xi$-, and hence $\mathcal{F}_{g_m(\xi)}$-measurable $\forall m$. Also, $g_n(Z) \notin (g_n(\xi))^\circ$, so $(B \oplus g_n(Z)) \cap g_m(\xi) = \emptyset \;\; \forall m \geq n$.

Denote the possible different configurations of $g_m(\xi)$ by $D_1, D_2, \ldots$ and the possible different values of $g_n(Z)$ by $z_1, z_2, \ldots$, and let $F \in \mathcal{F}_{g_m(\xi)}$.

$$\begin{aligned}
P(\{N_{B\oplus g_n(Z)} = k\} \cap F) &= \sum_h \sum_\ell P(\{N_{B\oplus g_n(Z)} = k\} \cap F \cap \{g_n(Z) = z_\ell\} \cap \{g_m(\xi) = D_h\}) \\
&= \sum_h \sum_\ell P(\{N_{B\oplus z_\ell} = k\} \cap F \cap \{g_n(Z) = z_\ell\} \cap \{g_m(\xi) = D_h\}) \\
&= \sum_h \sum_\ell P(\{N_{B\oplus z_\ell} = k\}) P(F \cap \{g_n(z) = z_\ell\} \cap \{g_m(\xi) = D_h\}) \qquad (14) \\
&= \sum_h \sum_\ell P(N_B = k) P(F \cap \{g_n(z) = z_\ell\} \cap \{g_m(\xi) = D_h\}) \qquad (15) \\
&= P(N_B = k)P(F).
\end{aligned}$$

$F \cap \{g_n(Z) = z_\ell\} \cap \{g_m(\xi) = D_h\} \in \mathcal{F}_{D_h}$ since $F \cap \{g_n(Z) = z_\ell\} \in \mathcal{F}_{g_m(\xi)}$, and so equation (14) follows from the independence of the increments of the Poisson process. Since the Poisson process is assumed to be homogeneous, equation (15) follows. This proves that for $m \geq n$,

$$P\left(N_{B\oplus g_n(Z)} = k | \mathcal{F}_{g_m(\xi)}\right) = P\left(N_{B\oplus g_n(Z)} = k\right) = P(N_B = k). \tag{16}$$

Since $\cap_m \mathcal{F}_{g_m(\xi)} = \mathcal{F}_\xi$ ([13], p. 31), we may apply the reverse martingale convergence theorem to obtain

$$\lim_{m\to\infty} P\left(N_{B\oplus g_n(Z)} = k | \mathcal{F}_{g_m(\xi)}\right) = P\left(N_{B\oplus g_n(Z)} = k | \mathcal{F}_\xi\right) \tag{17}$$

almost surely. Finally, equation (13) follows from equations (16) and (17) since almost surely,

$$\lim_{n\to\infty} P\left(N_{B\oplus g_n(Z)} = k | \mathcal{F}_\xi\right) = P(N_{B\oplus Z} = k | \mathcal{F}_\xi).$$

If: To prove the converse, assume that N satisfies the WTP. We shall show that N is stationary and has independent increments, and as already observed, this implies that N is a homogeneous Poisson process.

We show first that N has independent increments on the class of disjoint left-open, right-closed rectangles of the form $(z, z']$. Since the class of rectangles is a dissecting system which generates the Borel sets, this is enough (cf. [5]).

Without loss of generality (by suitably adding rectangles if necessary), we may assume that $R_1 = (z_1, z_1'], \ldots, R_n = (z_n, z_n']$ are disjoint left-open, right-closed rectangles such that for every $i, 1 \leq i \leq n$,

$$D_i := \overline{\cup_{h=1}^i R_h}$$

is a lower layer (note that $N(D_i) = N(\cup_{h=1}^i R_h)$, since no jump points of N can have a coordinate equal to 0). For $1 \leq h \leq n-1$, z_{h+1} must be an exposed point on the boundary of D_h and so by the WTP applied to the deterministic lower layers D_h,

$$P\left(N_{R_{h+1}} = k | \mathcal{F}_{D_h}\right) = P\left(N_{R_{h+1}\ominus z_{h+1}} = k\right) = P(N((0, z_{h+1}' - z_{h+1}]) = k), \tag{18}$$

and so $N_{R_{h+1}}$ is independent of $\mathcal{F}_{D_h}$. Now it is easily seen that

$$\begin{aligned} P\left(N_{R_1} = k_1, \ldots, N_{R_n} = k_n\right) &= P\left(N_{R_1} = k_1\right) P\left(N_{R_2} = k_2 | N_{R_1} = k_1\right) \cdots \\ &\quad \times P\left(N_{R_n} = k_n | N_{R_1} = k_1, \ldots, N_{R_{n-1}} = k_{n-1}\right) \\ &= P\left(N_{R_1} = k_1\right) P\left(N_{R_2} = k_2 | \mathcal{F}_{D_1}\right) \times \cdots \times P\left(N_{R_n} = k_n | \mathcal{F}_{D_{n-1}}\right) \\ &= \prod_{h=1}^n P\left(N_{R_h} = k_h\right). \end{aligned}$$

Therefore, N has independent increments.

Stationarity is now an immediate consequence of independence of the increments and equation (13), letting $Z = t$ and $\xi = D_t$. This completes the proof. □

We have shown the equivalence of properties 1, 2 and 3, and now move to properties 4 and 5. Since a point process with independent increments and no fixed atoms is necessarily Poisson, 4 and 5 are equivalent. Thus, it is enough to prove the following theorem:

THEOREM 6.4. A Poisson process N on T is renewal if and only if it is homogeneous.

We will require Lemma 6.5 for the proof.

LEMMA 6.5. Let N be a strictly simple point process with independent increments and let $\{\xi_i\}$ be the associated rll's. For $A \in \mathcal{A}$ arbitrary but fixed, let $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ be the exposed points of $\xi_i \cap A$ numbered lexicographically. Given $\mathcal{F}_{\xi_i}$, the rlls $(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A) \ominus \tau_j^{(i)}$ are conditionally independent with conditional distributions that depend only on $(\tau_j^{(i)}, j = 1, 2, \ldots)$.

Proof. The sets $(E_{\tau_j^{(i)}})^\circ \cap \xi_i^+ \cap A$ are disjoint and given $\mathcal{F}_{\xi_i}$, they are deterministic and depend only on $(\tau_j^{(i)}, j = 1, 2, \ldots)$. The rll $(E_{\tau_j^{(i)}} \cap \xi_{i+1} \cap A) \ominus \tau_j^{(i)}$ is determined entirely by the restriction of N to the set $(E_{\tau_j^{(i)}})^\circ \cap \xi_i^+ \cap A$, and so the lemma follows from the fact that N has independent increments, using an argument similar to that in the proof of Theorem 6.3. □

Proof of Theorem 6.4. First assume that N is a Poisson renewal process with mean measure Λ. By independence of the increments and equation (10), if $t \in T$

$$
\begin{aligned}
&I\Big(t \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times \exp\Big(-\Lambda\Big(\Big(\tau_j^{(i)}, t + \tau_j^{(i)}\Big)\Big)\Big) \\
&\quad = I\Big(t \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P\Big(N\Big(\Big(\tau_j^{(i)}, \tau_j^{(i)} + t\Big)\Big) = 0 | \mathcal{F}_{\xi_i}\Big) \\
&\quad = I\Big(t \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times P(N((0, t)) = 0) \\
&\quad = I\Big(t \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big) \times \exp(-\Lambda((0, t))).
\end{aligned}
$$

It follows that

$$
\exp\Big(-\Lambda\Big(\Big(\tau_j^{(i)}, t + \tau_j^{(i)}\Big)\Big)\Big) = \exp(-\Lambda((0, t)))
$$

on

$$
\Big\{\Big(t \in \Big[\Big(E_{\tau_j^{(i)}} \cap \xi_i^+ \cap A\Big) \ominus \tau_j^{(i)}\Big]\Big)\Big\}.
$$

Given $\tau_j^{(i)}$, this event has positive probability provided that $t + \tau_j^{(i)} \in A$, and so Λ must be a homogeneous measure. Hence, N is a homogeneous Poisson process.

Conversely, assume that N is a homogeneous Poisson process. By Lemma 6.5, the first condition in Definition 4.1 is satisfied. Condition $2'$, is an immediate consequence of an application of the WTP to the stopping set $\xi_i \cap A$, the $\mathcal{F}_{\xi_i \cap A}$-random element $\tau_j^{(i)}$ and the set $B = \cup_{h=1}^n (0, t_h)$. Then by observing that we may condition on $\mathcal{F}_{\xi_i \cap A'}$ where A' is any set in $\mathcal{A}$ containing A, we may pass to the limit and condition on $\mathcal{F}_{\xi_i}$. This proves that N is renewal. □

To summarize, we have proven that conditions 1–5 are equivalent. In particular, the characterizations of the homogeneous Poisson process via the WTP and the renewal property are new in dimension greater than one.

7. Compensators

Martingale methods in the theory of point processes on $\mathbf{R}_+$ have proven to be an indispensable tool, and more recently have been successfully applied to point processes on partially ordered spaces (cf. [13] and [15]). The first step is the calculation of a *compensator* for the point process in question. On $\mathbf{R}_+$, the compensator is defined to be the unique predictable increasing process in the Doob–Meyer decomposition of the process (N_t) regarded as a submartingale, and explicit formulas are well known. On partially ordered spaces, there are various definitions of the martingale property. The existence of a Doob–Meyer decomposition has been established only under particular conditions (refer to [13]), and specific formulas are calculated on a case-by-case basis. In this section, we will consider two different definitions for the compensator of a point process and show that in the case of a renewal process, the question of existence reduces to existence of a compensator for a single line process. These results will then be applied to the examples of the partial sum and Poisson processes and as well to superpositions of independent renewal processes.

The compensator of a point process N indexed by $\mathcal{A}$ must be defined in terms of its behaviour on increments; that is, its value on the generalized rectangles $\mathcal{C}$, the class of sets of the form $A\backslash\cup_{i=1}^{k}A_i$, where $A, A_1, \ldots, A_k \in \mathcal{A}$, $k < \infty$. Given a ($\mathcal{A}$-indexed) filtration $\mathcal{F}$, we may associate two histories with $C = A\backslash\cup_{i=1}^{k}A_i$:

$$\mathcal{G}_C = \cap_{A'\in\mathcal{A};A'\cap C\neq\emptyset}\mathcal{F}_{A'}, \text{ and } \mathcal{G}^*_C = \vee_{A'\in\mathcal{A};A'\cap C=\emptyset}\mathcal{F}_{A'}.$$

If $C = A \in \mathcal{A}$, we define $\mathcal{G}_C = \mathcal{G}^*_C = \mathcal{F}_0$. We observe that $\mathcal{G}_C \subseteq \mathcal{G}^*_C$, and that both $\mathcal{G}$ and $\mathcal{G}^*$ decrease as C increases. We refer to $\mathcal{G}_C$ and $\mathcal{G}^*_C$, respectively, as the strict past and the wide past of $C \in \mathcal{C}$.

Any $\mathcal{A}$-indexed stochastic process X can be extended to an additive process on $\mathcal{C}$ by the usual inclusion–exclusion formula. If X is adapted to a filtration $\mathcal{F}$, X is a weak (sub-) martingale if

$$E[X_C|\mathcal{G}_C] = (\geq)0, \quad \forall C \in \mathcal{C}.$$

X is a strong (sub-)martingale if

$$E[X_C|\mathcal{G}^*_C] = (\geq)0, \quad \forall C \in \mathcal{C}.$$

Clearly, any point process N is both a weak and a strong submartingale with respect to any filtration to which it is adapted. We call any random measure $\hat{N}$ a compensator for N if

$$E[N_C - \hat{N}_C|\mathcal{G}_C] = 0, \quad \forall C \in \mathcal{C}. \tag{19}$$

We call any random measure $\hat{N}^*$ a $*$-compensator for N if

$$E[N_C - \hat{N}^*_C|\mathcal{G}^*_C] = 0, \quad \forall C \in \mathcal{C}. \tag{20}$$

We do not require $\hat{N}(\hat{N}^*)$ to be adapted (although, if it is, $N - \hat{N}$ $(N - \hat{N}^*)$ is a weak (strong) martingale), and it is trivial that any $*$-compensator is a compensator. It is clear that a ($*$-) compensator need not be unique (N is trivially both a compensator and a $*$-compensator for itself). However, uniqueness of the ($*$-) compensator is proven in Ref. [13] if a certain predictability property is satisfied. In addition, in Ref. [13] it is shown that the Poisson process is characterized by the fact that its mean measure is a $*$-compensator, but it is important to note that there are many (non-Poisson) point processes whose mean measure is a compensator. Thus, in general the $*$-compensator is more useful. In particular, under certain conditions the asymptotic behaviour of a sequence of point processes is determined by the asymptotic behaviour of the corresponding sequence of $*$-compensators (cf. [13]). This will be discussed briefly in Example 7.5.

In what follows, we shall see that a ($*$-)compensator of a renewal process N can be expressed in terms of the ($*$-)compensator of M_1, the single line process associated with ξ_1. Although, conceptually this result may appear obvious, the mathematical formalism is somewhat delicate. Care must be taken to ensure that the appropriate measurability problems are addressed. First, we begin with some comments about compensators of single line processes, noting that by additivity it is enough to consider the conditional expectations in equations (19) and (20) for sets of the form $C = (z, z']$.

Let M be the single line process associated with a rll ξ and assume that $\mathcal{F}$ is the minimal filtration generated by M. Because M is a single line process, both compensators will have mass only on ξ (otherwise, if $(z, z'] \subseteq \xi^c$, then $M_z \geq 1$ and since none of the jump points of M are comparable, no more jump points can occur in $(z, z']$). We will be considering the compensators of the point process restricted to a lower layer B:

$$M|_B(\cdot) = M(\cdot \cap B).$$

Denote by $\mathcal{F}|_B(\cdot)$ the minimal filtration generated by $M|_B$ and define $\mathcal{G}|_B(\cdot)$ and $\mathcal{G}^*|_B(\cdot)$ as before, using $\mathcal{F}|_B(\cdot)$.

Note 1. For $C = (z, z']$, $E[M(z, z']|\mathcal{G}_C] = E[M(z, z']|\mathcal{F}_z]$. This conditional expectation is 0 on $\{M_z \neq 0\}$, and on $\{M_z = 0\}$ it is deterministic since $\{M_z = 0\}$ is an atom of $\mathcal{F}_z$. Thus, there is a deterministic measure Λ_z on (z, ∞) such that

$$E[M(z, z']|\mathcal{G}_C] = I(M_z = 0)\Lambda_z(z, z'].$$

Also, if $z \in B$, $\mathcal{G}|_B(z, z'] = \mathcal{F}|_B(z) = \mathcal{F}_z = \mathcal{G}(z, z']$, and so it follows that if $\hat{M}$ is any compensator of M, then $\widehat{M|_B} = \hat{M}|_B$ is a compensator of $M|_B$.

Note 2. For $C = (z, z']$, it is not true that $\mathcal{G}^*|_B(z, z'] = \mathcal{G}^*(z, z']$ when $z \in B$, and so in general we cannot conclude that $\hat{M}^*|_B$ is a $*$-compensator of $M|_B$ if $\hat{M}^*$ is a $*$-compensator of M, and we must distinguish between $\hat{M}^*|_B$ and $\widehat{M|_B}^*$ (this is because $\mathcal{G}^*|_B(z, z']$ contains information about jump points of M that lie in $D_z \cap B$, whereas $\mathcal{G}^*(z, z']$ contains information about all jump points of M in D_z). Now,

$$E\left[M|_B(z, z']|\mathcal{G}^*|_B(z, z']\right]$$

is 0 on $\{M_z \neq 0\}$, and on $\{M_z = 0\}$ it is a function of the random set $\xi \cap D_z \cap B$. It is a trivial observation that the support of $\widehat{M|_B}^*$ is B.

Henceforth, we shall assume that $(\Omega, \mathcal{F})$ is the *canonical space* of nonexplosive integer-valued measures on T with $\mathcal{F}$ the σ-algebra generated by the mappings $\mu \in \Omega \rightarrow \mu[0, t], t \in T$. As shown in Section 2, given a point process N on the canonical space, the associated rll's $\xi_i, i \geq 1$ are measurable mappings from $(\Omega, \mathcal{F})$ to $(\mathcal{L}, \mathcal{F}_{\mathcal{L}})$. Given a point process $N = N(\mu)$, we shall consider the *path compensator* $\hat{N} = \hat{N}(\mu)$ (respectively, the path $*$-compensator $\hat{N}^*(\mu)$). This is a function of μ that depends on the distribution of N and the filtration $\mathcal{F}$.

Now we return to a general renewal process N (with rll's $\xi_1, \xi_2, \dots$) and the decomposition $N = \sum_{i=1}^{\infty} M_i$ of Proposition 2.7. Recall that M_i is a single line process corresponding to a rll ρ_i, that the exposed points of ρ_i are all jump points of N, and $\xi_i = \rho_i \cap \xi_{i-1}^{+}$. Fix $A \in \mathcal{A}$ and suppose that $\alpha = \tau_J^{(i-1)}$ is an exposed point of $\xi_{i-1} \cap A$ (where we will always assume that the exposed points are numbered lexicographically to ensure that α is $\mathcal{F}_{\xi_{i-1} \cap A}$-measurable). Let $\rho_i^{\alpha} = \rho_i \cup D_{\alpha}$. Then since the exposed points of ρ_i^{α} are precisely the minimum jump points of N contained in the set $(E_{\alpha})^{\circ} \cap \xi_{i-1}^{+}$ and these sets are all incomparable, by Lemma 2.6 we have

$$\rho_i \cap A = \cap_{\alpha \in \varepsilon(\xi_{i-1} \cap A)} \rho_i^{\alpha} \cap A$$

and $M_i(A) = \sum_{\alpha \in \varepsilon(\xi_{i-1} \cap A)} M_i^{\alpha}(A)$ where M_i^{α} is the single line process associated with ρ_i^{α}. It is important to note that each of the random elements $\xi_i, M_i, \rho_i, \alpha$ is a measurable mapping on $(\Omega, \mathcal{F})$, although, the dependence on $\mu \in \Omega$ has been suppressed in the notation.

Next we consider a (*-)compensator of M_1 with respect to its minimal filtration $\mathcal{F}^{M_1}$; we assume that the path compensators $\hat{M}_1$ and $\widehat{M_1|_B}^{*}$ are defined for each $B \in \mathcal{L}$. These are functions of μ and $\mu|_B$, respectively, that depend only on the law of M_1, and not on the law of N. Given a rll $\xi = \xi(\mu)$, denote for $\mu \in \Omega$ and $\mu \to \mu'$ a measurable mapping from Ω to itself,

$$\hat{M}_1|_{\xi}(\mu') := \hat{M}_1|_{\xi(\mu)}(\mu') \tag{21}$$

$$\widehat{M_1|_{\xi}}^{*}(\mu') := \widehat{M_1|_{\xi(\mu)}}^{*}(\mu'). \tag{22}$$

Next, for $t \in T$ and $\alpha = \tau_j^{(i-1)}$ an exposed point of $\xi_{i-1} \cap A_t$, let

$$\hat{M}_i^{\alpha}(t) := \hat{M}_1|_{\xi_{\alpha}}(\mu_{\alpha})[0, t-\alpha] = I_{\{\alpha \in A_t\}} \hat{M}_1|_{\xi_{\alpha}}(\mu_{\alpha})[0, t-\alpha] \tag{23}$$

$$\hat{M}_i^{\alpha^{*}}(t) := \widehat{M_1|_{\xi_{\alpha}}}^{*}(\mu_{\alpha})[0, t-\alpha] = I_{\{\alpha \in A_t\}} \widehat{M_1|_{\xi_{\alpha}}}^{*}(\mu_{\alpha})[0, t-\alpha] \tag{24}$$

where $\xi_{\alpha} = (E_{\alpha} \cap \xi_{i-1}^{+}) \ominus \alpha$ and $\mu_{\alpha} = (\mu|_{E_{\alpha}^{\circ} \cap \xi_{i-1}^{+}}) \ominus \alpha$ (i.e. each of the jump points of μ on $E_{\alpha}^{\circ} \cap \xi_{i-1}^{+}$ is translated by $-\alpha$).

In order to keep the notation concise in what follows, "$M^{(*)}$" should be read as "M, respectively, M^{*}". Likewise, the notation $\mathcal{G}^{(*)}$ means "$\mathcal{G}$, respectively, $\mathcal{G}^{*}$". Also, we reintroduce superscripts on the filtration to distinguish between $\mathcal{F}^{M_1}$ and $\mathcal{F}^{N}$.

THEOREM 7.1. Let N be a renewal point process with decomposition $N = \sum_{i=1}^{\infty} M_i$ (as in Proposition 2.7). Assume that path compensators $\hat{M}_1$ and $\widehat{M_1|_B}^{*}$ (with respect to $\mathcal{F}^{M_1}$) are defined for each $B \in \mathcal{L}$ and that $\hat{M}_1^{*}(\mu, t, B) := \widehat{M_1|_B}^{*}(\mu)(t)$ is jointly measurable as a map from $\Omega \times T \times \mathcal{L}$ to the Borel sets of $\mathbf{R}_+$. Then for $i \geq 2$,

$$\hat{M}_i^{(*)}(t) \sum_{\alpha \in \varepsilon(\xi_{i-1} \cap A_t)} = \hat{M}_i^{\alpha^{(*)}}(t) \tag{25}$$

is a path (*-) compensator for M_i with respect to $\mathcal{F}^{N}$, where $\hat{M}_i^{\alpha^{(*)}}(t)$ is defined in equations (23) and (24).

Proof. We begin by observing that the measurability of the process defined in equation (25) follows from the assumption of joint measurability of $\hat{M}_1^{*}(\mu, t, B)$ in the case of the *-compensator. Such an assumption is not required in the case of the compensator, since by definition $\hat{M}_1$ is a random measure and $\widehat{M_1|_B} = \hat{M}_1|_B$.

We will prove that for any rectangle $(z, z']$ and $\alpha = \tau_j^{(i-1)}$ an exposed point of $\xi_{i-1} \cap A_{z'}$,

$$E\Big[M_i^\alpha(z, z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}_{(z,z']}\Big] = E\Big[\hat{M}_i^\alpha(z, z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}_{(z,z']}\Big], \tag{26}$$

and

$$E\Big[M_i^\alpha(z, z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}^*_{(z,z']}\Big] = E\Big[\hat{M}_i^{\alpha^*}(z, z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}^*_{(z,z']}\Big] \tag{27}$$

where $\mathcal{G}, \mathcal{G}^*$ are defined using $\mathcal{F}^N$. Then equations (19) and (20) follow immediately from equations (26) and (27).

Next note that by the definition of the renewal process, given $\mathcal{F}^N_{\xi_{i-1}}$, the exposed points $\alpha \in \varepsilon(\xi_{i-1} \cap A_{z'})$ are known and the conditional laws of M_i^α are independent with

$$M_i^\alpha(t) =_{\mathcal{D}} M|_{(E_\alpha \cap \xi^+_{i-1}) \ominus \alpha}((t \vee \alpha) - \alpha),$$

where M is an independent copy of M_1, independent of $\mathcal{F}^N_{\xi_{i-1}}$. Therefore, for a fixed rectangle $(z, z']$, the conditional law of $M_i^\alpha(z, z']$ given $\mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}^{(*)}_{(z,z']}$, is the same as the conditional law given $\mathcal{F}^N_{\xi_{i-1}} \vee \sigma(M_i^\alpha|_{(A_z \cap E_\alpha \cap \xi^+_{i-1})}(\cdot))$ in the case of $\mathcal{G}_{(z,z']}$ and given $\mathcal{F}^N_{\xi_{i-1}} \vee \sigma(M_i^\alpha|_{(D_z \cap E_\alpha \cap \xi^+_{i-1})}(\cdot))$ in the case of $\mathcal{G}^*_{(z,z']}$. In both cases, the conditional law is equal to the corresponding conditional law of $M|_{(E_\alpha \cap \xi^+_{i-1}) \ominus \alpha}((z \vee \alpha) - \alpha), (z' \vee \alpha) - \alpha)]$ and the jump points of M on $(A_z \cap E_\alpha \cap \xi^+_{i-1}) \ominus \alpha$ (respectively, $(D_z \cap E_\alpha \cap \xi^+_{i-1}) \ominus \alpha$) can be identified with those of M_i^α on $(A_z \cap E_\alpha \cap \xi^+_{i-1})$ (respectively, $(D_z \cap E_\alpha \cap \xi^+_{i-1})$), each translated by $-\alpha$. In other words, for $t \in T$,

$$M_i^\alpha|_{(A_z \cap E_\alpha \cap \xi^+_{i-1})}(t + \alpha) = M|_{(A_z \cap E_\alpha \cap \xi^+_{i-1}) \ominus \alpha}(t) \tag{28}$$

and

$$M_i^\alpha|_{(D_z \cap E_\alpha \cap \xi^+_{i-1})}(t + \alpha) = M|_{(D_z \cap E_\alpha \cap \xi^+_{i-1}) \ominus \alpha}(t). \tag{29}$$

First, we will consider the compensator. By the preceding discussion and by independence of M and $\mathcal{F}^N_{\xi_{i-1}}$,

$$\begin{aligned}
E\Big[M_i^\alpha(z,z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}_{(z,z']}\Big] &= E\Big[M_i^\alpha(z,z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \sigma\Big(M_i^\alpha|_{(A_z \cap E_\alpha \cap \xi^+_{i-1})}(\cdot)\Big)\Big] \\
&= E\Big[M((z \vee \alpha) - \alpha, (z' \vee \alpha) - \alpha] | \mathcal{F}^N_{\xi_{i-1}} \vee \sigma\Big(M|_{(A_z \cap E_\alpha \cap \xi^+_{i-1}) \ominus \alpha}(\cdot)\Big)\Big] \\
&= E\Big[\hat{M}_1|_{\xi_\alpha}((z \vee \alpha) - \alpha, (z' \vee \alpha) - \alpha] | \mathcal{F}^N_{\xi_{i-1}} \vee \sigma\Big(M|_{(A_z \cap E_\alpha \cap \xi^+_{i-1}) \ominus \alpha}(\cdot)\Big)\Big] \\
&= E\Big[\hat{M}_i^\alpha(z,z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \sigma\Big(M_i^\alpha|_{(A_z \cap E_\alpha \cap \xi^+_{i-1})}(\cdot)\Big)\Big] \\
&= E\Big[\hat{M}_i^\alpha(z,z'] | \mathcal{F}^N_{\xi_{i-1}} \vee \mathcal{G}_{(z,z']}\Big].
\end{aligned}$$

(Recall that $\xi_\alpha = (E_\alpha \cap \xi^+_{i-1}) \ominus \alpha$).

Similarly, for the $*$-compensator

$$\begin{aligned}E\Big[M_i^{\alpha}(z,z']|\mathcal{F}^N_{\xi_{i-1}}\vee\mathcal{G}^*_{(z,z']}\Big] &= E\Big[M_i^{\alpha}(z,z']|\mathcal{F}^N_{\xi_{i-1}}\vee\sigma\Big(M_i^{\alpha}|_{(D_z\cap E_\alpha\cap\xi^+_{i-1})}(\cdot)\Big)\Big]\\ &= E\Big[M((z\vee\alpha)-\alpha,(z'\vee\alpha)-\alpha]|\mathcal{F}^N_{\xi_{i-1}}\vee\sigma(M|_{(D_z\cap E_\alpha\cap\xi^+_{i-1})\ominus\alpha}(\cdot))\Big]\\ &= E\Big[\widehat{M_1|_{\xi_\alpha}}^*((z\vee\alpha)-\alpha,(z'\vee\alpha)-\alpha]|\mathcal{F}^N_{\xi_{i-1}}\\ &\quad\vee\sigma\Big(M|_{(D_z\cap E_\alpha\cap\xi^+_{i-1})\ominus\alpha}(\cdot)\Big)\Big]\\ &= E\Big[\hat{M}_i^{\alpha^*}(z,z']|\mathcal{F}^N_{\xi_{i-1}}\vee\sigma\Big(M_i^{\alpha}|_{(D_z\cap E_\alpha\cap\xi^+_{i-1})}(\cdot)\Big)\Big]\\ &= E\Big[\hat{M}_i^{\alpha^*}(z,z']|\mathcal{F}^N_{\xi_{i-1}}\vee\mathcal{G}^*_{(z,z']}\Big].\end{aligned}$$

Thus, equations (26) and (27) have been demonstrated, completing the proof. □

COROLLARY 7.2. Under the assumptions of Theorem 7.1, the renewal point process N has ($*$-) compensator

$$\hat{N}^{(*)}(t)=\sum_{i=1}^{\infty}\hat{M}_i^{(*)}(t)$$

with respect to $\mathcal{F}^N$.

Proof. This follows immediately from Proposition 2.7 and Theorem 7.1. □

Example 7.3. The partial sum process:

Recall that $\tau_1,\tau_2,\ldots$ are i.i.d. T-valued random variables with distribution F, and that

$$N_t=\sum_{i=1}^{\infty}M_i(t)=\sum_{i=1}^{\infty}I_{\{\tau^{(i)}\in A_t\}},$$

where $\tau^{(k)}=\sum_{i=1}^{k}\tau_i$. We have $\xi_i=D_{\tau^{(i)}}$ and $\xi_i^+=T$, and so in the notation of equations (23) and (24), $\xi_{\tau^{(i)}}=T$. It is straightforward to see that (cf. [13])

$$\hat{M}_1(t)=\int_{[0,t]}I_{\{\tau_1\notin A_u\}}(1-F(u))^{-1}\mathrm{d}F(u),$$

and

$$\hat{M}_1^*(t)=\int_{[0,t]}I_{\{\tau_1\notin D_u\}}(1-F(D_u))^{-1}\mathrm{d}F(u).$$

Therefore, according to equations (23) and (24), for $i\geq 2$

$$\begin{aligned}\hat{M}_i(t) &= I_{\{\tau^{(i-1)}\in A_t\}}\int_{[0,t-\tau^{(i-1)}]}I_{\{\tau_i\notin A_u\}}(1-F(u))^{-1}\mathrm{d}F(u)\\ &= I_{\{\tau^{(i-1)}\in A_t\}}\int_{[0,t-\tau^{(i-1)}]\cap D_{\tau_i}}(1-F(u))^{-1}\mathrm{d}F(u),\end{aligned}$$

and

$$\hat{M}_i^*(t) = I_{\{\tau^{(i-1)} \in A_t\}} \int_{[0,t-\tau^{(i-1)}]} I_{\{\tau_i \notin D_u\}} (1 - F(D_u))^{-1} \mathrm{d}F(u)$$
$$= I_{\{\tau^{(i-1)} \in A_t\}} \int_{[0,t-\tau^{(i-1)}] \cap [0,\tau_i]} (1 - F(D_u))^{-1} \mathrm{d}F(u).$$

Finally, by Corollary 7.2 we obtain (letting $\tau^{(0)} = 0$)

$$\hat{N}(t) = \sum_{i=1}^{N(t)} \int_{[0,t-\tau^{(i-1)}] \cap D\tau_i} \frac{\mathrm{d}F(u)}{1 - F(u)} + \int_{[0,t-\tau^{(N(t))}]} \frac{\mathrm{d}F(u)}{1 - F(u)},$$

and

$$\hat{N}^*(t) = \sum_{i=1}^{N(t)} \int_{[0,\tau_i]} \frac{\mathrm{d}F(u)}{1 - F(D_u)} + \int_{[0,t-\tau^{(N(t))}] \cap [0,\tau_{N(t)+1}]} \frac{\mathrm{d}F(u)}{1 - F(D_u)}.$$

The approach used here provides a very simple and direct technique for calculating the compensator of the renewal process on the line without appealing to conditional intensities (cf. [17]): for $T = \mathbf{R}_+$,

$$\hat{N}(t) = \hat{N}^*(t) = \sum_{i=1}^{N(t)} \int_{[0,\tau_i]} \frac{\mathrm{d}F(u)}{1 - F(u)} + \int_{[0,t-\tau^{N(t)}]} \frac{\mathrm{d}F(u)}{1 - F(u)}.$$

Example 7.4. The Poisson process:

Let N be a homogeneous Poisson process with intensity λ. By independent increments it is easily seen that

$$\hat{M}_1(t) = \hat{M}_1^*(t) = \int_{[0,t]} I_{\{N(u)=0\}} \lambda \, \mathrm{d}\, u = \lambda \times \mu([0,t] \cap \xi_1).$$

Therefore, from equations (23) to (25), for $i \geq 2$

$$\hat{M}_I^{(*)}(t) = \sum_{\alpha \in \varepsilon(\xi_{i-1} \cap A_t)} \hat{M}_I^{\alpha^{(*)}}(t) = \sum_{\alpha \in \varepsilon(\xi_{i-1} \cap A_t)} \int_{[0,t-\alpha]} I_{\{M_i^\alpha(\alpha+u)=0\}} \lambda \, \mathrm{d}u$$
$$= \sum_{\alpha \in \varepsilon(\xi_{i-1} \cap A_t)} \lambda \times \mu((\alpha,t] \cap \xi_i).$$

The final equality above follows since $M_i^\alpha(\alpha + u) = 0$ if and only if $u + \alpha \in \xi_i$. Finally, since $[0,t] = ([0,t] \cap \xi_1) \cup \cup_{i=2}^\infty \cup_{\alpha \in \varepsilon(\xi_{i-1} \cap A_t)} ((\alpha,t] \cap \xi_i)$ and the sets in the union are all disjoint, it follows that

$$\hat{N}^{(*)}(t) = \lambda \times \mu([0,t] \cap \xi_1) + \sum_{i=2}^\infty \sum_{\alpha \in \varepsilon(\xi_{i-1} \cap A_t)} \lambda \times \mu((\alpha,t] \cap \xi_i) = \lambda \times \mu[0,t].$$

Example 7.5. Superpositions of renewal processes:

For renewal processes on $\mathbf{R}_+$, it is well known that the superposition of a finite number of independent stationary renewal processes is Poisson if and only if each of the processes is Poisson (cf. [5]). It is also known that under a suitable renormalization, the superposition of n

i.i.d. renewal processes converges weakly to a Poisson process as $n \to \infty$. This can be extended under certain conditions to the multi-parameter renewal case using martingale methods, since the $*$-compensator of a superposition of independent point processes is the sum of the corresponding $*$-compensators. It is proven in Theorem 8.2.2 of [13] that if the $*$-compensators of a sequence of point processes converges in a suitable sense to a continuous deterministic limit, the point processes converge to a Poisson process. However, it should be noted that the situation is somewhat delicate in the multiparameter setting and not all such superpositions of renewal processes have Poisson limits. This will be the subject of a separate publication.

8. Open questions

Several questions and directions for further research beyond the scope of this paper arise naturally from our definition of a multi-parameter renewal process.

1. In this article we have defined a renewal process in the positive quadrant using the usual translation operator on the Euclidean space $\mathbf{R}^d$. Is it possible to extend the definition of a renewal process to a general set-indexed framework: i.e. to point processes indexed by a class of subsets $\mathcal{A}$ of an arbitrary space T (cf. [13])? We conjecture that this is possible under the hypothesis that there exists a translation operation which is consistent with the partial order on T induced by the collection of sets $\mathcal{A}$. A first step in this direction would be to consider the entire space $\mathbf{R}^d$, permitting renewals simultaneously in all 2^d quadrants.
2. As we have seen, a Poisson process is a renewal process if and only if it is homogeneous. What can be said about Cox processes (doubly-stochastic Poisson processes)? When is a Cox process a renewal process? For the one-dimensional case, this problem was solved by Kingman and Daley: a Cox process is renewal if and only if the directing measure associated with the process is regenerative and takes only the values zero and a positive constant (see [18] for references to this problem). This is a non-trivial result even on the line; the multi-parameter problem seems much more difficult and leads naturally to the next open question.
3. Regenerative processes: A regenerative process is an important generalization of a renewal process. How does one define multi-parameter regenerative processes?
4. It is clear that the compensator does not determine the law of the renewal process. Indeed, the first line of the "diagonal Poisson" process has the same compensator as the first line of the Poisson process (see Counterexample 2 of [13], p. 92), and following the construction of Theorem 4.2, the resulting renewal processes have the same deterministic compensator, although, the "diagonal Poisson" is not a Poisson process. However, in light of the fact that the $*$-compensator characterizes the Poisson process, it is natural to ask whether there are circumstances under which the $*$-compensator determines the law of a general renewal process. We conjecture that under an assumption of conditional independence (sometimes referred to as the "F4 Property"), the answer is positive.
5. Limit theorems: For applications, the study of the asymptotic behaviour of renewal processes on $\mathbf{R}_+$ has been crucial; in particular the almost sure convergence of the classic renewal theorem and the L^1 convergence of Blackwell's theorem. It is essential to have such results for multi-parameter renewal processes. Attacking this problem directly seems difficult; one possibility is to study these processes along certain increasing paths

(called “flows”). Some results exist, see [1–4,7–12,19,21,26,27,29–31]. However, the definitions of the renewal property in the cited articles are different from ours.

6. Renewal on configurations: Instead of considering renewals with respect to the exposed points of the different stopping lines, we can consider some interesting configurations, for example a fixed number of jump points which are not necessarily non-ordered, and try to define a renewal with respect to this special configuration. This kind of problem may have interesting applications in environmental sciences (cf. [6]).
7. Renewals under appropriate “time-changes”: It is known that under certain conditions, any strictly simple point process N can be “time-changed” via random sets into a Poisson process, provided the rll’s ξ_i associated with N each have an infinite number of exposed points. Does there exist a similar result for multidimensional renewal processes? This problem is related to the study of renewals along flows (increasing paths).
8. Sequences of renewal processes: Is the weak limit (in an appropriate topology) of a sequence of renewal processes still a renewal process?

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