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Operator topologies and reflexive representability

Michael G. Megrelishvili

Abstract. Using the concept of fragmentability, we show that weakly continuous group representations are frequently strongly continuous. We show that if a Banach (or, even, Frechet) space X has the Radon-Nikodym property RNP, then the weak and strong operator topologies coincide on every bounded (respectively, equicontinuous) subgroup G of $GL(X)$. We also strengthen a result of Shtern on reflexive representability of topological groups.

1. Fragmentable subsets of locally convex spaces

It is now well known that the concept of fragmentability in the sense of Jayne and Rogers [14, 13, 29, 9] is very powerful in various aspects of Banach space theory. In [22] we deal with continuity problems of linear semigroup representations using as a main tool fragmentability and its natural generalizations. As in [22], we say that a subset A of a locally convex space (in short, l.c.s.) X is *fragmented* if, for every non-empty subset B of A and every element ε of the natural uniform structure of X, there is a weakly open subset W of X such that $B \cap W$ is non-empty and ε-small. If X is a Banach space then we obtain the original definition of Jayne and Rogers [14].

Among various applications of Namioka's joint continuity theorem [28], we recall that every relatively weakly compact subset of a Banach space is fragmented [29]. We need the following locally convex version.

Lemma 1.1. *Every relatively weakly compact subset A of an l.c.s. X is fragmented in X.*

Proof. See [22, Proposition 3.5]. ∎

We say that an l.c.s. X is *bound-fragmented* (in short, BF) if every bounded subset A of X is fragmented. For instance, by Lemma 1.1, every semireflexive l.c.s. X is BF. In order to formulate a stronger result, we need a locally convex version of Rieffel's concept of a dentable set. A non-empty subset A of an l.c.s. X is said to be *dentable* if for every neighborhood V of 0, there exists a point a in A such that

$$a \notin cl(con(A \setminus (a+V))),$$

where cl(con) denotes the closed convex hull. A space X is called *dentable* if every bounded subset of X is dentable. For the extent of the class of dentable spaces see [7, section 2]. It follows that Frechet spaces with the Radon-Nikodym property (in short, RNP) and semireflexive l.c.s. are dentable. It is easy to show (for the case of a Banach space X, see [14, section 3]) that every dentable subset is fragmented. Therefore we obtain the following result.

Lemma 1.2. *Every dentable l.c.s. X (e.g., Frechet space with RNP) is BF.*

It is well-known (see [20]) that a Banach space is dentable if and only if it has RNP.

Recall that a Banach space X is said to have *the point of continuity property* (in short, PCP) if each bounded weakly closed subset C of X admits a point of continuity of the identity map $(C, \text{weak}) \to (C, \text{norm})$.

Here we reformulate a known result in terms of BF.

Lemma 1.3. *[14, p. 55] Let X be a Banach space. Then X is BF if and only if X satisfies PCP.*

2. When does "weak imply strong?"

Let X be an l.c.s. By $End(X)$ $(GL(X))$ we denote the set of all continuous linear endomorphisms (resp., automorphisms) of X. The dual Banach space of a Banach space $(X, \|\cdot\|)$ will be denoted by X^*. We use the following notation:

$$B(X) := \{x \in X \mid \|x\| \leq 1\} \quad Cont(X) := \{s \in L(X) \mid \|s\| \leq 1\}.$$

The group of all linear isometries of X will be denoted by $Is(X)$. The *strong, strong** and *weak operator topology* (denote T_s, T_{s^*} and T_w) on $L(X)$ is the weakest topology generated respectively by the following systems of maps:

- $\{\tilde{x} : L(X) \to X, \ \tilde{x}(s) = sx \mid x \in X\}$,
- $\{\tilde{f} : L(X) \to X^*, \ \tilde{f}(s)(x) = f(sx) \mid f \in X^*\}$,
- $\{\psi_{x,f} : L(X) \to \mathbb{R}, \ \psi_{x,f}(s) = f(sx) \mid x \in X, f \in X^*\}$.

If a subset P of $L(X)$ is endowed with one of the following subspace topologies $T_s \big|_P, T_{s^*} \big|_P, T_w \big|_P$, then we indicate this by writing P_s, P_{s^*} and P_w, respectively. A subset A of X endowed with its usual weak topology is denoted by A_w. By [37] the unitary group $Is(H)$ is weakly dense in $Cont(H)$.

Fact 2.1. (Banach-Bourbaki Theorem) A Banach space X is reflexive iff $B(X)_w$ is compact.

Fact 2.2. The semitopological semigroup $Cont(X)_w$ is compact iff X is reflexive.

Proof. The compactness of $Cont(X)_w$ for a reflexive X is well-known (see, for instance [17, Th. 3.1]). The converse implication follows from Fact 2.1, taking into account that for any fixed vector x_0 with $\|x_0\| = 1$, the map

$$Cont(X)_w \to B(X)_w, \quad s \mapsto sx_0$$

is continuous and *onto*. Indeed, take a continuous functional f on X such that $f(x_0) = 1$ and $\|f\| = 1$. For every $z \in B(X)$ assign to the pair (f, z) the following linear operator

$$A_{f,z} : X \to X, \quad A_{f,z}(x) = f(x)z$$

Clearly, $A_{f,z}$ is a contraction of X moving x_0 into z. ∎

Remark 2.3. In general, $T_s \mid_{Is(X)} \neq T_{s^*} \mid_{Is(X)}$. Indeed, there are many continuous norm invariant continuous linear group actions on Banach spaces X such that the corresponding dual actions on X^* are not continuous. Consider, for example, $X = l_1$ and define a subgroup $S(\mathbb{N})$ of $Is(X)_s$ consisting of all permutations of "coordinates." Then the dual action of $S(\mathbb{N})$ on $l_1^* = m$ is not continuous. Hence, it is not even true that $T_{s^*} \mid_{Is(X)} \subseteq T_s \mid_{Is(X)}$. Note also that if X is *Asplund* (by Stegall's result [33], it is equivalent to saying that the dual X^* satisfies the Radon-Nikodym property), then, necessarily, $T_{s^*} \mid_{Is(X)} \subseteq T_s \mid_{Is(X)}$ (cf. [22, Corollary 6.9]).

Definition 2.4. We say that a subgroup G of $GL(X)$ is *light* if the weak and strong operator topologies coincide on G.

It is a standart fact that for every Hilbert space H, the unitary group $Is(H)$ is light. Moreover, every bounded amenable subgroup G of $GL(H)$ is light. Indeed, it suffices to remark that every bounded Hilbert representation of an amenable group is equivalent to a unitary representation. In general, $Is(X)$ may not be light (see Remarks 2.6). However, we show below (Theorem 2.4) that every bounded subgroup, amenable or not, G of $GL(X)$ is light for Banach spaces X with PCP (e.g., with RNP).

The general question about sufficient conditions for lightnesss of subgroups G in $GL(X)$ is closely related to the classical problem: find sufficient conditions which guarantee that a given weakly continuous group representation is strongly continuous. From the extensive literature on this topic, we mention here only [27, 11, 12, 18, 16]. Roughly speaking, under strong restrictions on G and X (for example, if G is topologically a Čech-complete, or Namioka space, enabling the use of Namioka's theorem) "weak implies strong." In this section, we obtain results that show that in many cases the "goodness" of X is enough.

Throughout the paper, all spaces are assumed to be Hausdorff and all linear spaces are real. The filter of all neighbourhoods of a point x is denoted by N_x. Let $\pi : G \times X \to X$, $\pi(g, x) := gx$ be an action of a group G on a linear space X. Then the *orbit maps* $\tilde{x} : G \to X$ and translations $\tilde{g} : X \to X$ are defined by the rules $\tilde{x}(g) = gx = \tilde{g}(x)$. The action π is said to be *linear* if $\tilde{g}$ is linear for every $g \in G$. In this case π induces a homomorphism $G \to GL(X)$, a group representation of G in X.

Recall that a vector $z \in X$ is said to be (*weakly*) *continuous* if the orbit map $\tilde{z} : G \to X$ is (*weakly*) *continuous*. Following [6, 17] we say that $z \in X$ is *weakly almost periodic* (*wap*) if the orbit $Gz := \{gz | g \in G\}$ is relatively weakly compact. We will say that z is *locally weakly almost periodic (lwap)* if the set $Vz := \{gz | g \in V\}$ is relatively weakly compact for a certain neighbourhood V of the identity $e \in G$.

A subset B of G is called (uniformly) equicontinuous if the family $\tilde{B} := \{\tilde{g} : X \to X | g \in B\}$ is (uniformly) equicontinuous. Recall that a topologized group G is said to be *left topological* if every left translation $L_g : G \to G,\ L_g(h) = gh$ is continuous.

Theorem 2.5. *Let $\pi : G \times X \to X$ be a (not necessarily linear) action of a left topological group G on an l.c.s. X, and let $z \in X$ such that:*

(a) For every $g \in G$ there exists a neighbourhood U of g such that the set U^{-1} is uniformly equicontinuous;

(b) There exists a neighbourhood V of the identity $e \in G$ such that Vz is a fragmented subset of X;

(c) z is a weakly continuous vector.

Then z is a continuous vector.

Proof. We have to show that $\tilde{z} : G \to X$ is continuous at arbitrary fixed $g_0 \in G$. For a fixed element ε of the standard uniformity μ on X, we will find an neighbourhood O of g_0 such that $\tilde{z}(O) = Oz$ is ε-small. Since algebraically G is a group, it follows from condition (a) that the g_0-translation $\tilde{g}_0 : X \to X$ is μ-uniformly continuous. Again by (a), we can pick a neighbourhood U of e such that U^{-1} is uniformly equicontinuous. Hence g_0U^{-1} is uniformly equicontinuous, too. Therefore, for a fixed $\varepsilon \in \mu$ there exists an element $\delta \in \mu$ such that $(gx, gy) \in \varepsilon$ for every $(x, y) \in \delta$ and every $g \in g_0U^{-1}$. By condition (b), there exists an open neighbourhood V of e such that Vz is fragmented in X. Every non-empty subset of a fragmented subset is fragmented as well. Therefore, without any loss of generality, we may suppose, in addition, that $V \subseteq U$. By the definition, the fragmentability of Vz implies that for a certain weakly open subset W of X the intersection $Vz \cap W$ is non-empty and δ-small. Then the identities

$$\tilde{z}(V \cap \tilde{z}^{-1}(W)) = \tilde{z}(V) \cap W = Vz \cap W$$

imply, in particular, that the set $V \cap \tilde{z}^{-1}(W)$ is non-empty. Moreover, this set is open in G by weak continuity of $\tilde{z}$. Choose arbitrarily $v_0 \in V \cap \tilde{z}^{-1}(W)$ and set $p := g_0v_0^{-1}$, $O := p(V \cap \tilde{z}^{-1}(W))$. Since G is a left topological group, O is also open in G. Moreover, by the construction, O is a neighbourhood of g_0. Clearly, $\tilde{z}(O) = g_0v_0^{-1}\tilde{z}(V \cap \tilde{z}^{-1}(W))$. Since $V \in U$, we have $p = g_0v_0^{-1} \in g_0U^{-1}$. On the other hand, $\tilde{z}(V \cap \tilde{z}^{-1}(W))$ is δ-small. By the choice of δ, we obtain that $\tilde{z}(O)$ is ε-small. ■

Remark 2.6. (1) If z is an lwap vector (under a not necessarily linear action), then the condition (b) is clearly satisfied, by Lemma 1.1. Applying Theorem 2.5, in particular, to the case of wap representations, we obtain a generalization of [36, Corollary 5.3].

(2) If G is locally compact, then (b) can be omitted.

(3) Recall [18] that if G is a locally compact group and X is Banach (for barreled X, see [27]), then every linear weakly continuous representation is strongly

continuous. The original proof in its main part is essentially based on Haar integration, convolution and the linearity of the representation. A transparent proof follows from Theorem 2.5. Moreover, we may suppose that π *is not linear*, provided of course that π is locally uniformly equicontinuous. For linear π, the latter is guaranteed (as in [27]) by the generalized principle of uniform boundedness.

Theorem 2.7. *Let X be an l.c.s. and BF, and let G be a subgroup of $GL(X)$. Suppose that there exists a weakly open neighbourhood U of e in G such that U and U^{-1} are each equicontinuous. Then G is a light subgroup of $GL(X)$.*

Proof. It suffices to show that for an arbitrary fixed element z of X, the orbit map $\tilde{z} : G \to X$ is continuous whenever G is endowed with the weak operator topology. We can apply Theorem 2.5. Indeed, the condition (c) is trivially satisfied. Choose $U \in N_e$ such that U and U^{-1} are equicontinuous, and hence uniformly equicontinuous, since the given action is linear. Therefore, (a) holds. In order to check (b), observe that Uz is fragmented since it is a bounded subset of a bound-fragmented l.c.s. X. ■

Theorem 2.8. *A subgroup G of $GL(X)$ is light under each of the following conditions:*

- *X is a Banach space with PCP (e.g., reflexive, or with RNP) and G is norm-bounded.*
- *X is a dentable l.c.s. (e.g., Frechet space with RNP) and G is equicontinuous.*

Proof. Apply Theorem 2.7 making use of Lemmas 1.3 and 1.2. ■

Remark 2.9. (1) Some boundedness condition on G is necessary even for the case of the Hilbert space $X := \ell_2$. Indeed, $GL(\ell_2)$ is not light.

(2) The group G may not be replaced, in general, by semigroups. Indeed, the semigroup $Cont(l_2) := \{g \in GL(l_2) \mid \|g\| \leq 1\}$ of all linear contractions endowed with the weak operator topology is compact in contrast to the strong operator topology.

(3) If the geometric behavior of X is "bad," then even $Is(X)$ may not be light in $GL(X)$ as the following example shows.

The following general construction, providing counterexamples, is based on an idea of Helmer [11]. Let Y be a compact space, and let G be a subgroup of the group $H(Y)$ of all autohomeomorphisms of Y. Denote by G_p the group G endowed with the topology of pointwise convergence. Then the evaluation map $\alpha : G_p \times Y \to Y$ is separately continuous. Assume that α is not jointly continuous and that G_p is a k-space (for example, in [11, Example 13], Y is the Euclidean 2-cell $[-1, 1]^2$ and G_p is a topological group homeomorphic to the space of all rationals). Consider the induced action

$$\pi : G_p \times C(Y) \to C(Y), \quad \pi(g, f)(y) = f(g^{-1}y)$$

and the induced injective group homomorphism $j : G_p \to Is(X)_w$, where $X := C(Y)$. By a classical result of Grothendieck [10, Theorem 5], for bounded subsets of $C(Y)$, pointwise compactness and weakly compactness are equivalent. This implies that for every compact subset K of G_p and every function $f \in C(Y)$ the restriction of the orbit map $\tilde{f} : G_p \to X$ on K is weakly continuous. Since G_p is a k-space, we can conclude that $\tilde{f}$ is weakly continuous. Therefore, j is continuous. Moreover, it is easy to see that j is a topological embedding. Now we can establish that the subgroup $j(G)$ (and, hence, $Is(X)$) is not light in $GL(X)$. Indeed, otherwise we will obtain that the restricted dual action

$$\pi_B^* : G_p \times B(X^*)_{w^*} \to B(X^*)_{w^*}, \quad \pi_B^*(g, \psi)(f) = \psi(g^{-1} \circ f)$$

is jointly continuous, where $B(X^*)_{w^*}$ is the unit ball of X^* endowed with the weak* topology. Then the original action α (being canonically equivalent to a subaction of π_B^*) is jointly continuous, too. This contradicts our assumption on α.

3. Compact semitopological semigroups "live" in reflexive spaces

Let S be a semitopological semigroup, that is, a topologized semigroup with a separately continuous multiplication. We will denote by $C(S)$ the commutative algebra of all bounded continuous real valued functions on S. For each $s \in S$, the right translation maps R_s of $C(S)$ into itself are defined by

$$R_s f(x) = f(xs) \quad \forall \quad x \in S.$$

Recall some basic facts about weak almost periodicity (see [6, 17, 3, 31]). A function $f \in C(S)$ is *weakly almost periodic* if the *orbit* of f, that is, the set

$$Sf := \{R_s f \mid s \in S\}$$

is relatively weakly compact in $C(S)$. The set $WAP(S)$ of all such functions is a closed S-invariant subalgebra of $C(S)$. If S is compact, then $WAP(S) = C(S)$. Moreover, the compactification $u : S \to S^w$, induced by the algebra $WAP(S)$, is the *universal semitopological semigroup compactification* of S.

For every reflexive Banach space X, the semitopological semigroup $Cont(X)_w$ is compact (see [17]). Hence every closed subsemigroup of $Cont(X)_w$ (with reflexive X) is a compact Hausdorff semitopological semigroup as well. Conversely, arbitrary compact Hausdorff semitopological semigroup can be obtained in this way. This fact was first proved by Shtern [32]. We reproduce here another proof of this result (see also [30]) using a factorization procedure discovered by Davis, Figiel, Johnson and Pelczynski [4, 9].

Theorem 3.1. *([32, 23]) Let (S, τ) be a compact Hausdorff semitopological semigroup. Then there exists a reflexive Banach space X such that (S, τ) is a subsemigroup of the compact semitopological semigroup $Cont(X)_w$.*

Proof. Without loss of generality, we can assume that S has the identity e. Consider the natural monoid action

$$\alpha : S \times C(S) \to C(S), \quad \alpha(s, f) = sf = R_s(f).$$

Clearly, each s-translation $\alpha^s = R_s$ is continuous. Moreover, by [10, Theorem 5], each orbit map $\alpha_f : S \to C(S)$ $(s \mapsto sf)$ is weakly continuous (see [17]). Therefore, for every fixed $f \in C(S)$, the orbit Sf is weakly compact. Denote by E_f the Banach subspace of $C(S) = WAP(S)$ linearly and topologically generated by Sf. Since a^s is continuous for every $s \in S$, E_f is S-invariant. By the Hahn-Banach Theorem, the weak topology of E_f is the same as its relative weak topology as a subset of $C(S)$. In particular, Sf is weakly compact in E_f. By the Krein-Smulian Theorem, the convex hull $co(-Sf \cup Sf) = W$ of the weakly compact symmetric subset $-Sf \cup Sf$ is relatively weakly compact. That is, the (weak) closure of W in E_f is weakly compact. Since W is a convex, bounded and symmetric subset of E_f, we can apply the factorization procedure by [4]. For each natural n, set

$$U_n = 2^n W + 2^{-n} B(E_f).$$

Let $\| \, \|_n$ be the gauge of the set U_n. That is,

$$\|x\|_n = \inf \{\lambda > 0 \mid x \in \lambda U_n\}.$$

Then, using [4, ch. 2] we obtain:

(1) $\| \, \|_n$ is a norm on E_f equivalent to the given norm $\| \, \|$ of E_f;

(2) For $x \in E_f$, let

$$N(x) = \left(\sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2}, \quad X_f = \{x \in E_f \mid N(x) < \infty\}.$$

Denote by $j : X_f \to E_f$ the inclusion map;

(3) $f \in Sf \subseteq W \subseteq B(X_f)$;

(4) (X_f, N) is a Banach space and $j : X_f \to E_f$ is a continuous linear injection;

(5) X_f is reflexive;

(6) The restriction of $j : X_f \to E_f$ on each bounded subset A of X_f induces a homeomorphism of A and $j(A)$ in the weak topologies.

Proof. Consider the weak closure $cl_w(A)$ of A in X_f. By the reflexivity of X_f, the set $cl_w(A)$ is weakly compact. Hence, j, being weakly continuous and injective, induces a homeomorphism of $cl_w(A)$ and $j(cl_w(A))$ with respect to the weak topologies. This proves assertion (6).

(7) $N(sx) \leq N(x)$ for every $x \in X_f$ and every $s \in S$.

Proof. It suffices to show that $\|sx\|_n \leq \|x\|_n$ for every $n \in \mathbb{N}$. By our construction $sW \subseteq W$ and $sB(E_f) \subseteq B(E_f)$ (R_s is a contraction of E_f). Then, from $x \in \lambda(2^n W + 2^{-n} B(E_f))$ we obtain that $sx \in \lambda(2^n(sW) + 2^{-n} s(B(E_f))) \subseteq \lambda(2^n W + 2^{-n} B(E_f))$. Hence, $\|sx\|_n \leq \|x\|_n$, as required. This proves assertion (7).

As a corollary, we get that X_f is an S-invariant subset of E_f. Therefore, the restricted action $\alpha_f : S \times X_f \to X_f$ is well-defined.

(8) For every $z \in X_f$, the orbit map $\tilde{z} : S \to X_f$ $(\tilde{z}(s) = sz)$ is weakly continuous.

Proof. Indeed, by assertion (7), the orbit $\tilde{z}(S) = Sz$ is a bounded in N-norm subset in X_f. Our assertion follows from (6) (for $A = Sz$), taking into account that $\tilde{z} : S \to E_f$ is weakly continuous.

By (7), for every $s \in S$, the translation map $\alpha_f^s : X_f \to X_f$ is a linear contraction of (X_f, N). Therefore, we get the map $\gamma_f : S \to Cont(X_f)$ $(\gamma_f(s) = \alpha_f^s)$.

Now, directly from (8) we obtain the following assertion.

(9) $\gamma_f : S \to Cont(X_f)_w$ is a continuous monoid homomorphism.

Now we are ready to construct the desired reflexive Banach space X. Consider the family $F = \{X_f \mid f \in C(S)\}$ of reflexive Banach spaces and the family

$$\{\gamma_f : S \to Cont(X_f) \mid f \in C(S)\}$$

of monoid homomorphisms . Define X as the ℓ_2-product (cf. [5, p. 35]) $X = \prod_2 X_f$ of the family F. Recall that it is the space of all functions $x = (x_f)$ such that $x_f \in X_f$ for each $f \in C(S)$, and the norm on X is defined by

$$\|x\| = \left(\sum_{f \in C(S)} \|x_f\|^2 \right)^{1/2} < \infty.$$

Then $(X, \| \ \|)$ is reflexive. Moreover, $(\prod_2 X_f)^* = \prod_2 X_f^*$ and the corresponding pairing for $x = (x_f) \in \prod_2 X_f$, $h = (h_f) \in \prod_2 X_f^*$ is defined by

$$h(x) = \sum_{f \in C(S)} h_f(x_f).$$

Now we define a linear representation of S in X as the ℓ_2-product of old representations. Precisely, we define

$$\gamma : S \to Cont(X) \quad , \quad \gamma(s)(x_f) = (sx_f).$$

First observe that by assertion (7), X is well-defined. Clearly γ is a monoid homomorphism. By assertion (9) and the above-mentioned description of X^*, it is easy to show that γ is weakly continuous. In order to establish that γ is the desired embedding, by the compactness of S, we have only to show that γ is injective. Equivalently, it suffices to check that $\{\gamma_f \mid f \in C(S)\}$ separates the points of S. Let s_1, s_2 be distinct points of S. Choose a continuous function $f \in C(S)$ with $f(s_1) \neq f(s_2)$. Since $(s_1 f)(e) = f(s_1)$ and $(s_2 f)(e) = f(s_2)$, it follows that $s_1 f$ and $s_2 f$ are distinct elements of $C(S)$ and of E_f. Moreover, by our construction, $X_f \subseteq E_f$ and $s_1 f, s_2 f$ both belong to X_f (see assertion (3)). Therefore, $\gamma_f(s_1) \neq \gamma_f(s_2)$. This implies that $\gamma(s_1) \neq \gamma(s_2)$, as required. ∎

Remark 3.2. In Theorem 3.1 we may choose X as having the same topological weight as S. That is, $w(X) = w(S)$. Indeed, we can easily modify the second part of the proof, taking the family $\{X_f \mid f \in P\}$, where P separates the points of S.

4. Reflexively representable groups

Recall that by a result of Lawson [15, Corollary 6.3] every subgroup of a compact Hausdorff semitopological semigroup is always a topological group.

Theorem 4.1. *Let G be a topological group. The following conditions are equivalent:*

(i) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_s$ (say, G is reflexively representable);

(ii) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_w$;

(iii) $WAP(G)$ separates points and closed subsets;

(iv) $j : G \to G^w$ is a topological embedding;

(v) G is a topological subgroup of a Hausdorff compact semitopological semigroup.

Proof. The equivalence of (iii), (iv) and (v), is well known [31, 3].

The part (ii) $\Longrightarrow$ (iii) follows from the well-known fact that for every reflexive Banach space E and a norm-bounded semigroup S of linear operators on E the *generalized matrix coefficients*

$$\{m_{v,f} : E \to \mathbb{R}\}_{f \in E^*, v \in E} \quad m_{v,f}(s) = f(sv)$$

all are wap.

The part (iii) $\Longrightarrow$ (ii), is a direct consequence of Theorem 3.1.

For the equivalence (i) $\Longleftrightarrow$ (ii), note that by Theorem 2.8, strong and weak operator topologies coincide on $Is(X)$ for all reflexive Banach spaces. ■

It is a standard fact that every Hausdorff topological group G is a subgroup (up to topological isomorphisms) of $Is(X)_s$ for a suitable Banach space X (see [35]). It turns out that the situation is more complex for *reflexive* X. Not every Hausdorff topological group is reflexively representable. Recently, it has been proved in [25] that there exists a Hausdorff topological group G (namely, the group $G := H_+(I)$ of all orientation preserving selfhomeomorphisms of the closed interval endowed with the compact open topology) such that every wap function on G is constant. This result answers a question discussed by Ruppert in [31] and conjectured by Pestov [30]. Every continuous homomorphism $H_+(I) \to Is(X)_w$ is trivial for every reflexive Banach space X.

It is also remarkable that by [26] there exists a reflexively representable topological group (namely, the additive group of the classical Banach space L_4) which does not admit an embedding, as a topological subgroup, into a unitary group $Is(H)_s$ (where H is a Hilbert space). This disproves a conjecture of Shtern [32]. We refer to the survey paper of Pestov [30] for more comprehensive discussion about these questions.

The following result is a modificaton of [21, Counterexample 3.13].

Proposition 4.2. *Let G be a separable metrizable group and let $u_L(G)$ denote its left uniform structure. If G is reflexively representable, then $(G, u_L(G))$, as a uniform space, is embedded into a separable reflexive Banach space Y. Moreover, the unitary group $Is(l_2)_s$ is left-uniformly embedded into l_2.*

Proof. By Theorem 4.1, G is a topological subgroup of $Is(X)_s$ for a certain reflexive Banach space X. Without loss of generality we may suppose that X is separable. By the definition of the strong operator topology, the system of all orbit maps on G generates the uniformity $u_L(G)$. Since G is second countable, we may suppose that there exists a sequence z_n in X such that the sequence of orbit maps

$$\tilde{z_n} : G \to X \quad , \quad \tilde{z_n}(g) = gz_n$$

generates $u_L(G)$. Moreover, we may suppose that $\|z_n\| = 2^{-n}$. Now without loss of generality we may suppose that X is separable. Consider the ℓ_2-product $\prod_2 X_n$ of the family $\{(X_n, \| \ \|_n) \mid n \in \mathbb{N}\}$, where each $(X_n, \| \ \|_n)$ is a copy of $(X, \| \ \|)$. The *reflexive* Banach space $\prod_2 X_n$ is denoted by Y. Since $\|z_n\| = 2^{-n}$, it is easy to show that the diagonal product map

$$\gamma : G \to \prod_2 X_n = y, \qquad \gamma(g) = (gz_n)$$

provides the desired uniform embedding.

Actually the same proof works in the case of $Is(l_2)_s$. ■

We say that a separable Banach space U is *uniformly universal* if every separable Banach space, as a uniform space, can be uniformly embedded into U. In answer to a question by Yu. Smirnov, Enflo [8] found, in 1969, a countable metrizable uniform space which is not uniformly embedded into a Hilbert space. That is, ℓ_2 is not uniformly universal. However, it is not clear if "Hilbert" may be replaced by "reflexive." In [1] Aharoni proved that c_0 is uniformly universal. Therefore, it easily follows from the second part of Proposition 4.2, that c_0, as a topological group, is not embedded into $Is(H)_s$ for any Hilbert space H. By [2] for a certain closed topological subgroup H of ℓ_2 the quotient group ℓ_2/H is *exotic* (that is, does not admit nontrivial continuous unitary representations).

Question 4.3. Is it true that $C[0,1], c_0$ (or any other uniformly universal space) is a uniform subset of a certain reflexive Banach space ? Equivalently, is it true that there exists a uniformly universal reflexive Banach space ?

Note that there is no *linearly* universal separable reflexive Banach space [34]. Moreover, there is no *Lipschitz embedding* of c_0 into a reflexive Banach space [19].

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Department of Mathematics
Bar-Ilan University
52900 Ramat-Gan
Israel
megereli@macs.biu.ac.il
http://www.cs.biu.ac.il/~megereli