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A note on the precompactness of weakly almost periodic groups

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Abstract. An action of a group G on a compact space X is called weakly almost periodic if the orbit of every continuous function on X is weakly relatively compact in $C(X)$. We observe that for a topological group G the following are equivalent: (i) every continuous action of G on a compact space is weakly almost periodic; (ii) G is precompact. For monothetic groups the result was previously obtained by Akin and Glasner, while for locally compact groups it has been known for a long time.

1. Introduction

Let a group G act on a set X. Denote, as usual, by $l^\infty(X)$ the vector space of all bounded complex-valued functions on X equipped with the supremum norm. A bounded function f on X is called *weakly almost periodic* (*w.a.p.* for short) if the G-orbit of f is weakly relatively compact in the Banach space $l^\infty(X)$.

Now suppose that a group G acts by homeomorphisms on a compact space X. The action is called *weakly almost periodic* (or again *w.a.p.*) if every continuous function on X is weakly almost periodic.

It follows easily from well-known results that if G is a precompact topological group, then every continuous action of G on a compact space is weakly almost periodic. (Recall that a topological group is *precompact* if it is isomorphic to a subgroup of a compact group.) Our main goal is to establish the converse.

Main Theorem 4.5. *For an arbitrary topological group G the following conditions are equivalent.*

1. *Every continuous action of G on a compact space is weakly almost periodic.*
2. *Every bounded right uniformly continuous function on G is weakly almost periodic.*
3. *G is precompact.* ■

In the case where G is a *monothetic* group (that is, contains an everywhere dense cyclic subgroup), the result was established by Akin and Glasner [1].

Our proof is independent and different in nature from that in [1], and is exploiting theory of invariant means and a result by Pachl [14].

While surveying some basic notions and results from abstract topological dynamics upon which our proof hinges, we give a new and simpler proof of the Ellis-Lawson Joint Continuity Theorem (Theorem 2.2 below).

2. Weak almost periodicity and compactifications

We begin with some background material about weak almost periodicity, greatest ambits, and their significance in abstract topological dynamics. For more comprehensive information we refer the reader to e.g. [16].

For a topological space X, let $C^b(X)$ stand for the space of all bounded continuous functions on X. Notice that $C^b(X)\subset l^\infty(X)$ forms a (weakly) closed vector subspace. If G acts on X by homeomorphisms, then the orbit of every function $f\in C^b(X)$ is contained in $C^b(X)$. It follows that f is weakly almost periodic if and only if its orbit is relatively weakly compact in the Banach space $C^b(X)$.

In particular, considering the left and right actions of a group G on itself, we can define left and right weakly almost periodic functions on G. These two notions are actually equivalent [3, Corollary 1.12], so we can simply speak about w.a.p. functions on a group G.

For a topological group G, denote by $W(G)$ the space of all continuous w.a.p. functions on G. The space $W(G)$ is a commutative C^*-algebra and thus is isomorphic to the algebra $C(G^w)$ of continuous functions on a compact space G^w, the maximal ideal space of $W(G)$. Call G^w the *weakly almost periodic (w.a.p.) compactification* of the topological group G. There is a natural structure of a semitopological semigroup on G^w, where 'semitopological' means that the multiplication is separately continuous. The canonical semigroup homomorphism $G\to G^w$ is a universal object in the category of continuous homomorphisms of G to compact semitopological semigroups.

Remark 2.1. Under a *compactification* of a topological space X we mean a compact Hausdorff space K together with a continuous map $j:X\to K$ with a dense range. We do *not* require that j be a homeomorphic embedding. The long-standing problem of whether for every topological group G the canonical map $G\to G^w$ is a topological embedding has been recently solved in the negative by the first named author (M.G.M.). Namely, for the Polish group $G=\mathrm{Homeo}_+\mathbb{I}$ of all orientation-preserving homeomorphisms of the closed interval, equipped with the compact-open topology, the w.a.p. compactification G^w is a singleton (equivalently, every $f\in W(G)$ is constant) [13].

For a topological group G the *greatest ambit* $\mathcal{S}(G)$ is the compactification of G corresponding to the algebra $\mathrm{RUC}^b(G)$ of all bounded right uniformly continuous functions on G, that is, the maximal ideal space of the latter C^*-algebra. (A function $f\in C(G)$ is *right uniformly continuous* if

$$\forall\varepsilon>0\,\exists V\in\mathcal{N}(G)\,\forall x,y\in G\,(xy^{-1}\in V\implies |f(y)-f(x)|<\varepsilon),$$

where $\mathcal{N}(G)$ is the filter of neighbourhoods of unity.) There is a natural G-space structure on $\mathcal{S}(G)$. (By a G-space we mean a topological space X equipped with a jointly continuous action $G\times X\to X$.) The canonical map $i:G\to\mathcal{S}(G)$ is a homeomorphic embedding, and we will identify G with $i(G)$. Let $e \in G \subset \mathcal{S}(G)$ be the unity. The pair $(\mathcal{S}(G), e)$ has the following universal property: for every compact G-space X with a distinguished point p there exists a unique continuous G-map $\mathcal{S}(G) \to X$ which sends e to p. The multiplication on G extends to a multiplication on $\mathcal{S}(G)$ such that every right shift $r_a : \mathcal{S}(G) \to \mathcal{S}(G)$, $r_a(x) = xa$, is continuous. The shift r_a is the unique G-selfmap of $\mathcal{S}(G)$ such that $r_a(e) = a$.

Let X be a Banach space. Denote by $U(X)$ the semigroup of all linear operators on X of norm ≤ 1. If X is a reflexive space, then $U(S)$, equipped with the weak operator topology, becomes a compact semitopological semigroup. Every compact semitopological semigroup is isomorphic to a closed subsemigroup of $U(X)$ for some reflexive Banach space X [18, 13].

Let again X be a Banach space. For a subgroup G of the group $\mathrm{Is}\,(X)$ of all linear isometries of X denote by G_w and G_s the group G equipped with the weak and strong operator topology, respectively. For a wide class of Banach spaces X, including reflexive spaces, the two topologies actually coincide: $G_w = G_s$ [12]. If X is a Hilbert space then $\mathrm{Is}\,(X)$ is weak-dense in $U(X)$ [19]. Moreover, it follows from the results of [19] that in this case $U(X)$ is the w.a.p. compactification of $\mathrm{Is}\,(X)$.

Each of the following three assertions easily implies the other two:

(1) $W(G) \subset \mathrm{RUC}(G)$;

(2) there is a map $f : \mathcal{S}(G) \to G^w$ such that $j = fi$, where $i : G \to \mathcal{S}(G)$ and $j : G \to G^w$ are the canonical maps;

(3) the action of G on G^w, induced by the semigroup homomorphism $j : G \to G^w$, is jointly continuous.

All these are true for an arbitrary topological group G (see, for example, [16]), although this is not at all obvious. Actually, the question of whether every w.a.p. function is uniformly continuous appeared on the top of the list of open problems in [3]. The assertion (3) above follows from a more general result of J. Lawson, which in itself is a corollary of a fundamental *Ellis-Lawson Joint Continuity Theorem* [10]. We include here a new "soft" proof of this result.

Theorem 2.2. ([10], and [6] for $S = G$.) *Let S be a compact semitopological semigroup with unity. Let G be the group of invertible elements in S. Then G with the induced topology is a topological group, and the action of G on S by translations is jointly continuous.*

Proof. As we have noted before, there exists a reflexive Banach space X such that S is a subsemigroup of $U(X)$. We can therefore suppose that $S = U(X), G = \mathrm{Is}\,(X)$. We have $G_s = G_w$ [12]. Since G_s is a topological group, so is G_w, that is, G with the topology it inherits as a subspace of S. To see that the action of G on $U(X)$ is (jointly) continuous, it suffices to prove that the action of $G = \mathrm{Is}\,(X)_s$ on the dual Banach space X^* is continuous. This fact has been proved in [11, Corollary 6.9], and easily follows also by observing that the topological groups $\mathrm{Is}_{\,w}(X) = \mathrm{Is}_{\,s}(X)$ and $\mathrm{Is}_{\,w}(X^*) = \mathrm{Is}_{\,s}(X^*)$ are naturally isomorphic. ■

3. Invariant means

Let G be an arbitrary group and let A be a C^*-subalgebra of $l^\infty(G)$. A *mean* on A is a complex-valued linear functional ϕ on A that is positive (that is, $\phi(f) \geq 0$ whenever $f \geq 0$) and takes the constant function 1 to 1. (In C^*-algebra theory, one uses the term *state* instead.) Every mean is automatically bounded of norm one (cf. [17], Prop. 1.5.1). A mean ϕ is *left-invariant* if $\phi({}_g f) = \phi(f)$ for every $f \in A$ and every $g \in G$, where ${}_g f$ denotes the left translate of the function f by g. In a similar way one defines *right-invariant means* on A. If a mean ϕ is both left and right invariant, we will refer to it as a *bi-invariant* mean.

For instance, there is always a bi-invariant mean on the algebra $W(G)$, as the following well-known result asserts. (See e.g. [3, Corollary 1.26].)

Theorem 3.1. (Ryll-Nardzewski.) *For every topological group G, there is a unique bi-invariant mean on the algebra $W(G)$. Moreover, such a mean is the unique left-invariant mean on $W(G)$ as well.* ■

Remark 3.2. The cited corollary actually only asserts that there exists a unique two-sided invariant mean on $W(G)$. To show the uniqueness in the class of left-invariant means, note that for every $f \in W(G)$ there is a constant that can be uniformly approximated by convex combinations of left translates of f [3, Theorem 1.25 and Corollary 1.26]. The value of any left-invariant mean at f must be equal to such a constant, whence the uniqueness.

On the contrary, the larger algebra $\mathrm{RUC}^b(G)$ need not in general support even a left-invariant mean. If there is such a mean on $\mathrm{RUC}^b(G)$, the topological group G is called *amenable*. Equivalently, G is amenable if for every compact convex set K (lying in some locally convex space) and every continuous action of G on K by affine maps there exists a G-fixed point in K. See e.g. [15].

Left-invariant means on $\mathrm{RUC}^b(G)$ can be identified with left-invariant probability measures on the greatest ambit $\mathcal{S}(G)$. The collection of all such left-invariant means, equipped with the weak* topology, forms a compact space, which we will denote by LIM_G.

If LIM_G consists of a single point, we will call the topological group G *uniquely amenable.* Every compact group provides an obvious example of a uniquely amenable group, for which the unique invariant mean comes from the Haar measure. It is also obvious that every precompact group G is uniquely amenable as well, since the algebras $\mathrm{RUC}^b(G)$ and $\mathrm{RUC}^b(\widehat{G})$ are canonically isomorphic, where $\widehat{G}$ denotes the compact completion of G. This observation can be, at least partially, inverted, as the following result shows.

Theorem 3.3. (Pachl, [14].) *Let G be a separable metrizable group. If the compactum LIM_G contains a G_δ-point, then G is a precompact group.* ■

Corollary 3.4. *Every uniquely amenable separable metrizable topological group is precompact.* ■

It remains unclear if the above Corollary remains true for arbitrary topological groups.

Question 3.5. Is it true that every uniquely amenable topological group is precompact?

If the answer is in the affirmative, then our Main Theorem 4.5 below follows at once, and Proposition 4.3 and Lemma 4.4 are not needed. We can only provide a positive answer to Question 3.5 under the additional assumption that G is ω-bounded.

A topological group is called *ω-bounded* if for every $U \in \mathcal{N}(G)$ there exists a countable $A \subset G$ such that $AU = G$, or, equivalently, if G is isomorphic to a subgroup of the product of separable metrizable groups [2, 8].

Lemma 3.6. *The property of being uniquely amenable is preserved by continuous homomorphisms.*

Proof. If G is amenable and $f : K \to L$ is an affine onto G-map between compact convex G-spaces, then for every G-fixed point $y \in L$ there exists a G-fixed point $x \in K$ such that $f(x) = y$. Indeed, the set $f^{-1}(y)$ is a compact convex G-space and hence contains a G-fixed point.

Let $f : G \to H$ be an onto homomorphism. Assume that G is uniquely amenable. Applying the remark of the previous paragraph to the sets of probability measures on $\mathcal{S}(G)$ and $\mathcal{S}(H)$, we see that every left-invariant measure on $\mathcal{S}(H)$ is the image under f of a left-invariant measure on $\mathcal{S}(G)$. Since the latter is unique, it follows that the left-invariant measure on $\mathcal{S}(H)$ is also unique. ∎

Combining this result with Theorem 3.3, we obtain:

Corollary 3.7. *Every uniquely amenable ω-bounded group is precompact.* ∎

4. The main result

As we have seen in Section 2., $W(G) \subset \mathrm{RUC}^b(G)$ for every topological group G. If G is compact, then $W(G) = \mathrm{RUC}^b(G)$. Indeed, it is well known that every continuous function f on a compact group is *almost periodic,* that is, the orbit of f under left (or right) translations is even *norm relatively compact* in $C(G)$. The same is true of every precompact group G.

The aim of this paper is to prove that the equality $W(G) = \mathrm{RUC}^b(G)$ is actually *equivalent* to G being precompact. Such fact is well known for locally compact groups; see [5] and [4] (the latter for the case of locally compact groups G having small invariant neighbourhoods). For monothetic groups (that is, those — not necessarily locally compact — topological groups containing an everywhere dense cyclic subgroup) this result was recently established by Akin and Glasner, whose preprint [1] has stimulated our present research.

We begin with the following auxiliary (and well known) statement.

Proposition 4.1. *For every topological group G the following are equivalent:*

1. *$W(G) = RUC^b(G)$, that is, every right uniformly continuous function on G is weakly almost periodic;*

2. *$\mathcal{S}(G) = G^w$;*

3. *the natural multiplication on $\mathcal{S}(G)$ is separately continuous (on both sides);*

4. *if X is a compact G-space and $f \in C(X)$, then f is weakly almost periodic.*

Proof. The preceding discussion shows that the conditions (1) – (3) are equivalent. Applying (4) to $X = \mathcal{S}(G)$, we see that (4) $\implies$ (1). The implication (1) $\implies$ (4) can be established directly. We omit the proof, since we show below (Theorem 4.5) that (1) actually implies that G is precompact, and for precompact groups (4) is straightforward: just note that the G-space structure on a compact space X extends to a $\widehat{G}$-space structure. Of course one has to keep in mind that in our proofs the implication (1) $\implies$ (4) is never used. ∎

Definition 4.2. Following [1], we will say provisionally that a topological group G is *weakly almost periodic* (w.a.p.) if it satisfies the equivalent conditions of Proposition 4.1.

Lemma 4.3. *The class of w.a.p. groups is closed under forming continuous homomorphic images and topological subgroups.*

Proof. Let G be a w.a.p. topological group, and let $h : G \to H$ be an onto homomorphism. If $f \in \mathrm{RUC}^b(H)$, then $fh \in \mathrm{RUC}^b(G) = W(G)$. The map $h^* : C(H) \to C(G)$ is an isometric embedding for the norm topologies and hence a homeomorphic embedding for the weak topologies. Since h^* sends the orbit of f to the orbit of fh and the orbit of fh is weakly relatively compact, so is the orbit of f. Thus $\mathrm{RUC}^b(H) \subset W(H)$.

Now let H be a subgroup of G and $f \in \mathrm{RUC}^b(H)$. A uniformly continuous bounded function defined on a subspace of a uniform space can be extended to a uniformly continuous bounded function defined on the entire space [7, 8.5.6], [9]. Since the right uniformity of H is induced by the right uniformity of G, there exists a function $g \in \mathrm{RUC}^b(G)$ such that $g \restriction H = f$. By assumption, $g \in W(G)$. The restriction map $C(G) \to C(H)$ sends weakly relatively compact subsets of $C(G)$ to weakly relatively compact subsets of $C(H)$, while the image of the G-orbit of g contains the H-orbit of f. We conclude that $f \in W(H)$. ∎

Lemma 4.4. *Let $\mathcal{P}$ be a class of topological groups closed under subgroups and homomorphic images. Unless all groups in $\mathcal{P}$ are precompact, there exists a countable metrizable group in $\mathcal{P}$ which is not precompact.*

Proof. Let $G \in \mathcal{P}$ be non-precompact. There exists an infinite set $A \subset G$ and a neighbourhood U of unity such that the family $\{Ua : a \in A\}$ is disjoint. The subgroup of G generated by A is countable and non-precompact. Thus we may assume that G is countable. Every countable group, being ω-bounded, is isomorphic to a subgroup of the product $\prod G_\alpha$ of countable metrizable groups. We may assume that the projections of G to the factors G_α are onto and hence all the G_α's are in $\mathcal{P}$. Since the class of precompact groups is closed under products and subgroups, at least one of the factors G_α's has to be non-precompact. ∎

We now turn to the main result of the paper.

Theorem 4.5. *For an arbitrary topological group G the following conditions are equivalent.*

1. *Every continuous action of G on a compact space is weakly almost periodic.*
2. *Every bounded right uniformly continuous function on G is weakly almost periodic.*
3. *G is precompact.*

Proof. Only (2) $\implies$ (3) requires a proof. Let $W(G) = \mathrm{RUC}^b(G)$. As an immediate consequence of this assumption and Ryll-Nardzewski Theorem 3.1, the topological group G is uniquely amenable.

Assume first that G is separable metrizable. According to Pachl's Corollary 3.4, G is precompact.

The general case is now being reduced to the case of a separable metrizable group with the aid of Proposition 4.3 and Lemma 4.4. ■

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