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OPERATOR TOPOLOGIES AND REFLEXIVE REPRESENTABILITY OF GROUPS

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Abstract. Using the concept of fragmentability, we show that weakly continuous group representations are frequently strongly continuous. We show that if a Banach (or, even, Frechet) space X has the Radon-Nikodym property (RNP), then the weak and strong operator topologies coincide on every bounded (respectively, equicontinuous) subgroup G of $GL(X)$. We also strengthen a result of Shtern on reflexive representability of topological groups. We show that three natural problems become closely related: a problem of Ruppert about the existence of sufficiently many wap functions, Archangelskii's problem on the existence of sufficiently many minimal topological groups, and a question about the existence of uniformly universal reflexive Banach spaces.

§1. Fragmentable subsets of locally convex spaces.

It is now well known that the concept of fragmentability in the sense of Jayne and Rogers [JR,JNR,Na2,Fa] is very powerful in various aspects of Banach space theory. In [Me2] we deal with continuity problems of linear semigroup representations using as a main tool fragmentability and its natural generalizations. As in [Me2, §3], we say that a subset A of a locally convex space (in short, l.c.s.) X is *fragmented* if, for every non-empty subset B of A and every element ε of the natural uniform structure of X, there is a weakly open subset W of X such that $B \cap W$ is non-empty and ε-small. If X is a Banach space then we obtain the original definition of Jayne and Rogers [JR].

Among various applications of Namioka's joint continuity theorem [Na1], we recall that every relatively weakly compact subset of a Banach space is fragmented [Na2]. We need the following locally convex version.

Lemma 1.1. *Every relatively weakly compact subset A of an l.c.s. X is fragmented in X.*

Proof. See [Me2; Proposition 3.5]. □

We say that an l.c.s. X is *bound-fragmented* (in short, (BF)) if every bounded subset A of X is fragmented. For instance, by Lemma 1.1 , every semireflexive l.c.s. X is (BF). In order to formulate a stronger result, we need a locally convex version of Rieffel's concept of a dentable set. A non-empty subset A of an l.c.s. X is said

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to be *dentable* if for every neighborhood V of 0, there exists a point a in A such that

$$a \notin cl(con(A \setminus (a+V)),$$

where cl(con) denotes the closed convex hull. A space X is called *dentable* if every bounded subset of X is dentable. For the extent of the class of dentable spaces see [Eg]. It follows from the results of [Ch] that every Frechet space with the Radon-Nikodym property (RNP) is dentable. It is easy to show (for the case of a Banach space X, see [JR; §3]) that every dentable subset is fragmented. Therefore we obtain

Lemma 1.2. *Every dentable l.c.s. X (e.g., Frechet space with (RNP)) is (BF).*

Recall that a Banach space X is said to have *the point of continuity property* (in short, (PCP) if each bounded weakly closed subset C of X admits a point of continuity of the identity map $(C, \text{weak}) \to (C, \text{norm})$.

Here we reformulate a known result in terms of (BF).

Lemma 1.3. [JR; p. 55] *Let X be a Banach space. Then X is (BF) if and only if X has the (PCP).*

§2. When does "weak imply strong?".

Let X be an l.c.s. By $GL(X)$ we denote the group of all linear automorphisms. The group of all linear isometries of a Banach space X is denoted by $Is(X)$. If $Is(X)$ is endowed with the weak (strong) operator topology, we sometimes indicate this by writing $Is(X)_w$ (respectively, $Is(X)_s$).

Definition 2.1. We say that a subgroup G of $GL(X)$ is *light* if the weak and strong operator topologies coincide on G.

It is a standart fact that for every Hilbert space H, the group $Is(H)$ of all unitary operators is light. Moreover, every bounded amenable subgroup G of $GL(H)$ is light. Indeed, it suffices to remark that every bounded Hilbert representation of an amenable group is equivalent to a unitary representation. In general, $Is(X)$ may not be light (see Remarks 2.6). However, we show below (Theorem 2.4) that every bounded subgroup (amenable or not) G of $GL(X)$ is light for Banach spaces X with (PCP) (e.g., with (RNP)).

The general question about sufficient conditions for lightnesss of subgroups G in $GL(X)$ is closely related to the classical question: when are weakly continuous group representations strongly continuous? From the extensive literature on this topic, we mention here only [La, He1, He2, Mo]. Roughly speaking, under strong restrictions on G and X (for example, if G is topologically a Čech-complete, or Namioka space, enabling the use of Namioka's theorem) "weak implies strong." In this section, we obtain results that show that in many cases the "goodness" of X is enough.

Throughout the paper, all spaces are assumed to be Hausdorff and all linear spaces are real. The filter of all neighborhoods (*nbd*'s) of a point x is denoted by

N_x. Let $\pi : G \times X \to X$, $\pi(g,x) := gx$ be an action of a group G on a linear space X. Then the *orbit maps* $\tilde{x} : G \to X$ and translations $\tilde{g} : X \to X$ are defined by the rules $\tilde{x}(g) = gx = \tilde{g}(x)$. The action π is said to be *linear* if $\tilde{g}$ is linear for every $g \in G$. In this case π induces a homomorphism $G \to GL(X)$, a group representation of G in X.

Recall that a vector $z \in X$ is said to be (*weakly*) *continuous* if the orbit map $\tilde{z} : G \to Z$ is (*weakly*) *continuous*. Following [Eb,LG1], we say that $z \in X$ is *weakly almost periodic* (*wap*) if the orbit $Gz := \{gz | g \in G\}$ is relatively weakly compact. We will say that z is *locally weakly almost periodic (lwap)* if the set $Vz := \{gz | g \in V\}$ is relatively weakly compact for a certain nbd V of the identity $e \in G$.

A subset B of G is called (uniformly) equicontinuous if the family $\tilde{B} := \{\tilde{g} : X \to X | g \in B\}$ is (uniformly) equicontinuous. Recall that a topologized group G is said to be *left topological* if every left translation $L_g : G \to G$, $L_g(h) = gh$ is continuous.

Theorem 2.2. *Let $\pi : G \times X \to X$ be a (not necessarily linear) action of a left topological group G on an l.c.s. X, and let $z \in X$ such that:*

(a) *For every $g \in G$ there exists an nbd U of g such that the set U^{-1} is uniformly equicontinuous;*
(b) *There exists an nbd V of the identity $e \in G$ such that Vz is a fragmented subset of X;*
(c) *z is a weakly continuous vector.*

Then z is a continuous vector.

Proof. We have to show that $\tilde{z} : G \to X$ is continuous at arbitrary fixed $g_0 \in G$. For a fixed element ε of the standard uniformity μ on X, we will find an nbd O of g_0 such that $\tilde{z}(O) = Oz$ is ε-small. Since algebraically G is a group, it follows from condition (a) that the g_0-translation $\tilde{g}_0 : X \to X$ is μ-uniformly continuous. Again by (a), we can pick an nbd U of e such that U^{-1} is uniformly equicontinuous. Hence g_0U^{-1} is uniformly equicontinuous, too. Therefore, for a fixed $\varepsilon \in \mu$ there exists an element $\delta \in \mu$ such that $(gx, gy) \in \varepsilon$ for every $(x,y) \in \delta$ and every $g \in g_0U^{-1}$. By condition (b), there exists an open nbd V of e such that Vz is fragmented in X. Every non-empty subset of a fragmented subset is fragmented as well. Therefore, without any loss of generality, we may suppose, in addition, that $V \subseteq U$. By the definition, the fragmentability of Vz implies that for a certain weakly open subset W of X the intersection $Vz \cap W$ is non-empty and δ-small. Then the identities

$$\tilde{z}(V \cap \tilde{z}^{-1}(W)) = \tilde{z}(V) \cap W = Vz \cap W$$

imply, in particular, that the set $V \cap \tilde{z}^{-1}(W)$ is non-empty. Moreover, this set is open in G by weak continuity of $\tilde{z}$. Choose arbitrarily $v_0 \in V \cap \tilde{z}^{-1}(W)$ and set $p := g_0v_0^{-1}$, $O := p(V \cap \tilde{z}^{-1}(W))$. Since G is a left topological group, O is also open in G. Moreover, by the construction, O is an nbd of g_0. Clearly, $\tilde{z}(O) = g_0v_0^{-1}\tilde{z}(V \cap \tilde{z}^{-1}(W))$. Since $V \in U$, we have $p = g_0v_0^{-1} \in g_0U^{-1}$. On the other hand, $\tilde{z}(V \cap \tilde{z}^{-1}(W))$ is δ-small. By the choice of δ, we obtain that $\tilde{z}(O)$ is ε-small. $\square$

Remark 2.3. (1) If z is an lwap vector (under a not necessarily linear action), then the condition (b) is clearly satisfied, by Lemma 1.1. Applying Theorem 2.2, in particular, to the case of wap representations, we obtain a generalization of [Tr; Corollary 5.3].

(2) If G is locally compact, then (b) can be omitted.

(3) Recall [LG2] that if G is a locally compact group and X is Banach (for barreled X, see [Mo]), then every linear weakly continuous representation is strongly continuous. The original proof in its main part is essentially based on Haar integration, convolution and the linearity of the representation. A transparent proof follows from Theorem 2.2. Moreover, we may suppose that π *is not linear*, provided of course that π is locally uniformly equicontinuous. For linear π, the latter is guaranteed (as in [Mo]) by the generalized principle of uniform boundedness.

Theorem 2.4. *Let X be an l.c.s. and (BF), and let G be a subgroup of $GL(X)$. Suppose that there exists a weakly open nbd U of e in G such that U and U^{-1} are each equicontinuous. Then G is a light subgroup of $GL(X)$.*

Proof. It suffices to show that for an arbitrary fixed element z of X, the orbit map $\tilde{z} : G \to X$ is continuous whenever G is endowed with the weak operator topology. We can apply Theorem 2.2. Indeed, the condition (c) is trivially satisfied. Choose $U \in N_e$ such that U and U^{-1} are equicontinuous, and hence uniformly equicontinuous, since the given action is linear. Therefore, (a) holds. In order to check (b), observe that Uz is fragmented since it is a bounded subset of a bound-fragmented l.c.s. X. $\square$

Theorem 2.5. *A subgroup G of $GL(X)$ is light under each of the following conditions:*

(1) *X is a Banach space with (PCP) and G is norm-bounded.*
(2) *X is a Frechet l.c.s. with (RNP) and G is equicontinuous.*

Proof. Apply Theorem 2.4 making use of Lemmas 1.3 and 1.2. $\square$

Remarks 2.6.

(1) Some boundedness condition on G is necessary even for the case of the Hilbert space $X := \ell_2$. Indeed, $GL(\ell_2)$ is not light.

(2) The group G may not be replaced, in general, by semigroups. Indeed, the semigroup $\operatorname{cont}(\ell_2) := \{g \in GL(\ell_2) \mid \|g\| \leq 1\}$ of all linear contractions endowed with the weak operator topology is compact in contrast to the strong operator topology.

(3) If the geometric behavior of X is "bad," then even $Is(X)$ may not be light in $GL(X)$. The following general construction, providing counterexamples, is based on an idea of Helmer [He1]. Let Y be a compact space, and let G be a subgroup of the group $H(Y)$ of all autohomeomorphisms of Y. Denote by G_p the group G endowed with the topology of pointwise convergence. Then the evaluation map $\alpha : G_p \times Y \to Y$ is separately continuous. Assume that α is not jointly continuous

and that G_p is a k-space (for example, in [He1, Example 13], Y is the Euclidean 2-cell $[-1,1]^2$ and G_p is a topological group homeomorphic to the space of all rationals). Consider the induced action

$$\pi : G_p \times C(Y) \to C(Y), \quad (g \circ f)(y) = f(g^{-1}y)$$

and the induced injective group homomorphism $j : G_p \to Is(X)_w$, where $X := C(Y)$. By a classical result of Grothendieck [Gr, Theorem 5], for bounded subsets of $C(Y)$, pointwise compactness and weakly compactness are equivalent. This implies that for every compact subset K of G_p and every function $f \in C(Y)$ the restriction of the orbit map $\tilde{f} : G_p \to X$ on K is weakly continuous. Since G_p is a k-space, we can conclude that $\tilde{f}$ is weakly continuous. Therefore, j is continuous. Moreover, it is easy to see that j is a topological embedding. Now we can establish that the subgroup $j(G)$ (and, hence, $Is(X)$) is not light in $GL(X)$. Indeed, otherwise we will obtain that the restricted dual action

$$\pi^*_B : G_p \times B(X^*)_{w^*} \to B(X^*)_{w^*}, \quad (g \circ \psi)(f) = \psi(g^{-1} \circ f)$$

is jointly continuous, where $B(X^*)_{w^*}$ is the unit ball of X^* endowed with the weak* topology. Then the original action α (being canonically equivalent to a subaction of π^*_B) is jointly continuous, too. This contradicts our assumption on α.

§3. Reflexive representability and related problems.

Recall [BJM] that for every semitopological semigroup S, the compactification $u\colon S \to S^w$ induced by the algebra $WAP(S)$ of all weakly almost periodic (wap) functions on S is just the *universal semitopological compactification* of S.

Definition 3.1. We say that G is an *Eberlein group* if one of the following equivalent conditions hold:

(i) $WAP(G)$ separates the points from the closed sets;
(ii) $u\colon G \to G^w$ is a topological embedding;
(iii) G is a topological subgroup of a compact semitopological semigroup.

For instance, every Hausdorff locally compact group is Eberlein.

Problem 3.2. (Ruppert [Ru1; p. 114]) *Find a Hausdorff topological group which is not Eberlein.*

This problem is open even for a case when G algebraically is isomorphic to $\mathbb{Z}$ (cf. [Ru2]).

Every topological group G is a topological subgroup of $Is(X)_s$ for a certain Banach space X. Take, for example (as in Teleman [Te]) $X = C_r(G)$, the space of all bounded right uniformly continuous functions on G. The natural question is whether X may be taken to be Asplund, reflexive or even Hilbert. The last case, that is the case of unitary representations, in contrast to the case of a reflexive X, is extensively studied. There are, in particular, many counterexamples (see for example [Ba]). It turns out that the case of a reflexive X gives a characterization of Eberlein groups.

The following theorem strengthens a result of A. Shtern [Sh].

Theorem 3.3. *For every topological group G, the following conditions are equivalent:*

(i) *G is an Eberlein group;*
(ii) *G is a topological subgroup of $Is(X)_s$ for a certain reflexive Banach space X.*

Proof. Every reflexive Banach space has the (RNP). Therefore, by Theorem 2.5, the group $Is(X)$ is light for every reflexive Banach space X. Hence $Is(X)_w$ and $Is(X)_s$ are the same. On the other hand, by a result of Shtern [Sh], a topological group G can be embedded into $Is(X)_w$ for a certain reflexive Banach space X if and only if G is a topological subgroup of some Hausdorff compact semitopological semigroup. Now use the equivalence of the conditions (i), (ii) and (iii) in Definition 3.1. □

The following Proposition is a modificaton of [Me1; Counterexample 3.13].

Proposition 3.4. *Let G be a separable metrizable group and let $L(G)$ denote its left uniform structure. If G is Eberlein, then $(G, L(G))$, as a uniform space, is embedded into a separable reflexive Banach space Y. Moreover, the unitary group $Is(l_2)_s$ is left-uniformly embedded into l_2.*

Proof. By Theorem 3.3, G is a topological subgroup of $Is(X)_s$ for a certain reflexive Banach space X. Without loss of generality we may suppose that X is separable. By the definition of the strong operator topology, the system of all orbit maps on G generates the uniformity $L(G)$. Since G is second countable, we may suppose that there exists a sequence z_n in X such that the sequence of orbit maps

$$\tilde{z}_n \colon G \to X \quad , \quad \tilde{z}_n(g) = gz_n$$

generates $L(G)$. Moreover, we may suppose that $\|z_n\| = 2^{-n}$. Now without loss of generality we may suppose that X is separable. Consider the ℓ_2-product (see [Da, p. 35]) $\prod_2 X_n$ of the family $\{(X_n, \| \, \|_n) \mid n \in \mathbb{N}\}$, where each $(X_n, \| \, \|_n)$ is a copy of $(X, \| \, \|)$. The *reflexive* Banach space $\prod_2 X_n$ is denoted by Y. Since $\|z_n\| = 2^{-n}$, it is easy to show that the diagonal product map

$$\gamma \colon G \to \prod_2 X_n = y, \qquad \gamma(g) = (gz_n)$$

provides the desired uniform embedding.

Actually the same proof works in the case of $Is(l_2)_s$. □

We say that a separable Banach space U is *uniformly universal* if every separable Banach space, as a uniform space, can be uniformly embedded into U. In answer to a question by Yu. Smirnov, Enflo [En] found, in 1969, a countable metrizable uniform space which is not uniformly embedded into a Hilbert space. That is, ℓ_2 is not uniformly universal. However, it is not clear if "Hilbert" may be replaced by "reflexive." In [Ah] Aharoni proved that c_0 is uniformly universal. Therefore, it easily follows from the second part of Proposition 3.4, that c_0, as a topological group, is not embedded into $Is(H)_s$ for any Hilbert space H.

Question 3.5. *Is it true that $C[0,1], c_0$ (or any other uniformly universal space) is a uniform subset of a certain reflexive Banach space? Equivalently, is it true that there exists a uniformly universal reflexive Banach space?*

Note that there is no *linearly* universal separable reflexive Banach space [Sz]. Note also that there is no *Lipshitz embedding* of c_0 into a reflexive Banach space [Ma]. For additional results in the theory of uniform (or Lipschitz) Banach embeddings, we refer to [Li, AMM].

In order to explain another link to Problem 3.2, let us recall that a Hausdorff topological group G is said to be *minimal* [St, Di] if there is no strictly coarser Hausdorff group topology on G.

Question 3.6. *(Arhangel'skii (cf.* [Ar; problem 6.6] *or* [Di; Question 2.9]*) Is it true that every Hausdorff topological group G is a quotient of a minimal topological group?*

The following proposition implies a positive answer in the case of an Eberlein group G.

Proposition 3.7. *Every Eberlein group is a group retract of a minimal topological group.*

Proof. In [Me2; Theorem 6.12], we proved that for every Asplund Banach space X every topological subgroup G of $Is(X)_s$ is a group retract of a minimal topological group. Every reflexive Banach space is Asplund. Therefore, it suffices now to apply Theorem 3.3. □

If the answer to Arhangel'skii's question or to Question 3.5 is negative, then this immediately will provide the solution to Ruppert's problem.

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