# Operator Topologies And Reflexive Representability Of Groups

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# Operator topologies and reflexive representability

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**Abstract.** Using the concept of fragmentability, we show that weakly continuous group representations are frequently strongly continuous. We show that if a Banach (or, even, Frechet) space $X$ has the Radon-Nikodym property RNP, then the weak and strong operator topologies coincide on every bounded (respectively, equicontinuous) subgroup $G$ of $GL(X)$. We also strengthen a result of Shtern on reflexive representability of topological groups.

## 1. Fragmentable subsets of locally convex spaces

It is now well known that the concept of fragmentability in the sense of Jayne and Rogers [14, 13, 29, 9] is very powerful in various aspects of Banach space theory. In [22] we deal with continuity problems of linear semigroup representations using as a main tool fragmentability and its natural generalizations. As in [22], we say that a subset $A$ of a locally convex space (in short, l.c.s.) $X$ is *fragmented* if, for every non-empty subset $B$ of $A$ and every element $\varepsilon$ of the natural uniform structure of $X$, there is a weakly open subset $W$ of $X$ such that $B \cap W$ is non-empty and $\varepsilon$-small. If $X$ is a Banach space then we obtain the original definition of Jayne and Rogers [14].

Among various applications of Namioka's joint continuity theorem [28], we recall that every relatively weakly compact subset of a Banach space is fragmented [29]. We need the following locally convex version.

**Lemma 1.1.** *Every relatively weakly compact subset $A$ of an l.c.s. $X$ is fragmented in $X$.*

**Proof.** See [22, Proposition 3.5]. ■

We say that an l.c.s. $X$ is *bound-fragmented* (in short, BF) if every bounded subset $A$ of $X$ is fragmented. For instance, by Lemma 1.1, every semireflexive l.c.s. $X$ is BF. In order to formulate a stronger result, we need a locally convex version of Rieffel's concept of a dentable set. A non-empty subset $A$ of an l.c.s. $X$ is said to be *dentable* if for every neighborhood $V$ of 0, there exists a point $a$ in $A$ such that

$$a \notin cl(con(A \setminus (a + V)),$$

where cl(con) denotes the closed convex hull. A space $X$ is called *dentable* if every bounded subset of $X$ is dentable. For the extent of the class of dentable spaces see [7, section 2]. It follows that Frechet spaces with the Radon-Nikodym property (in short, RNP) and semireflexive l.c.s. are dentable. It is easy to show (for the case of a Banach space $X$, see [14, section 3]) that every dentable subset is fragmented. Therefore we obtain the following result.

**Lemma 1.2.** *Every dentable l.c.s. $X$ (e.g., Frechet space with RNP) is BF.*

It is well-known (see [20]) that a Banach space is dentable if and only if it has RNP.

Recall that a Banach space $X$ is said to have *the point of continuity property* (in short, PCP) if each bounded weakly closed subset $C$ of $X$ admits a point of continuity of the identity map $(C, \text{weak}) \to (C, \text{norm})$.

Here we reformulate a known result in terms of BF.

**Lemma 1.3.** *[14, p. 55] Let $X$ be a Banach space. Then $X$ is BF if and only if $X$ satisfies PCP.*

## 2. When does "weak imply strong?"

Let $X$ be an l.c.s. By $End(X)$ ($GL(X)$) we denote the set of all continuous linear endomorphisms (resp., automorphisms) of $X$. The dual Banach space of a Banach space $(X, \|\cdot\|)$ will be denoted by $X^*$. We use the following notation:

$$B(X) := \{x \in X \mid \|x\| \leq 1\} \quad Cont(X) := \{s \in L(X) \mid \|s\| \leq 1\}.$$

The group of all linear isometries of $X$ will be denoted by $Is(X)$. The *strong, strong** and *weak operator topology* (denote $T_s, T_{s^*}$ and $T_w$) on $L(X)$ is the weakest topology generated respectively by the following systems of maps:

- $\{\tilde{x} : L(X) \to X,\ \tilde{x}(s) = sx \mid x \in X\}$,
- $\{\tilde{f} : L(X) \to X^*,\ \tilde{f}(s)(x) = f(sx) \mid f \in X^*\}$,
- $\{\psi_{x,f} : L(X) \to \mathbb{R},\ \psi_{x,f}(s) = f(sx) \mid x \in X, f \in X^*\}$.

If a subset $P$ of $L(X)$ is endowed with one of the following subspace topologies $T_s\big|_P, T_{s^*}\big|_P, T_w\big|_P$, then we indicate this by writing $P_s, P_{s^*}$ and $P_w$, respectively. A subset $A$ of $X$ endowed with its usual weak topology is denoted by $A_w$. By [37] the unitary group $Is(H)$ is weakly dense in $Cont(H)$.

**Fact 2.1.** (Banach-Bourbaki Theorem) A Banach space $X$ is reflexive iff $B(X)_w$ is compact.

**Fact 2.2.** The semitopological semigroup $Cont(X)_w$ is compact iff $X$ is reflexive.

**Proof.** The compactness of $Cont(X)_w$ for a reflexive $X$ is well-known (see, for instance [17, Th. 3.1]). The converse implication follows from Fact 2.1, taking into account that for any fixed vector $x_0$ with $\|x_0\| = 1$, the map

$$Cont(X)_w \to B(X)_w, \quad s \mapsto sx_0$$

is continuous and *onto*. Indeed, take a continuous functional $f$ on $X$ such that $f(x_0) = 1$ and $\|f\| = 1$. For every $z \in B(X)$ assign to the pair $(f, z)$ the following linear operator

$$A_{f,z} : X \to X, \quad A_{f,z}(x) = f(x)z$$

Clearly, $A_{f,z}$ is a contraction of $X$ moving $x_0$ into $z$. ■

**Remark 2.3.** In general, $T_s \big|_{Is(X)} \neq T_{s^*} \big|_{Is(X)}$. Indeed, there are many continuous norm invariant continuous linear group actions on Banach spaces $X$ such that the corresponding dual actions on $X^*$ are not continuous. Consider, for example, $X = l_1$ and define a subgroup $S(\mathbb{N})$ of $Is(X)_s$ consisting of all permutations of "coordinates." Then the dual action of $S(\mathbb{N})$ on $l_1^* = m$ is not continuous. Hence, it is not even true that $T_{s^*} \big|_{Is(X)} \subseteq T_s \big|_{Is(X)}$. Note also that if $X$ is *Asplund* (by Stegall's result [33], it is equivalent to saying that the dual $X^*$ satisfies the Radon-Nikodym property), then, necessarily, $T_{s^*} \big|_{Is(X)} \subseteq T_s \big|_{Is(X)}$ (cf. [22, Corollary 6.9]).

**Definition 2.4.** We say that a subgroup $G$ of $GL(X)$ is *light* if the weak and strong operator topologies coincide on $G$.

It is a standart fact that for every Hilbert space $H$, the unitary group $Is(H)$ is light. Moreover, every bounded amenable subgroup $G$ of $GL(H)$ is light. Indeed, it suffices to remark that every bounded Hilbert representation of an amenable group is equivalent to a unitary representation. In general, $Is(X)$ may not be light (see Remarks 2.6). However, we show below (Theorem 2.4) that every bounded subgroup, amenable or not, $G$ of $GL(X)$ is light for Banach spaces $X$ with PCP (e.g., with RNP).

The general question about sufficient conditions for lightnesss of subgroups $G$ in $GL(X)$ is closely related to the classical problem: find sufficient conditions which guarantee that a given weakly continuous group representation is strongly continuous. From the extensive literature on this topic, we mention here only [27, 11, 12, 18, 16]. Roughly speaking, under strong restrictions on $G$ and $X$ (for example, if $G$ is topologically a Čech-complete, or Namioka space, enabling the use of Namioka's theorem) "weak implies strong." In this section, we obtain results that show that in many cases the "goodness" of $X$ is enough.

Throughout the paper, all spaces are assumed to be Hausdorff and all linear spaces are real. The filter of all neighbourhoods of a point $x$ is denoted by $N_x$. Let $\pi : G \times X \to X$, $\pi(g, x) := gx$ be an action of a group $G$ on a linear space $X$. Then the *orbit maps* $\tilde{x} : G \to X$ and translations $\tilde{g} : X \to X$ are defined by the rules $\tilde{x}(g) = gx = \tilde{g}(x)$. The action $\pi$ is said to be *linear* if $\tilde{g}$ is linear for every $g \in G$. In this case $\pi$ induces a homomorphism $G \to GL(X)$, a group representation of $G$ in $X$.

Recall that a vector $z \in X$ is said to be (*weakly*) *continuous* if the orbit map $\tilde{z} : G \to X$ is (*weakly*) *continuous*. Following [6, 17] we say that $z \in X$ is *weakly almost periodic* (*wap*) if the orbit $Gz := \{gz | g \in G\}$ is relatively weakly compact. We will say that $z$ is *locally weakly almost periodic (lwap)* if the set $Vz := \{gz | g \in V\}$ is relatively weakly compact for a certain neighbourhood $V$ of the identity $e \in G$.

A subset $B$ of $G$ is called (uniformly) equicontinuous if the family $\tilde{B} := \{\tilde{g} : X \to X | g \in B\}$ is (uniformly) equicontinuous. Recall that a topologized group $G$ is said to be *left topological* if every left translation $L_g : G \to G,\ L_g(h) = gh$ is continuous.

**Theorem 2.5.** *Let $\pi : G \times X \to X$ be a (not necessarily linear) action of a left topological group $G$ on an l.c.s. $X$, and let $z \in X$ such that:*

*(a) For every $g \in G$ there exists a neighbourhood $U$ of $g$ such that the set $U^{-1}$ is uniformly equicontinuous;*

*(b) There exists a neighbourhood $V$ of the identity $e \in G$ such that $Vz$ is a fragmented subset of $X$;*

*(c) $z$ is a weakly continuous vector.*

*Then $z$ is a continuous vector.*

**Proof.** We have to show that $\tilde{z} : G \to X$ is continuous at arbitrary fixed $g_0 \in G$. For a fixed element $\varepsilon$ of the standard uniformity $\mu$ on $X$, we will find an neighbourhood $O$ of $g_0$ such that $\tilde{z}(O) = Oz$ is $\varepsilon$-small. Since algebraically $G$ is a group, it follows from condition (a) that the $g_0$-translation $\tilde{g}_0 : X \to X$ is $\mu$-uniformly continuous. Again by (a), we can pick a neighbourhood $U$ of $e$ such that $U^{-1}$ is uniformly equicontinuous. Hence $g_0U^{-1}$ is uniformly equicontinuous, too. Therefore, for a fixed $\varepsilon \in \mu$ there exists an element $\delta \in \mu$ such that $(gx, gy) \in \varepsilon$ for every $(x, y) \in \delta$ and every $g \in g_0U^{-1}$. By condition (b), there exists an open neighbourhood $V$ of $e$ such that $Vz$ is fragmented in $X$. Every non-empty subset of a fragmented subset is fragmented as well. Therefore, without any loss of generality, we may suppose, in addition, that $V \subseteq U$. By the definition, the fragmentability of $Vz$ implies that for a certain weakly open subset $W$ of $X$ the intersection $Vz \cap W$ is non-empty and $\delta$-small. Then the identities

$$\tilde{z}(V \cap \tilde{z}^{-1}(W)) = \tilde{z}(V) \cap W = Vz \cap W$$

imply, in particular, that the set $V \cap \tilde{z}^{-1}(W)$ is non-empty. Moreover, this set is open in $G$ by weak continuity of $\tilde{z}$. Choose arbitrarily $v_0 \in V \cap \tilde{z}^{-1}(W)$ and set $p := g_0v_0^{-1},\ O := p(V \cap \tilde{z}^{-1}(W))$. Since $G$ is a left topological group, $O$ is also open in $G$. Moreover, by the construction, $O$ is a neighbourhood of $g_0$. Clearly, $\tilde{z}(O) = g_0v_0^{-1}\tilde{z}(V \cap \tilde{z}^{-1}(W))$. Since $V \in U$, we have $p = g_0v_0^{-1} \in g_0U^{-1}$. On the other hand, $\tilde{z}(V \cap \tilde{z}^{-1}(W))$ is $\delta$-small. By the choice of $\delta$, we obtain that $\tilde{z}(O)$ is $\varepsilon$-small. ∎

**Remark 2.6.** (1) If $z$ is an lwap vector (under a not necessarily linear action), then the condition (b) is clearly satisfied, by Lemma 1.1. Applying Theorem 2.5, in particular, to the case of wap representations, we obtain a generalization of [36, Corollary 5.3].

(2) If $G$ is locally compact, then (b) can be omitted.

(3) Recall [18] that if $G$ is a locally compact group and $X$ is Banach (for barreled $X$, see [27]), then every linear weakly continuous representation is strongly

continuous. The original proof in its main part is essentially based on Haar integration, convolution and the linearity of the representation. A transparent proof follows from Theorem 2.5. Moreover, we may suppose that $\pi$ *is not linear*, provided of course that $\pi$ is locally uniformly equicontinuous. For linear $\pi$, the latter is guaranteed (as in [27]) by the generalized principle of uniform boundedness.

**Theorem 2.7.** *Let $X$ be an l.c.s. and BF, and let $G$ be a subgroup of $GL(X)$. Suppose that there exists a weakly open neighbourhood $U$ of $e$ in $G$ such that $U$ and $U^{-1}$ are each equicontinuous. Then $G$ is a light subgroup of $GL(X)$.*

**Proof.** It suffices to show that for an arbitrary fixed element $z$ of $X$, the orbit map $\tilde{z} : G \to X$ is continuous whenever $G$ is endowed with the weak operator topology. We can apply Theorem 2.5. Indeed, the condition (c) is trivially satisfied. Choose $U \in N_e$ such that $U$ and $U^{-1}$ are equicontinuous, and hence uniformly equicontinuous, since the given action is linear. Therefore, (a) holds. In order to check (b), observe that $Uz$ is fragmented since it is a bounded subset of a bound-fragmented l.c.s. $X$. ∎

**Theorem 2.8.** *A subgroup $G$ of $GL(X)$ is light under each of the following conditions:*

- *$X$ is a Banach space with PCP (e.g., reflexive, or with RNP) and $G$ is norm-bounded.*
- *$X$ is a dentable l.c.s. (e.g., Frechet space with RNP) and $G$ is equicontinuous.*

**Proof.** Apply Theorem 2.7 making use of Lemmas 1.3 and 1.2. ∎

**Remark 2.9.** (1) Some boundedness condition on $G$ is necessary even for the case of the Hilbert space $X := \ell_2$. Indeed, $GL(\ell_2)$ is not light.

(2) The group $G$ may not be replaced, in general, by semigroups. Indeed, the semigroup $Cont(l_2) := \{g \in GL(l_2) \mid \|g\| \leq 1\}$ of all linear contractions endowed with the weak operator topology is compact in contrast to the strong operator topology.

(3) If the geometric behavior of $X$ is "bad," then even $Is(X)$ may not be light in $GL(X)$ as the following example shows.

The following general construction, providing counterexamples, is based on an idea of Helmer [11]. Let $Y$ be a compact space, and let $G$ be a subgroup of the group $H(Y)$ of all autohomeomorphisms of $Y$. Denote by $G_p$ the group $G$ endowed with the topology of pointwise convergence. Then the evaluation map $\alpha : G_p \times Y \to Y$ is separately continuous. Assume that $\alpha$ is not jointly continuous and that $G_p$ is a $k$-space (for example, in [11, Example 13], $Y$ is the Euclidean 2-cell $[-1, 1]^2$ and $G_p$ is a topological group homeomorphic to the space of all rationals). Consider the induced action

$$\pi : G_p \times C(Y) \to C(Y), \quad \pi(g, f)(y) = f(g^{-1}y)$$

and the induced injective group homomorphism $j : G_p \to Is(X)_w$, where $X := C(Y)$. By a classical result of Grothendieck [10, Theorem 5], for bounded subsets of $C(Y)$, pointwise compactness and weakly compactness are equivalent. This implies that for every compact subset $K$ of $G_p$ and every function $f \in C(Y)$ the restriction of the orbit map $\tilde{f} : G_p \to X$ on $K$ is weakly continuous. Since $G_p$ is a $k$-space, we can conclude that $\tilde{f}$ is weakly continuous. Therefore, $j$ is continuous. Moreover, it is easy to see that $j$ is a topological embedding. Now we can establish that the subgroup $j(G)$ (and, hence, $Is(X)$) is not light in $GL(X)$. Indeed, otherwise we will obtain that the restricted dual action

$$\pi_B^* : G_p \times B(X^*)_{w^*} \to B(X^*)_{w^*}, \quad \pi_B^*(g, \psi)(f) = \psi(g^{-1} \circ f)$$

is jointly continuous, where $B(X^*)_{w^*}$ is the unit ball of $X^*$ endowed with the weak* topology. Then the original action $\alpha$ (being canonically equivalent to a subaction of $\pi_B^*$) is jointly continuous, too. This contradicts our assumption on $\alpha$.

## 3. Compact semitopological semigroups "live" in reflexive spaces

Let $S$ be a semitopological semigroup, that is, a topologized semigroup with a separately continuous multiplication. We will denote by $C(S)$ the commutative algebra of all bounded continuous real valued functions on $S$. For each $s \in S$, the right translation maps $R_s$ of $C(S)$ into itself are defined by

$$R_s f(x) = f(xs) \quad \forall \quad x \in S.$$

Recall some basic facts about weak almost periodicity (see [6, 17, 3, 31]). A function $f \in C(S)$ is *weakly almost periodic* if the *orbit* of $f$, that is, the set

$$Sf := \{R_s f \mid s \in S\}$$

is relatively weakly compact in $C(S)$. The set $WAP(S)$ of all such functions is a closed $S$-invariant subalgebra of $C(S)$. If $S$ is compact, then $WAP(S) = C(S)$. Moreover, the compactification $u : S \to S^w$, induced by the algebra $WAP(S)$, is the *universal semitopological semigroup compactification* of $S$.

For every reflexive Banach space $X$, the semitopological semigroup $Cont(X)_w$ is compact (see [17]). Hence every closed subsemigroup of $Cont(X)_w$ (with reflexive $X$) is a compact Hausdorff semitopological semigroup as well. Conversely, arbitrary compact Hausdorff semitopological semigroup can be obtained in this way. This fact was first proved by Shtern [32]. We reproduce here another proof of this result (see also [30]) using a factorization procedure discovered by Davis, Figiel, Johnson and Pelczynski [4, 9].

**Theorem 3.1.** *([32, 23]) Let $(S, \tau)$ be a compact Hausdorff semitopological semigroup. Then there exists a reflexive Banach space $X$ such that $(S, \tau)$ is a subsemigroup of the compact semitopological semigroup $Cont(X)_w$.*

**Proof.** Without loss of generality, we can assume that $S$ has the identity $e$. Consider the natural monoid action

$$\alpha : S \times C(S) \to C(S), \quad \alpha(s,f) = sf = R_s(f).$$

Clearly, each $s$-translation $\alpha^s = R_s$ is continuous. Moreover, by [10, Theorem 5], each orbit map $\alpha_f : S \to C(S)$ $(s \mapsto sf)$ is weakly continuous (see [17]). Therefore, for every fixed $f \in C(S)$, the orbit $Sf$ is weakly compact. Denote by $E_f$ the Banach subspace of $C(S) = WAP(S)$ linearly and topologically generated by $Sf$. Since $a^s$ is continuous for every $s \in S$, $E_f$ is $S$-invariant. By the Hahn-Banach Theorem, the weak topology of $E_f$ is the same as its relative weak topology as a subset of $C(S)$. In particular, $Sf$ is weakly compact in $E_f$. By the Krein-Smulian Theorem, the convex hull $co(-Sf \cup Sf) = W$ of the weakly compact symmetric subset $-Sf \cup Sf$ is relatively weakly compact. That is, the (weak) closure of $W$ in $E_f$ is weakly compact. Since $W$ is a convex, bounded and symmetric subset of $E_f$, we can apply the factorization procedure by [4]. For each natural $n$, set

$$U_n = 2^n W + 2^{-n} B(E_f).$$

Let $\| \, \|_n$ be the gauge of the set $U_n$. That is,

$$\|x\|_n = \inf \{\lambda > 0 \mid x \in \lambda U_n\}.$$

Then, using [4, ch. 2] we obtain:

(1) $\| \, \|_n$ is a norm on $E_f$ equivalent to the given norm $\| \, \|$ of $E_f$;

(2) For $x \in E_f$, let

$$N(x) = \left( \sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2}, \quad X_f = \{x \in E_f \mid N(x) < \infty\}.$$

Denote by $j : X_f \to E_f$ the inclusion map;

(3) $f \in Sf \subseteq W \subseteq B(X_f)$;

(4) $(X_f, N)$ is a Banach space and $j : X_f \to E_f$ is a continuous linear injection;

(5) $X_f$ is reflexive;

(6) The restriction of $j : X_f \to E_f$ on each bounded subset $A$ of $X_f$ induces a homeomorphism of $A$ and $j(A)$ in the weak topologies.

Proof. Consider the weak closure $cl_w(A)$ of $A$ in $X_f$. By the reflexivity of $X_f$, the set $cl_w(A)$ is weakly compact. Hence, $j$, being weakly continuous and injective, induces a homeomorphism of $cl_w(A)$ and $j(cl_w(A))$ with respect to the weak topologies. This proves assertion (6).

(7) $N(sx) \leq N(x)$ for every $x \in X_f$ and every $s \in S$.

Proof. It suffices to show that $\|sx\|_n \leq \|x\|_n$ for every $n \in \mathbb{N}$. By our construction $sW \subseteq W$ and $sB(E_f) \subseteq B(E_f)$ ($R_s$ is a contraction of $E_f$). Then, from $x \in \lambda(2^n W + 2^{-n} B(E_f))$ we obtain that $sx \in \lambda(2^n (sW) + 2^{-n} s(B(E_f))) \subseteq \lambda(2^n W + 2^{-n} B(E_f))$. Hence, $\|sx\|_n \leq \|x\|_n$, as required. This proves assertion (7).

As a corollary, we get that $X_f$ is an $S$-invariant subset of $E_f$. Therefore, the restricted action $\alpha_f : S \times X_f \to X_f$ is well-defined.

(8) For every $z \in X_f$, the orbit map $\tilde{z} : S \to X_f$ $(\tilde{z}(s) = sz)$ is weakly continuous.

Proof. Indeed, by assertion (7), the orbit $\tilde{z}(S) = Sz$ is a bounded in $N$-norm subset in $X_f$. Our assertion follows from (6) (for $A = Sz$), taking into account that $\tilde{z} : S \to E_f$ is weakly continuous.

By (7), for every $s \in S$, the translation map $\alpha_f^s : X_f \to X_f$ is a linear contraction of $(X_f, N)$. Therefore, we get the map $\gamma_f : S \to Cont(X_f)$ $(\gamma_f(s) = \alpha_f^s)$.

Now, directly from (8) we obtain the following assertion.

(9) $\gamma_f : S \to Cont(X_f)_w$ is a continuous monoid homomorphism.

Now we are ready to construct the desired reflexive Banach space $X$. Consider the family $F = \{X_f \mid f \in C(S)\}$ of reflexive Banach spaces and the family

$$\{\gamma_f : S \to Cont(X_f) \mid f \in C(S)\}$$

of monoid homomorphisms . Define $X$ as the $\ell_2$-product (cf. [5, p. 35]) $X = \prod_2 X_f$ of the family $F$. Recall that it is the space of all functions $x = (x_f)$ such that $x_f \in X_f$ for each $f \in C(S)$, and the norm on $X$ is defined by

$$\|x\| = \left( \sum_{f \in C(S)} \|x_f\|^2 \right)^{1/2} < \infty.$$

Then $(X, \|\ \|)$ is reflexive. Moreover, $(\prod_2 X_f)^* = \prod_2 X_f^*$ and the corresponding pairing for $x = (x_f) \in \prod_2 X_f$, $h = (h_f) \in \prod_2 X_f^*$ is defined by

$$h(x) = \sum_{f \in C(S)} h_f(x_f).$$

Now we define a linear representation of $S$ in $X$ as the $\ell_2$-product of old representations. Precisely, we define

$$\gamma : S \to Cont(X) \quad , \quad \gamma(s)(x_f) = (sx_f).$$

First observe that by assertion (7), $X$ is well-defined. Clearly $\gamma$ is a monoid homomorphism. By assertion (9) and the above-mentioned description of $X^*$, it is easy to show that $\gamma$ is weakly continuous. In order to establish that $\gamma$ is the desired embedding, by the compactness of $S$, we have only to show that $\gamma$ is injective. Equivalently, it suffices to check that $\{\gamma_f \mid f \in C(S)\}$ separates the points of $S$. Let $s_1, s_2$ be distinct points of $S$. Choose a continuous function $f \in C(S)$ with $f(s_1) \neq f(s_2)$. Since $(s_1 f)(e) = f(s_1)$ and $(s_2 f)(e) = f(s_2)$, it follows that $s_1 f$ and $s_2 f$ are distinct elements of $C(S)$ and of $E_f$. Moreover, by our construction, $X_f \subseteq E_f$ and $s_1 f, s_2 f$ both belong to $X_f$ (see assertion (3)). Therefore, $\gamma_f(s_1) \neq \gamma_f(s_2)$. This implies that $\gamma(s_1) \neq \gamma(s_2)$, as required. ∎

**Remark 3.2.** In Theorem 3.1 we may choose $X$ as having the same topological weight as $S$. That is, $w(X) = w(S)$. Indeed, we can easily modify the second part of the proof, taking the family $\{X_f \mid f \in P\}$, where $P$ separates the points of $S$.

## 4. Reflexively representable groups

Recall that by a result of Lawson [15, Corollary 6.3] every subgroup of a compact Hausdorff semitopological semigroup is always a topological group.

**Theorem 4.1.** *Let $G$ be a topological group. The following conditions are equivalent:*

*(i) There exists a reflexive Banach space $E$ such that $G$ is embedded as a topological subgroup into $Is(E)_s$ (say, $G$ is reflexively representable);*

*(ii) There exists a reflexive Banach space $E$ such that $G$ is embedded as a topological subgroup into $Is(E)_w$;*

*(iii) $WAP(G)$ separates points and closed subsets;*

*(iv) $j : G \to G^w$ is a topological embedding;*

*(v) $G$ is a topological subgroup of a Hausdorff compact semitopological semigroup.*

**Proof.** The equivalence of (iii), (iv) and (v), is well known [31, 3].

The part (ii) $\Longrightarrow$ (iii) follows from the well-known fact that for every reflexive Banach space $E$ and a norm-bounded semigroup $S$ of linear operators on $E$ the *generalized matrix coefficients*

$$\{m_{v,f} : E \to \mathbb{R}\}_{f \in E^*, v \in E} \quad m_{v,f}(s) = f(sv)$$

all are wap.

The part (iii) $\Longrightarrow$ (ii), is a direct consequence of Theorem 3.1.

For the equivalence (i) $\Longleftrightarrow$ (ii), note that by Theorem 2.8, strong and weak operator topologies coincide on $Is(X)$ for all reflexive Banach spaces. ■

It is a standard fact that every Hausdorff topological group $G$ is a subgroup (up to topological isomorphisms) of $Is(X)_s$ for a suitable Banach space $X$ (see [35]). It turns out that the situation is more complex for *reflexive* $X$. Not every Hausdorff topological group is reflexively representable. Recently, it has been proved in [25] that there exists a Hausdorff topological group $G$ (namely, the group $G := H_+(I)$ of all orientation preserving selfhomeomorphisms of the closed interval endowed with the compact open topology) such that every wap function on $G$ is constant. This result answers a question discussed by Ruppert in [31] and conjectured by Pestov [30]. Every continuous homomorphism $H_+(I) \to Is(X)_w$ is trivial for every reflexive Banach space $X$.

It is also remarkable that by [26] there exists a reflexively representable topological group (namely, the additive group of the classical Banach space $L_4$) which does not admit an embedding, as a topological subgroup, into a unitary group $Is(H)_s$ (where $H$ is a Hilbert space). This disproves a conjecture of Shtern [32]. We refer to the survey paper of Pestov [30] for more comprehensive discussion about these questions.

The following result is a modificaton of [21, Counterexample 3.13].

**Proposition 4.2.** *Let $G$ be a separable metrizable group and let $u_L(G)$ denote its left uniform structure. If $G$ is reflexively representable, then $(G, u_L(G))$, as a uniform space, is embedded into a separable reflexive Banach space $Y$. Moreover, the unitary group $Is(l_2)_s$ is left-uniformly embedded into $l_2$.*

**Proof.** By Theorem 4.1, $G$ is a topological subgroup of $Is(X)_s$ for a certain reflexive Banach space $X$. Without loss of generality we may suppose that $X$ is separable. By the definition of the strong operator topology, the system of all orbit maps on $G$ generates the uniformity $u_L(G)$. Since $G$ is second countable, we may suppose that there exists a sequence $z_n$ in $X$ such that the sequence of orbit maps

$$\tilde{z_n} : G \to X \quad , \quad \tilde{z_n}(g) = gz_n$$

generates $u_L(G)$. Moreover, we may suppose that $\|z_n\| = 2^{-n}$. Now without loss of generality we may suppose that $X$ is separable. Consider the $\ell_2$-product $\prod_2 X_n$ of the family $\{(X_n, \| \ \|_n) \mid n \in \mathbb{N}\}$, where each $(X_n, \| \ \|_n)$ is a copy of $(X, \| \ \|)$. The *reflexive* Banach space $\prod_2 X_n$ is denoted by $Y$. Since $\|z_n\| = 2^{-n}$, it is easy to show that the diagonal product map

$$\gamma : G \to \prod_2 X_n = y, \qquad \gamma(g) = (gz_n)$$

provides the desired uniform embedding.

Actually the same proof works in the case of $Is(l_2)_s$. ■

We say that a separable Banach space $U$ is *uniformly universal* if every separable Banach space, as a uniform space, can be uniformly embedded into $U$. In answer to a question by Yu. Smirnov, Enflo [8] found, in 1969, a countable metrizable uniform space which is not uniformly embedded into a Hilbert space. That is, $\ell_2$ is not uniformly universal. However, it is not clear if "Hilbert" may be replaced by "reflexive." In [1] Aharoni proved that $c_0$ is uniformly universal. Therefore, it easily follows from the second part of Proposition 4.2, that $c_0$, as a topological group, is not embedded into $Is(H)_s$ for any Hilbert space $H$. By [2] for a certain closed topological subgroup $H$ of $\ell_2$ the quotient group $\ell_2/H$ is *exotic* (that is, does not admit nontrivial continuous unitary representations).

**Question 4.3.** Is it true that $C[0,1], c_0$ (or any other uniformly universal space) is a uniform subset of a certain reflexive Banach space ? Equivalently, is it true that there exists a uniformly universal reflexive Banach space ?

Note that there is no *linearly* universal separable reflexive Banach space [34]. Moreover, there is no *Lipschitz embedding* of $c_0$ into a reflexive Banach space [19].

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