Topology Proceedings (To Appear)

REFLEXIVELY BUT NOT UNITARILY REPRESENTABLE TOPOLOGICAL GROUPS

MICHAEL G. MEGRELISHVILI

ABSTRACT. We show that there exists a topological group G (namely, $G := L_4[0,1]$) such that for a certain *reflexive* Banach space X the group G can be represented as a topological subgroup of $Is(X)$ (the group of all linear isometries endowed with the strong operator topology) and such an X never may be Hilbert. This answers a question of V. Pestov and disproves a conjecture of A. Shtern.

1. INTRODUCTION

Let X be a real Banach space. Denote by $Is(X)_s$ ($Is(X)_w$) the group of all linear isometries of X endowed with the strong (resp., weak) operator topologies.

A *representation* of a Hausdorff topological group G in X is a continuous group homomorphism $G \to Is(X)_s$. Let $\mathbf{K}$ be a subclass of the class $\mathbf{Ban}$ of all Banach spaces. We say that G is $\mathbf{K}$-*representable* if for a certain $X \in \mathbf{K}$ there exists a topological group embedding $G \hookrightarrow Is(X)_s$. Denote by $\mathbf{K_R}$ the class of all $\mathbf{K}$-representable groups. For instance, this leads to the definitions of the following classes $\mathbf{Ban_R}$, $\mathbf{Ref_R}$, and $\mathbf{Hilb_R}$, where $\mathbf{Ref}$ and $\mathbf{Hilb}$ denote all reflexive and all Hilbert spaces, respectively. We say that G is *reflexively representable* (*unitarily representable*) if $G \in \mathbf{Ref_R}$ (resp., $G \in \mathbf{Hilb_R}$). Denote by $\mathbf{TopGr}$ the class of all Hausdorff topological groups. We have

$$\mathbf{TopGr} = \mathbf{Ban_R} \supseteq \mathbf{Ref_R} \supseteq \mathbf{Hilb_R}.$$

Indeed, the inclusions are trivial. As to $\mathbf{TopGr} = \mathbf{Ban_R}$, recall that every Hausdorff topological group G can be embedded into the group $Is(E)_s$ of all linear isometries of a suitable Banach space E endowed with the strong operator topology. As in the paper of Teleman [23], take for example $X := C_r^b(G)$, the Banach space of all bounded right uniformly continuous functions on G.

It is also well known that $\mathbf{TopGr} \neq \mathbf{Hilb_R}$. Moreover, there are many examples of so-called *exotic* groups (that is, the groups whose unitary representations are trivial), see Herer-Christensen [12] and Banasczyk [3]. Among many interesting examples from [3], note that l_2/D is exotic for some discrete subgroup D of l_2. Not every separable Banach space (as a topological group) is unitarily representable. It is well known that l_2 is not uniformly universal in the class of separable metrizable uniform spaces (Enflo [10], Aharoni [1]). These arguments lead to the fact that $C[0,1], c_0 \notin \mathbf{Hilb_R}$ (see Proposition 3.7 below).

Recently, it has been proved in [18] that $\mathbf{TopGr} \neq \mathbf{Ref_R}$. Denote by $H_+[0,1]$ the group of all orientation preserving selfhomeomorphisms of the closed interval

2000 Mathematical Subject Classification. 22A25, 54H15, 43A65, 46B99.

endowed with the compact open topology. It turns out that every (weakly) continuous representation of $H_+[0,1]$ in a reflexive Banach space by linear isometries is trivial. This result answers a question discussed by Ruppert [20] and conjectured by V. Pestov.

A natural question arises about coincidence of $\mathbf{Ref_R}$ and $\mathbf{Hilb_R}$. This question is posed in the recent paper by V. Pestov [19]. Earlier the positive answer was conjectured by A. Shtern [22]. The main result of the present work disproves this conjecture. Theorem 3.1 below implies that $L_4[0,1] \in \mathbf{Ref_R}$ and $L_4[0,1] \notin \mathbf{Hilb_R}$.

2. Weakly almost periodic functions

Recall that a continuous bounded function $f \in C^b(G)$ on a topological group G is called *weakly almost periodic* (in short: *wap*) in the sense of Eberlein [9, 8] if the orbit of f in $C^b(G)$ is relatively weakly compact. The subset $WAP(G)$ of all wap functions in $C^b(G)$ forms a closed (left and right) translation-invariant subalgebra. Every positive definite function is wap [5].

A semigroup S is *semitopological* if the multiplication $S \times S \to S$ is separately continuous. The compactification $j : G \to G^w$ induced by the algebra $WAP(G)$ is the *universal semitopological compactification* of G.

For every reflexive Banach space E, the semigroup

$$\Theta(E)_w := \{s \in L(E,E) : \|s\| \leq 1\}$$

of all contractive linear operators forms a compact semitopological semigroup in the weak operator topology [8]. Hence, the same is true for its closed subsemigroups. Conversely, an arbitrary compact Hausdorff semitopological semigroup can be obtained in this way (see Fact 2.2). By a result of Lawson [7, Corollary 6.3] every subgroup of a compact Hausdorff semitopological semigroup is a topological group.

Fact 2.1. *Let G be a Hausdorff topological group. Then the following conditions are equivalent:*

(i) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_s$ (equivalently, G is reflexively representable);

(ii) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_w$;

(iii) The algebra $WAP(G)$ separates points and closed subsets;

(iv) The canonical map $j : G \to G^w$ is a topological embedding;

(v) G is a topological subgroup of a Hausdorff compact semitopological semigroup.

The equivalence of (iii), (iv) and (v) is well known [20, 5].

The part (ii) $\Longrightarrow$ (iii) follows from the fact that for every reflexive Banach space E and a norm-bounded semigroup S of linear operators on E the *generalized matrix coefficients*

$$\{m_{v,f} : S \to \mathbb{R}\}_{f \in E^*, v \in E} \qquad m_{v,f}(s) = f(sv)$$

all are wap.

The part (iii) $\Longrightarrow$ (ii), is a direct consequence of the following result of A. Shtern.

Fact 2.2. *([22, 15, 17]) The following conditions are equivalent:*
(a) $WAP(S)$ separates points and closed subsets;
(b) S can be embedded into $\Theta(E)_w$ for a certain reflexive Banach space E.

As to the equivalence (i) $\Longleftrightarrow$ (ii), note that by [16, 17], strong and weak operator topologies coincide on $Is(X)$ for a wide class of Banach spaces with PCP (the *point of continuity property*) including the class of all reflexive Banach spaces.

3. Main Results

Let (Ω, B, μ) be a measure space. The corresponding standard space $L_p(\Omega, B, \mu)$ $(1 \leq p < \infty)$ will be denoted simply by $L_p(\mu)$.

Theorem 3.1. (i) $L_{2k}(\mu) \in \mathbf{Ref_R}$ *for every natural* $k \in \mathbb{N}$*;*
(ii) $L_p[0,1] \notin \mathbf{Hilb_R}$ *for every* $2 < p < \infty$*.*

We say that a function $F : A \times B \to \mathbb{R}$ has *Double Limit Property* (in short: DLP) if for every pair of sequences $\{a_n\}, \{b_m\}$ in A and B respectively,

$$\lim_n \lim_m F(a_n, b_m) = \lim_m \lim_n F(a_n, b_m)$$

whenever both of the limits exist.

We need *Grothendieck's characterization of wap* in terms of Double Limit Property.

Fact 3.2. *([5, 20]) A function $f \in C^b(G)$ is wap iff the induced map $F : G \times G \to \mathbb{R}$ defined by $F(g,h) := f(gh)$ has DLP, that is, for every pair of sequences $\{g_n\}, \{h_m\}$ in G,*

$$\lim_n \lim_m f(g_n h_m) = \lim_m \lim_n f(g_n h_m)$$

whenever both of the limits exist.

Lemma 3.3. *Let a topological group G admit a left-invariant metric d with DLP. Then G is reflexively representable.*

Proof. Define the norm $||g|| := d(e,g)$. Then $||g_n h_m|| = d(g_n^{-1}, h_m)$.

By Grothendieck's characterization (Fact 3.2), the bounded function

$$\phi_e : G \to \mathbb{R} \quad g \mapsto \frac{1}{1 + ||g||}$$

is wap. Then its left or right translations are also wap. Therefore, for every fixed $z \in G$ the function

$$\phi_z : G \to \mathbb{R} \quad g \mapsto \frac{1}{1 + ||gz||}$$

is a wap function. Since the norm generates the original topology on G, the family $\{\phi_z\}_{z \in G}$ of wap functions separates points and closed subsets of G. Hence, by Fact 2.1 we can conclude that $G \in \mathbf{Ref_R}$. $\square$

Lemma 3.4. *The norm in the Banach space $L_{2k}(\mu)$ $(k \in \mathbb{N})$ has DLP.*

Proof. We have to show that for every pair $\{u_n(t)\}, \{v_m(t)\}$ of sequences in $L_{2k}(\mu)$

$$\lim_n \lim_m \|u_n + v_m\| = \lim_m \lim_n \|u_n + v_m\|$$

whenever both of the limits exist. We can suppose that the sequences are norm-bounded. Computing the norm $\|u_n + v_m\|$, we get

$$\|u_n + v_m\| = \left(\int_\Omega (u_n(t) + v_m(t))^{2k} d\mu\right)^{\frac{1}{2k}}$$

$$= \left(\|u_n\|^{2k} + \sum_{i=1}^{2k-1} C_{2k}^i < u_n^{2k-i}, v_m^i > + \|v_m\|^{2k}\right)^{\frac{1}{2k}}$$

where

$$< u_n^{2k-i}, v_m^i > = \int_\Omega u_n^{2k-i}(t) v_m^i(t) d\mu$$

and

$$u_n^{2k-i} \in L_{\frac{2k}{2k-i}}(\mu), \qquad v_m^i \in L_{\frac{2k}{i}}(\mu) = L^*_{\frac{2k}{2k-i}}(\mu).$$

Passing to subsequences if necessary, we may assume that there exist

$$\lim_n \|u_n\|, \qquad \lim_m \|v_m\|.$$

By reflexivity, every bounded subset of $L_p(\mu)$ $(p > 1)$ is relatively weakly compact and hence, relatively sequentially compact (as it follows by the classical Eberlein-$\check{S}$mulian theorem). Therefore, we can suppose in addition (again by passing to subsequences) that there exist weak limits

$$weak - \lim_n u_n^{2k-i}, \qquad weak - \lim_m v_m^i$$

for every $i \in \{1, 2, \cdots, 2k-1\}$.

Now, in order to complete the proof of the lemma it remains to observe that for every *reflexive* Banach space X and bounded subsets $A \subset X$, $B \subset X^*$, the canonical duality

$$A \times B \to \mathbf{R}, \qquad < x, f > = f(x)$$

has DLP. This fact easily follows using once again the Eberlein-$\check{S}$mulian theorem. $\square$

By Lemmas 3.4 and 3.3 we get $\boxed{L_{2k}(\mu) \in \mathbf{Ref_R} \quad \forall k \in N}$.

Now, we prove the second part of Theorem 3.1.

We need the following fundamental fact.

Fact 3.5. *(Aharoni-Maurey-Mityagin* [2]*)*
For $2 < p < \infty$, an infinite-dimensional $L_p(\mu)$ space is not uniformly embedded into a Hilbert space.

Additional information about uniform embeddings into Hilbert spaces can be found in [4].

The following Lemma is inspired by [14, Counterexample 2.13].

Lemma 3.6. *The uniform space $(Is(l_2)_s, \mathcal{L})$, where $\mathcal{L}$ denotes the left uniformity on $Is(l_2)_s$, can be uniformly embedded into l_2.*

Proof. Since $Is(l_2)_s$ is separable and metrizable, there exists a sequence $\{v_n\}$ in l_2 such that:

(a): $\|v_n\| = \frac{1}{2^n}$
(b): $\{\tilde{v}_n : Is(l_2) \to l_2, \quad g \mapsto gv_n\}_n$ generates the left uniformity on $Is(l_2)$.

Denote by $B_{\frac{1}{2^n}}$ the closed $\frac{1}{2^n}$-ball centered at the origin with its usual uniformity. Then the uniform product $\prod_n B_{\frac{1}{2^n}}$ is uniformly embedded in a natural way into the l_2-sum $(\sum_n (l_2)_n)_{l_2}$ of countably many copies of the Hilbert space l_2. Eventually we have

$$Is(l_2)_s \overset{unif}{\hookrightarrow} \prod_n B_{\frac{1}{2^n}} \overset{unif}{\hookrightarrow} (\sum_n (l_2)_n)_{l_2} \longleftrightarrow l_2$$

□

Now we prove the second part (ii) of Theorem 3.1. Assuming the contrary, suppose that $L_p[0,1]$ $(2 < p < \infty)$ is unitarily representable in the (infinite-dimensional) Hilbert space H. Then, using the separability of $L_p[0,1]$ and passing to an appropriate *separable* infinite-dimensional closed linear subspace E of H, we can suppose even that there exists a topological group embedding $L_p[0,1] \hookrightarrow Is(E)_s$. Since $Is(E)_s$ and $Is(l_2)_s$ are topologically isomorphic, it is clear that Fact 3.5 and Lemma 3.6 will lead to a contradiction.

Therefore, we have $\boxed{L_p[0,1] \notin \mathbf{Hilb_R} \quad (2 < p < \infty)}$.

Theorem 3.1 is proved.

Proposition 3.7. $C[0,1], c_0 \notin \mathbf{Hilb_R}$.

Proof. Follows from Lemma 3.6 because l_2 is not uniformly universal space for separable Banach spaces (Enflo [10]) in contrast to c_0 (Aharoni [1]) or $C[0,1]$ (Banach-Mazur). □

4. QUESTIONS

Question 4.1. Is it true that every Banach space X, as a topological group, is reflexively representable ? Or, equivalently, does $WAP(X)$ separate points and closed subsets ?

By Lemma 3.4, the answer is yes for $L_p(\mu)$ spaces, where $p = 2k$ is an arbitrary even integer. Using results of Shoenberg [21], we easily can extend this result to the case of $1 \leq p \leq 2$. Indeed, the function $f(v) = e^{-\|x\|^p}$ is positive definite (and hence, wap) on $L_p(\mu)$ spaces for every $1 \leq p \leq 2$. Moreover, Chaatit [6] proved that every *stable Banach space* in the sense of Krivine-Maurey [13], in particular, every separable $L_p(\mu)$ space $(1 \leq p < \infty)$, is reflexively representable.

By Lemma 3.3, a closely related question is: for which Banach spaces does the original norm (or its some renorming) satisfy DLP ?

It is easy to show that the original norm of the Banach space c_0 does not satisfy DLP. Indeed, define $u_n := e_n$ (the standard basis vectors) and $v_m := \Sigma_{i=1}^m e_i$. Then the corresponding double limits are 1 and 2.

Lemma 3.3 suggests also the following questions:

Question 4.2. Let G be a reflexively representable group. Is it true that the topology of G is generated by a family of left-invariant pseudometrics with DLP ?

Question 4.3. Which (non-locally compact) metrizable topological groups admit a left-invariant metric with DLP ?

For instance, it would be interesting to know for which metric spaces (X, d) does the corresponding *Graev's metric* ([11, 24]) on the free group $F(X)$ satisfy DLP ?

Lemma 3.6 leads to the following natural question.

Question 4.4. Suppose a topological group G is such that the left uniformity of G admits a uniform embedding into l_2. Is then G unitarily representable ?

Acknowledgment. We thank to S. Gao, G. Godefroy and Y. Tseitlin for helpful suggestions. We are grateful to the referee for several improvements.

References

1. I. Aharoni, *Every separable Banach space is Lipschitz equivalent to a subset of c_0*, Israel J. of Math., **19** (1974), 284-291.
2. I. Aharoni, B. Maurey, B.S. Mityagin, *Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces*, Israel J. Math., 52:3, 1985, 251-265.
3. W. Banasczyk, *Additive subgroups of topological vector spaces*, Lecture Notes in Math. 1466, Springer-Verlag, 1991.
4. Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis, vol. 1, Amer. Math. Soc., Colloquium Publications **48**, Providence, Rhode Island, 2000.
5. J.F. Berglund, H.D. Junghenn, P. Milnes, *Analysis on semigroups*, Canad. Math. Soc., Wiley-Interscience Publication, 1989.
6. F. Chaatit, *A representation of stable Banach spaces*, Arch. Math., **67** (1996), 59-69.
7. J.D. Lawson, *Joint continuity in semitopological semigroups*, Illinois J. Math., **18** (1974), 275–285.
8. K. de Leeuw, I. Glicksberg, *Applications of almost periodic compactification*, Acta Math. **105** (1961), 63–97.
9. W.F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc., **67** (1949), 217–240.
10. P.Enflo, *On a problem of Smirnov*, Ark. Math., **8** (1969), 107-109.
11. M.I. Graev, *Theory of topological groups I*, Uspekhi Mat. Nauk, (N.S.) **5** (1950), 3-56 (in Russian).
12. W. Herer, J.P.R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann., **213** (1975), 203-210.
13. J.L. Krivine, B. Maurey, *Espaces de Banach stables*, Israel J. Math., **39** (1981), 273-295.
14. M.G. Megrelishvili, *Group representations and construction of minimal topological groups*, Topology and its applications, **62** (1995), 1-19.
15. M.G. Megrelishvili, *Eberlein groups and compact semitopological semigroups*, Bar-Ilan University preprint, 1998.

16. M.G. Megrelishvili, *Operator topologies and reflexive representability of groups*, BIUMCS 98/52, Bar-Ilan University preprint, 1998.
17. M.G. Megrelishvili, *Operator topologies and reflexive representability*, In: "Nuclear groups and Lie groups" , Research and Exposition in Math. series, vol. 24, Helderman Heldermann–Verlag, Berlin, 197-208.
18. M.G. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial*, Semigroup Forum, **63:3** (2001), 357-370.
19. V. Pestov, *Topological groups: where to from here ?*, - in: Proceedings 14th Summer Conf. on General Topology and its Appl. (C.W. Post Campus of Long Island University, August 1999), Topology Proceedings 24 (1999), 421–502.
20. W. Ruppert, *Compact semitopological semigroups*, Lecture Notes in Math. 1079, Springer-Verlag, 1984.
21. I.J. Shoenberg, Metric spaces and positive definite functions, Trans. Amer. Math. Soc. **44**, 1938, 522-536.
22. A. Shtern, *Compact semitopological semigroups and reflexive representability of topological groups*, Russ. J. Math. Physics 2 (1994), 131-132.
23. S. Teleman, *Sur la représentation linéaire des groupes topologiques*, Ann. Sci. Ecole Norm. Sup. **74**(1957), 319-339.
24. V.V. Uspenskij, *Free topological groups of metrizable spaces*, Math. USSR-Izvestiya, **37** (1991), 657-680.

Internet address: http://www.cs.biu.ac.il/˜megereli

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: megereli@macs.biu.ac.il