Free Topological G-Groups

Michael Megrelishvili*

Department of Mathematics and Computer Science,

Bar-Ilan University, 52900 Ramat-Gan, Israel

Necessary and sufficient conditions for the equivariant embeddability into: topological G-groups, G-linear spaces, homogeneous spaces, are obtained. In particular, free topological G-groups are investigated. A "regionally-proximal type" relation enables us to find compact homogeneous G-spaces G/H such that the free topological G-group over G/H is trivial. We construct also a G-group which is not G-linearizable into a locally convex linear G-space. Using J. West's theorem about extension of actions from Z-subspaces to an action on the Hilbert cube, we prove that for every $\aleph_0$-bounded group G and a G-compactificable space X there exists an equivariant embedding of $\langle G, X\rangle$ into $\langle H(I^\tau), I^\tau\rangle$ where $\tau \leq w(X)\cdot w(G)$. In particular, Uspenskiĭ 's theorem on the universality of $H(I^{\aleph_0})$ is improved.

1. Conventions and Known Facts.

In this paper, all topological spaces are Tychonoff (i.e., are completely regular and Hausdorff) and all linear spaces are real locally convex. A *topological transformation group* (a *ttg*) or for short: a *G-space* is a system $\langle G, X, \alpha\rangle$ in which G is a topological group, X is a topological space and $\alpha\colon G\times X\to X$ is a continuous action. As usual, we write gx instead of $\alpha(g,x)$. A *g-transition* is the mapping $\alpha^g\colon X\to X, \alpha^g(x)=gx$ and an *x-orbit mapping* is the mapping $\alpha_x\colon G\to X, \alpha_x(g)=gx$. Denote by Comp^G and $[\mathrm{Tych}]^G$ the classes of all compact and all Tychonoff G-spaces respectively. A G-space X is called *G-automorphic* (or: a *G-group*) if X is a topological group and each α^g is an automorphism. In the case of a linear space X and linear $\alpha^g - s$, we obtain the definition of a *linear* G-space [31].

Let H be a closed subgroup of G and G/H be the space of all left cosets endowed with the quotient topology. The triple $\langle G, G/H, \alpha_\ell\rangle$, where $\alpha_\ell(g, sH) = gsH$ is called *homogeneous.*

*Partially supported by the Israel Ministry of Sciences, Grant No. 3505.
1991 *Mathematics Subject Classification.* 22A05, 54H15.

A *ttg* $\langle G, X, \alpha\rangle$ will be called:

(a) *G-compactificable*, or *G-Tychonoff* if X is a G subspace of a compact G-space;

(b) *G-homogenizable*, if there exists an equivariant embedding of $\langle G, X, \alpha\rangle$ into a homogeneous triple $\langle P, P/H, \alpha_\ell\rangle$ (i.e., there exists a topological group embedding $\varphi\colon G \hookrightarrow P$ and a topological embedding $f\colon X \hookrightarrow P/H$ such that $\alpha_\ell \mid_{\varphi(G)\times f(X)} = \alpha$).

(c) *G-automorphizable*, if X is a G-subspace of an automorphic G-space.

(d) *G-linearizable*, if X is a G-subspace of a linear G-space.

Denote the corresponding classes by $\mathrm{Tych}^G, \mathrm{HTych}^G, \mathrm{ATych}^G$ and LTych^G respectively. Our aim will the comparision of these classes.

Lemma 1.1. *Let $\langle G, X, \alpha\rangle$ be an automorphic triple, and let $X\lambda_\alpha G$ denote the corresponding topological semidirect product. Then $\langle G, X, \alpha\rangle$ is embedded into the homogeneous triple $\langle X\lambda_\alpha G, (X\lambda_\alpha G)/G,\ \alpha_\ell\rangle$, where G is identified with the subgroup $\{e\}\times G$.*

Proof. Consider the mapping $q\colon X \to (X\lambda_\alpha G)/G$ defined by the rule $q(x) = xG$. According to [25, Prop. 6.17(a)], q is a homeomorphism. Moreover, it can be easily checked that the restriction of α_ℓ on $G\times X := G\times q(X)$ is α. $\square$

Actually the same trick was used (non-explicitly) by Ta-Sun Wu in [33, p.512].

Lemma 1.2. *(J. de Vries [28]) Every homogeneous triple $\langle G, G/H, \alpha_\ell\rangle$ is G-Tychonoff.*

Proposition 1.3. *For every topological group G holds:*

$$\mathrm{LTych}^G \subset \mathrm{ATych}^G \subset \mathrm{HTych}^G \subset \mathrm{Tych}^G \subset [\mathrm{Tych}]^G.$$

Proof. Non-trivial inclusions follow from Lemmas 1.1 and 1.2. $\square$

If G is locally compact ($G \in LC$), then every G-space is G-linearizable (see for example [26, 31]), and, hence all classes from Proposition 1.3 coincide.

Example 1.4. [16]. In general, $\mathrm{Tych}^G \neq [\mathrm{Tych}]^G$. There exists a continuous action α of a group G on $J(\aleph_0)$ (the so-called *hedgehog space of spininess* $\aleph_0$ [11]) such that

$\langle G, J(\aleph_0), \alpha \rangle$ is not G-Tychonoff. Note that in this example G and $J(\aleph_0)$ are both *Polish* (i.e., are separable complete-metrizable).

Our main examples 3.7 and 4.7 will show that, in general, $\mathrm{ATych}^G \neq \mathrm{HTych}^G$ and $\mathrm{LTych}^G \neq \mathrm{ATych}^G$.

Furthermore, the filter of all neighborhoods (nbd's) at a point x in a space X, is denoted by $N_x(X)$. If μ is a compatible uniformity on a topological space X, then for every $\varepsilon \in \mu$ and $A \subset X$ denote by $\varepsilon(A)$ the set $\{y \in X \mid (x,y) \in \varepsilon, x \in A\}$. Subsets A, B will be called *ε-near* if $\varepsilon(A) \cap \varepsilon(B) \neq \emptyset$.

The finest compatible uniformity we denote by $\mu_{\max}$.

Due to [25] the *left, right, upper* and *lower* uniformities on a topological group will be denote by $\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}, \mathcal{L} \wedge \mathcal{R}$ respectively. An action α on a uniform space (X, μ) is *saturated* if each g-transition is uniformly continuous.

A system $\langle G, (X, \mu), \alpha \rangle$ is called:

(a) *bounded* [29] (or: *motion equicontinuous* [8]) if for every $\varepsilon \in \mu$ there exists $U \in N_e(G)$, satisfying $(x, gx) \in \varepsilon$ for every $(g, x) \in U \times X$.

(b) *quasibounded* [17] if for every $\varepsilon \in \mu$ there exists a pair $(\delta, U) \in \mu \times N_e(G)$ such that $(gx, gy) \in \varepsilon$ whenever $(x, y) \in \delta$ and $g \in U$.

For example, if $\mathcal{R}(G/H)$ denotes the right uniformity on the coset space G/H, then the action α_ℓ is $\mathcal{R}(G/H)$-bounded [28].

We will say that (X, μ) is a *uniform G-space* and write: $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$, (or: $(X, \mu) \in \mathrm{Unif}^G$) if α is μ-quasibounded, μ-saturated and continuous. By [8] every compact G-space is bounded with respect to its unique uniformity. Therefore, $\mathrm{Comp}^G \subset \mathrm{Unif}^G$. Recall some useful facts concerning the class Unif^G.

Fact 1.5. ([15, Th. 2.2], [18, Th. 1.5]). Let $\alpha\colon G \times X \to X$ be a continuous action. The following statements are equivalent:

(i) $\langle G, X, \alpha \rangle \in \mathrm{Tych}^G$;

(ii) $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$ for a certain compatible μ;

(iii) $\langle G, (X, \xi), \alpha \rangle$ is quasibounded for a certain compatible ξ.

Fact 1.6. ([17, Th. 1.2]). Let X be a topological group and $\alpha\colon G \times X \to X$ be a continuous action of a topological group G on X by automorphisms. Then $\langle G, (X,\mu), \alpha\rangle \in \mathrm{Unif}^G$ for every $\mu \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$.

Let G be a topological group and $U \in N_e(G)$. Denote by $\tilde{U}$ the set of all pairs $(x,y) \in G \times G$ such that $y \in UxU$. Then $\{\tilde{U}\colon U \in N_e(G)\}$ is a base of the lower uniformity on G (cf. [25, Def. 2.5]). Using this description, it is easy to check that Fact 1.6 is true also for $\mu = \mathcal{L} \wedge \mathcal{R}$.

For every topological group G there are several natural actions:

(a) $\langle G, G, \alpha_\ell\rangle, \alpha_\ell(g,x) = gx$;

(b) $\langle G, G, \alpha_r\rangle, \alpha_r(g,x) = xg^{-1}$;

(c) $\langle G, G, \alpha_\#\rangle, \alpha_\#(g,x) = gxg^{-1}$;

(d) $\langle G \times G, G, \pi\rangle, \pi((g_1,g_2),x) = g_1 x g_2^{-1}$.

Elementary verification shows that $\langle G \times G, (G,\mu), \pi\rangle \in \mathrm{Unif}^G$ for any $\mu \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}, \mathcal{L} \wedge \mathcal{R}\}$. This implies (by Fact 1.5) that for the actions $\alpha_\ell, \alpha_r, \alpha_\#$ there exists a *common* compact G-extension of G.

Denote by AUnif^G and LUnif^G the classes of all $\mathcal{L} \vee \mathcal{R}$-uniform G-subspaces in automorphic and linear G-spaces respectively. Then, $\mathrm{LUnif}^G \subset \mathrm{AUnif}^G \subset \mathrm{Unif}^G$. Indeed, $\mathrm{AUnif}^G \subset \mathrm{Unif}^G$ directly follows from Fact 1.6 and the second inclusion is trivial.

Fact 1.7. [17, Th. 1.10]. Suppose that G is Baire and $\alpha\colon G \times X \to X$ is a d-saturated action on a metrizable uniform space (X,d) such that for a certain dense subset $Y \subset X$ orbit mapping $\alpha_y\colon G \to X$ is continuous for every $y \in Y$. Then: (a) α is continuous; (b) $\langle G, (X,d), \alpha\rangle \in \mathrm{Unif}^G$; (c) X is G-Tychonoff.

Corollary 1.8. *Let G be Baire and $\alpha\colon G \times X \to X$ be an action on a metrizable group by continuous automorphisms. Suppose that orbit mapping $\alpha_y\colon G \to X$ is continuous for every $y \in Y$, where Y topologically generates X. Then α is continuous.*

The last result essentially improves Proposition 2 from [6, VIII, §2.1].

Fact 1.9. [18, Th. 3.1]. Let $\langle G,(X,\mu),\alpha\rangle \in \mathrm{Unif}^G$. Denote by $(\hat{X},\hat{\mu})$ and $\hat{G}$ the completions of (X,μ) and $(G,\mathcal{L}\vee\mathcal{R})$. Then there exists a continuous action $\hat{\alpha}\colon \hat{G}\times\hat{X}\to\hat{X}$ which extends α and satisfies $\langle\hat{G},(\hat{X},\hat{\mu}),\hat{\alpha}\rangle \in \mathrm{Unif}^G$.

2. West's Theorem and Equivariant Homogenization.

For each compact space X, the group $H(X)$ of all self-homeomorphisms, equipped with the compact-open topology, continuously acts on X by the rule $\pi(h,x)=h(x)$. For a metrizable compact X, the group $H(X)$ is Polish. If, in addition, π is transitive, then Effros's Theorem [9] yields that the triple $\langle H(X),X,\pi\rangle$ is homogeneous.

Lemma 2.1. *Let τ be an infinite cardinal and I^τ be the Tychonoff cube. Then the triple $\langle H(I^\tau),I^\tau,\pi\rangle$ is homogeneous.*

Proof. The arguments preceding the Lemma and Theorem 4.1 from [5, p.104] show that $\langle H(I^{\aleph_0}),I^{\aleph_0},\pi\rangle$ is homogeneous. By Exercise 3 of §5 in [25], the product *ttg* $\langle (H(I^{\aleph_0}))^\tau,(I^{\aleph_0})^\tau,\pi^\tau\rangle$, where π^τ is defined coordinate-wise, is also homogeneous. We can identify $(I^{\aleph_0})^\tau$ with I^τ and $(H(I^{\aleph_0}))^\tau$ with a topological subgroup of $H(I^\tau)$. Now the following claim completes the proof.

Claim. Let $\pi\colon G\times X\to X$ be a continuous transitive action and P be a topological subgroup of G such that $\langle P,X,\pi\restriction_{P\times X}\rangle$ is homogeneous. Then $\langle G,X,\pi\rangle$ is homogeneous too.

Proof of Claim. Let $x\in X$. By our assumption, the orbit mapping $\left(\pi\restriction_{P\times X}\right)_x : P\to X$ is open. For every nbd $U\in N_e(G)$, the set $\left(\pi\restriction_{P\times X}\right)_x(U)$ belongs to $N_x(X)$ and is contained in $\pi_x(U)$. Then $\pi_x(U)\in N_x(X)$. Therefore, the action π is *micro-transitive* in the sense of [1]. This implies the openness of π_x (see, for example the proof (a) $\Rightarrow$ (b) of Lemma 1 in [1]).

Remark 2.2. It should be mentioned that there exists (see [4]) a compact space X, such that the natural action $\pi\colon H(X)\times X\to X$ is transitive but non-Effros. Due to our terminology, this means that $\langle H(X), X, \alpha\rangle$ is not homogeneous. By the claim, the same is true for every topological subgroup G of $H(X)$ transitively acting on X. Moreover, by Theorem 3.2 from [12], we can suppose that G carries an arbitrary group topology that leaves $\alpha\,|_{G\times X}$ continuous.

Recall that a topological group G is called *τ-bounded* [14] (we write: $\ell^u(G) \leq \tau$) if for every $V \in N_e(G)$ there exists a subset S with the cardinality $|S| \leq \tau$ such that $SV = G$. As is well-known, G is τ-bounded iff it is a topological subgroup in a product of groups with weight $\leq \tau$.

Fact 2.3. [17]. For every G-Tychonoff space X there exists a compact G-extension Y of weight $w(Y) \leq w(X) \cdot \ell^u(G)$.

A weaker version (for a locally compact G, and replacing $\ell^u(G)$ by $\ell(G)$ (= the Lindelöf degree of G) is obtained in [30].

Every continuous action $\alpha\colon G \times X \to X$ induces the continuous action $G/\ker\alpha \times X \to X$, where $\ker\alpha = \{g \in G \mid gx = x \text{ for every } x \in X\}$. We say that α is *topologically exact* (for short: *t-exact*) [19], if α is algebraically exact (i.e., $\ker\alpha = \{e\}$) and there is no strictly coarser group topology on G, that makes α continuous.

Lemma 2.4. *Let $\langle G, X, \alpha\rangle \in \mathrm{Tych}^G$. Then there exists a ttg $\langle G, Y, \pi\rangle \in \mathrm{Comp}^G$ and a G-embedding $f\colon X \hookrightarrow Y$, such that $w(Y) = w(X) \cdot w(G)$ and π is t-exact.*

Proof. Consider the action $\alpha_\ell\colon G \times G \to G$ and the natural action of G on the disjoint sum $X \sqcup G$. Clearly, this action is t-exact. Since $\langle G, G, \alpha_\ell\rangle \in \mathrm{Tych}^G$ (see [8] or Lemma 1.2). Then, $X \sqcup G$ is G-Tychonoff as well. Finally, we apply Fact 2.3. $\square$

J. West's Theorem [32]. *Any continuous action of a topological group G on a closed Z-subset of the Hilbert cube $I^{\aleph_0}$ can be extended to a continuous action of G on $I^{\aleph_0}$.*

Theorem 2.5. *Let $\langle G, X, \alpha\rangle \in \mathrm{Tych}^G$ and $w(G) \cdot w(X) \leq \aleph_0$. Then $\langle G, X, \alpha\rangle$ is embedded into $\langle H(I^{\aleph_0}), I^{\aleph_0}, \pi\rangle$.*

Proof. By Lemma 2.4, without loss of generality, we may suppose that α is t-exact and X is a metrizable compact G-space. Embed X to $I^{\aleph_0}$ as a Z-space, and use West's Theorem. Then there exists a continuous action $\langle G, I^{\aleph_0}, \alpha^*\rangle$ such that algebraically G is a subgroup of $H(I^{\aleph_0})$ and $\alpha^* \mid_{G\times X} = \alpha$. Since α^* is continuous, then the given topology σ on G contains the compact-open topology σ^c defining with respect to the action α^*. The t-exactness of α (and hence of α^*) implies $\sigma = \sigma^c$. $\square$

Corollary 2.6. (Uspenskiĭ [27]). *$H(I^{\aleph_0})$ is a universal separable metrizable topological group.*

Theorem 2.7. *Let G be an $\aleph_0$-bounded group. Then for every G-Tychonoff space X there exists an equivariant embedding of $\langle G, X\rangle$ into $\langle H(I^\tau), I^\tau\rangle$ where $\tau \le w(X)\cdot w(G)$.*

Proof. It suffices to show that the given *ttg* $\langle G, X, \alpha\rangle$ is embedded into $\langle (H(I^{\aleph_0}))^\tau, (I^{\aleph_0})^\tau\rangle$ (see proof of Lemma 2.1). By Lemma 2.4 we can assume that α is t-exact and X is compact. Using the equivariant (generalization of Mardešic's) approximation Theorem [17, Th. 2.19], we can represent X as a G-limit of an inverse G-system $\{\langle G, X_i, \alpha_i\rangle \mid i \in M\}$ ($|M| \le \tau$), consisting of compact metrizable G-spaces X_i. Denote by f_i the i-th projection $X \to X_i$, and by φ_i the canonical homomorphism $G \to G/\ker\alpha_i$. As in the proof of Theorem 2.5, the topology σ_i of $G/\ker\alpha_i$ contains the compact-open topology σ_i^c, defining with respect to the action α_i. Since $w(X_i) \le \aleph_0$, then $w(G/\ker\alpha_i, \sigma_i^c) \le \aleph_0$. By Theorem 2.5, there exists an equivariant embedding

$$(\psi_i, h_i)\colon \langle (G/\ker\alpha_i, \sigma_i^c), X_i\rangle \hookrightarrow \langle H(I^{\aleph_0}), I^{\aleph_0}\rangle \qquad (i \in M).$$

Consider the diagonal products:

$$\tilde{h} = \prod_{i\in M}(h_i \circ f_i)\colon X \to (I^{\aleph_0})^\tau, \tilde{\psi} = \prod_{i\in M}(\psi_i \circ \varphi_i)\colon G \to (H(I^{\aleph_0}))^\tau.$$

Since $X = \varprojlim\{X_i \colon i \in M\}$, then $\tilde{h}$ is a topological embedding. Observe that the topological subgroup $\tilde{\psi}(G)$ acts continuously on $\tilde{h}(X)$. Since $\tilde{h}$ at the same time is equivariant, then the t-exactness of α implies that $\tilde{\psi}$ is a topological group embedding.

$\square$

Corollary 2.8. *For every $\aleph_0$-bounded group G holds* $\mathrm{Tych}^G = \mathrm{HTych}^G$.

Question 2.9. Is it true that $\mathrm{Tych}^G = \mathrm{HTych}^G$ for every G?

Probably, the most natural candidate to be a counterexample is the above-mentioned (Remark 2.2) example from [4].

Concerning Theorem 2.7, let us note that a related result for an LC σ-compact group G (but with additional important properties) is proved in [2].

3. Free Topological G-Groups.

As usual, for a topological space X denote by $F(X), A(X), L(X)$ the free topological group, the free Abelian topological group, and the free locally convex space respectively.

Let $\langle G, X, \alpha \rangle$ be a *ttg*. We will say that an automorphic triple $\langle G, F_\alpha(X), \tilde{\alpha} \rangle$ with a continuous G-mapping $i_\alpha \colon X \to F_\alpha(X)$ is the *free topological G-group* over X, if for every continuous G-mapping $\varphi \colon X \to P$ to an automorphic G-space P there exists an unique continuous G-homomorphism $\tilde{\varphi} \colon F_\alpha(X) \to P$ such that $\tilde{\varphi} \circ i_\alpha = \varphi$. If μ is an uniformity on X, then considering uniform G-mappings and the upper uniformities on topological groups, we obtain the definition of the *uniform free topological G-group* over (X, μ). The corresponding universal morphism is denoted by $i_\alpha \colon (X, \mu) \to F_\alpha(X, \mu)$.

The *(uniform) free locally convex G-space* $L_\alpha(X)$ (respectively : $L_\alpha(X, \mu)$) can be analogously defined.

An obvious *equivariant* generalization of the standard product procedure (see for example, [13]), shows that the just defined free G-objects always exist. However, it turns out that the *embedding problem* for i_α is much more complicated. Clearly, i_α is a topological (uniform) embedding iff $\langle G, X, \alpha \rangle \in \mathrm{ATych}^G$ (respectively : $\langle G, (X, \mu), \alpha \rangle \in \mathrm{AUnif}^G$). In this section we show that, in general, $\mathrm{HTych}^G \neq \mathrm{ATych}^G$. Our main tool will be a "regionally proximal-type" relation. We start with the well known (at least for $S = G$) definition from topological dynamics.

Let $\langle G, X, \alpha \rangle$ be a *ttg*, μ be a uniformity on X and $S \subset G$. A pair $(a, b) \in X \times X$ is called *regionally S-proximal* [7] and is indicated: $(a, b) \in Q_S$, if for every $\varepsilon \in \mu$ and arbitrary nbd's $O_1 \in N_a(X), O_2 \in N_b(X)$ there exists $g \in S$ such that gO_1 and gO_2 are ε-near. Otherwise, (a,b) is said to be regionally S-distal. X is called regionally S-distal if $Q_S = \Delta_X := \{(x,x) \mid x \in X\}$.

The following definition seems to be new.

Definition 3.1. We say that a pair $(a,b) \in X \times X$ is *regionally S-pseudoproximal* and write : $(a,b) \in Q_S^p$ (or: $(a,b) \in Q_S^p(X,\mu)$) if there exists a finite set $\{a = x_0, x_1, \dots, x_n = b\}$ with the following property:

($*^S$) for every $\varepsilon \in \mu$ and arbitrary nbd's $O_i \in N_{x_i}(X)$, $i \in \{0,1,\dots,n\}$ there exists $g \in S$ such that gO_i and gO_{i+1} are ε-near, for every $i \in \{0,1,\dots,n-1\}$.

A pair (a,b) will be called *regionally $*$-pseudoproximal* if $(a,b) \in Q_V^p$ for every $V \in N_e(G)$. This defines a relation $Q_*^p = \cap\{Q_V^p \mid V \in N_e(G)\}$. If $Q_*^p = X \times X$ or $Q_*^p = \Delta_X$, then we say that X is regionally $*$-pseudoproximal, or regionally $*$-pseudodistal respectively.

Obviously, Q_S^p and Q_*^p are reflexive symmetric relations on X and always $Q_S \subset Q_S^p, Q_*^p \subset Q_S^p$. If S is symmetric and acts uniformly μ-equicontinuously on X (i.e., if the family $\{\alpha^g \mid g \in S\}$ is uniformly μ-equicontinuous), then $Q_S^p = Q_S = \Delta_X$. Thus, if $G \in LC$ and X is compact then $Q_*^p = \Delta_X$. In general, $Q_S \neq Q_S^p$ and $Q_*^p \neq Q_S^p$.

Example 3.2. Let $G_n = \{h \in H(I) \mid h(x_i) = x_i, x_i = \frac{i}{n},\ i \in \{0,1,\dots,n\}\}$ be the topological subgroup of $H(I)$. Consider the *ttg* $\langle G_n, I, \alpha\rangle$ and the canonical uniformity on I. Then, for every natural $n \geq 3$, the elements 0 and 1 are regionally G_n-distal. On the other hand, every pair $(a,b) \in I \times I$ is regionally $*$-pseudoproximal. Therefore, Q_{G_n} is a proper subset of $Q_{G_n}^p$ for each $n \geq 3$.

Example 3.3. Define the homeomorphism $h : I \to I$ by the rule

$$h(x) = \begin{cases} 3x^2, & 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} + \frac{1}{3}\sqrt{3x-1}, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3x^2 - 4x + 2, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Consider the cyclic group $G = \{h^n\}_{n \in \mathbb{Z}}$ and the natural action $G \times I \to I$. Since

$0, \frac{1}{3}, \frac{2}{3}, 1$ are fixed then clearly $(0,1) \notin Q_G$. On the other hand, elementary computations show that $Q_G^p = I \times I$. Note also that $Q_*^p = \Delta_X$ if G is discrete.

Lemma 3.4. *If $f\colon (X_1, \mu_1) \to (X_2, \mu_2)$ is a uniform G-mapping, then $(f \times f)(Q_S^p(X_1, \mu_1)) \subset Q_S^p(X_2, \mu_2)$ and $(f \times f)(Q_*^p(X_1, \mu_1)) \subset Q_*^p(X_2, \mu_2)$. In particular, if (X, μ) is regionally $*$-pseudodistal, then every uniform G-subspace $(Y, \mu|_Y)$ is regionally $*$-pseudodistal.*

Theorem 3.5. *Every G-group $\langle G, (X, \mu), \alpha \rangle$ is regionally $*$-pseudodistal for each $\mu \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$.*

Proof. First we consider the case $\mu = \mathcal{R}$. Assuming the contrary, take a pair $(a, b) \in Q_*^p$ of distinct elements. Since X is a Hausdorff topological group and α is continuous, then we can choose nbd's $V_0 \in N_e(X)$, $U \in N_e(G)$ such that

$$V_0 \cap g(V_0 a b^{-1}) = \emptyset \qquad \forall\, g \in U \tag{1}$$

Since $Q_*^p \subset Q_U^p$, then $(a, b) \in Q_U^p$. Consider a finite set $\{a = x_0, x_1, \dots, x_n = b\}$ satisfying Definition 3.1. Choose symmetric nbd's $V_1, V_2 \in N_e(X)$ with the properties:

$$x_0 x_i^{-1} V_2^2 \subset V_1 x_0 x_i^{-1} \qquad \forall\, i \in \{0, 1, \dots, n\} \tag{2}$$

$$V_1^{n+1} \subset V_0 \tag{3}$$

Due to Definition 3.1, we pick for $\varepsilon\colon = \{(x, y) \in X \times X \mid xy^{-1} \in V_2\}$ an element $g \in U$ such that $g(V_2 x_i)$ and $g(V_2 x_{i+1})$ are V_2-near, with respect to the right uniformity $\mathcal{R}$ on X. More precisely, there exist finite sequences $\{p_0, p_1, \dots, p_{n-1}\}, \{q_1, q_2, \dots, q_n\}$ in V_2 such that $g(p_i x_i)(g(q_{i+1} x_{i+1}))^{-1} \in V_2$ for every $i \in \{0, 1, \dots, n-1\}$. Since α^g is an automorphism, then

$$g(p_i x_i x_{i+1}^{-1} q_{i+1}^{-1}) \in V_2 \qquad \forall\, i \in \{0, 1, \dots, n-1\} \tag{4}$$

Consider the element

$$z = g(p_0 x_0 x_1^{-1} q_1^{-1})\; g(p_1 x_1 x_2^{-1} q_2^{-1}) \cdots g(p_{n-1} x_{n-1} x_n^{-1} q_n^{-1}).$$

Since $V_2 \subset V_1$ by (2), then (4) and (3) imply $z \in V_2^n \subset V_1^n \subset V_0$. Eventually,

$$z = g(p_0 x_0 x_1^{-1}(q_1^{-1}p_1)x_1 x_2^{-1}(q_2^{-1}p_2)\cdots(q_{n-1}^{-1}p_{n-1})x_{n-1}x_n^{-1}q_n^{-1}).$$

Clearly, $q_i^{-1}p_i \in V_2^{-1}V_2 = V_2^2$ for each $i \in \{1,\dots,n-1\}$. Using (2) and the trivial cancellations of the form $x_0 x_i^{-1} x_i x_{i+1}^{-1} = x_0 x_{i+1}^{-1}$, $(1 \le i \le n-1)$ after $n-1$ steps we get

$$z \in g(p_0 V_1^{n-1} x_0 x_n^{-1} q_n^{-1}) \subset g(V_1^n x_0 x_n^{-1} q_n^{-1}).$$

Using (2) (for $i = n$), we obtain

$$z \in g(V_1^{n+1} x_0 x_n^{-1}) = g(V_1^{n+1} ab^{-1}) \subset g(V_0 ab^{-1}).$$

Thus, $z \in V_0 \cap g(V_0 ab^{-1})$, which contradicts (1). This proves the case $\mu = \mathcal{R}$.

For $\mu = \mathcal{L}$, use the G-unimorphism $(X,\mathcal{L}) \to (X,\mathcal{R}), x \to x^{-1}$ and if $\mu = \mathcal{L} \vee \mathcal{R}$ use Lemma 3.4 for the uniform G-mapping $f = 1_X \colon (X, \mathcal{L} \vee \mathcal{R}) \to (X, \mathcal{R})$. $\square$

Theorem 3.6. *Let (X,μ) be a $*$-pseudoproximal G-space. Then every uniform G-mapping $(X,\mu) \to (Y,\xi)$ into a G-group Y is trivial for each $\xi \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$. In particular, the free uniform G-group $F_\alpha(X,\mu)$ is cyclic and discrete (that is, as trivial as possible).*

Proof. Combine Lemma 3.4 and Theorem 3.5. $\square$

Example 3.7. Let $X = I^n$ be the n-dimensional cube, or let $X = \mathbb{S}^n$ be the n-dimensional sphere (in both cases $n \in \mathbb{N}$). Then $\langle H(X), X, \alpha \rangle$ is a regionally $*$-pseudoproximal *ttg* with respect to the unique uniformity on X. Then, by Theorem 3.5, the free topological G-group $F_\alpha(X)$ is cyclic and discrete. Since $\langle H(\mathbb{S}^n), \mathbb{S}^n, \alpha \rangle$ is a homogeneous triple, then we get that $\mathrm{HTych}^G \neq \mathrm{ATych}^G$ and $\mathrm{Unif}^G \neq \mathrm{AUnif}^G$ for $G\colon = H(\mathbb{S}^n)$. This example for $X = I$ answers the question posed by the author in [17, Problem 1.14].

Question 3.8. Under which conditions does the homogeneous triple $\langle G, G/H, \alpha_\ell \rangle$ belong to ATych^G?

This is so, for example, if H is a *neutral* [25] subgroup. Indeed, in such cases, Theorem 5.8 and Proposition 7.7 from [25] imply that the action α_ℓ of G on G/H is uniformly equicontinuous with respect to the quotient uniformity $\mathcal{L}/H$. Therefore, by [22, Th.1.2] (or by our Proposition 3.11) G/H is even G-linearizable.

Question 3.9. Under which conditions does the free uniform G-group $F_\alpha(X,\mu)$ coincide with the free uniform group $F(X,\mu)$ over X?

Fact 3.10 [17]. (For a stronger version see [18, Lemma 2.1]). Let an action $\alpha\colon G\times X\to X$ be quasibounded with respect to a uniformity μ on X and let each g-transition be continuous. Suppose that orbit mapping $\alpha_y\colon G\to X$ is continuous for each $y\in Y$, where Y is dense in X. Then α is continuous.

Proposition 3.11. *Let $\alpha\colon G\times X\to X$ be a saturated action on a uniform space (X,μ). Suppose that a certain $V\in N_e(G)$ acts μ-uniformly equicontinuous. Then $F_\alpha(X,\mu)=F(X,\mu)$ and $L_\alpha(X,\mu)=L(X,\mu)$.*

Proof. Let $\tilde\alpha\colon G\times F(X,\mu)\to F(X,\mu)$ be the lifted action. Clearly, each g-transition $\tilde\alpha^g$ is continuous. Since X algebraically generates $F(X,\mu)$, then the continuity of orbit mappings $\alpha_x\colon G\to X$ and of group operations in $F(X,\mu)$ imply that for each $w\in F(X,\mu)$ the orbit mapping $\tilde\alpha_w\colon G\to F(X,\mu)$ is continuous. From the constructive description of a neighborhood system of the identity in $F(X,\mu)$ [20, 23] follows that V acts $\mathcal{L}\vee\mathcal{R}$-uniformly equicontinuously on $F(X,\mu)$. In particular $\tilde\alpha$ is $\mathcal{L}\vee\mathcal{R}$-quasibounded. By Fact 3.10, $\tilde\alpha$ is continuous. Obviously, this implies that $F_\alpha(X,\mu)=F(X,\mu)$. Essentially the same proof works for $L(X,\mu)$ using [24]. $\square$

Pestov [21] proved the continuity of the associated action $\tilde\alpha\colon G\times F^b_\nu(X)\to F^b_\nu(X)$ for the uniformly equicontinuous action $\alpha\colon G\times X\to X$, where $F^b_\nu(X)$ denotes the free uniform *balanced* (i.e., $\mathcal{L}=\mathcal{R}$) group in a variety ν. For an analogous "lifting" Theorem for a modification of the free locally convex spaces, see [22, Th. 1.2].

Lemma 3.12. *If $G \in LC$ and $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$, then each compact subset of G acts μ-uniformly equicontinuous. In particular, this holds for $\mu = \mu_{\max}$.*

Proof. The standard compactness arguments prove the first assertion. In order to apply it to our second assertion, observe that $\langle G, (X, \mu_{\max}), \alpha \rangle \in \mathrm{Unif}^G$ (see [18, Pr. 3.7]). □

Proposition 3.13. *If $G \in LC$ and $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$, then $F_\alpha(X, \mu) = F(X, \mu)$ and $L_\alpha(X, \mu) = L(X, \mu)$. In particular, $F_\alpha(X) = F(X)$ and $L_\alpha(X) = L(X)$.*

Proof. Apply Lemma 3.12 and Proposition 3.11. □

Concerning the Proposition 3.13, it should be mentioned that the continuity of the lifted actions on $A(X)$ and $L(X)$ (under the assumption $G \in LC$) has been established earlier by Eisenberg [10] and de Vries [31] respectively.

Example 3.14. We give an example where $i_\alpha \colon X \to F_\alpha(X)$ is a topological embedding and the group $F_\alpha(X)$ only *algebraically* coincides with $F(X)$. Let $\mathbb{Q}$ be the topological group of all rational numbers. Consider the *ttg* $\langle \mathbb{Q}, (\mathbb{Q}, \mu_{\max}), \alpha_\ell \rangle$. Then $\mu_{\max}$ is complete and saturated, but there is no continuous non-trivial action of $\mathbb{R}$ on $\mathbb{Q}$. By Fact 1.9, α_ℓ is not $\mu_{\max}$-quasibounded. Then Fact 1.6 implies that $i_\alpha \colon (\mathbb{Q}, \mu_{\max}) \to (F_{\alpha_\ell}(\mathbb{Q}), \mathcal{L} \vee \mathcal{R})$ is not a uniform embedding. Thus, $F_{\alpha_\ell}(\mathbb{Q}) \neq F(\mathbb{Q})$. On the other hand, by Proposition 3.11, $F_{\alpha_\ell}(\mathbb{Q}, \mathcal{L}) = F(\mathbb{Q}, \mathcal{L})$. Since $F(\mathbb{Q}, \mathcal{L})$ is, in particular, algebraically free over $\mathbb{Q}$, then clearly $F_{\alpha_\ell}(\mathbb{Q})$ is also algebraically free over $\mathbb{Q}$.

Question 3.15. Is this true that if $\langle G, X, \alpha \rangle \in \mathrm{ATych}^G$, then $F_\alpha(X)$ is algebraically free over X?

4. G-Linearizations.

Definition 4.1. Let μ be the uniformity generated by a system $\mathcal{P} = \{\rho_i\}_{i \in M}$ of pseudometrics on a space X. We say that an action $\alpha \colon G \times X \to X$ is *weakly $\mathcal{P}$-Lipschitz* if for every $(i, g) \in M \times G$ there exists $(c, j, V) \in \mathbb{R} \times M \times N_g(G)$ such that

$\rho_i(gx, gy) \leq c\rho_j(x, y)$ whenever $\rho_i(x, y) \neq 0$ and $g \in V$. If ρ_j can be choosed in such a way that the inequality holds for every $(g, x, y) \in V \times X \times X$, then we say that the action is *$\mathcal{P}$-Lipschitz.* If α is continuous and $\mathcal{P}$-Lipschitz (resp.: weakly $\mathcal{P}$-Lipschitz) then we write $\langle G, (X, \mathcal{P}), \alpha\rangle \in \mathrm{Lip}^G$ (respectively: $\in \mathrm{Lip}^G_w$). If each ρ_i is a metric then these concepts coincide.

Theorem 4.2. *If $\langle G, (X, \mu), \alpha\rangle \in \mathrm{LUnif}^G$ then there exists a system $\mathcal{P} = \{\rho_i\}_{i\in M}$ of pseudometrics generating μ such that $\langle G, (X, \mathcal{P}), \alpha\rangle \in \mathrm{Lip}^G_w$.*

Proof. Let (X, μ) be a uniform G-subspace of a locally convex linear G-space E. Consider a system $\{p_i\}_{i\in M}$ of seminorms on E which generates the usual structure of E. We may assume that for every pair $i, j \in M$ there exists $k \in M$ such that $p_i + p_j \leq p_k$. Fix $g \in G$ and $i \in M$. From the continuity of the action $G \times E \to E$ at $(g, 0_E)$, where 0_E denotes the zero-element, it follows that for a certain $\delta > 0, V \in N_g(G)$ and $j \in M$, the inequality $p_j(a) \leq \delta$ implies $p_i(ga) \leq 1$ for each $(g, a) \in V \times E$. We can suppose that $p_i \leq p_j$. Then $p_i(x) \neq 0$ implies that $p_j(x) \neq 0$. Since $p_j\left(\frac{\delta}{p_j(x)}x\right) = \delta$, for every $x \in X$ with the property $p_i(x) \neq 0$, then $p_i\left(g\left(\frac{\delta}{p_j(x)}x\right)\right) \leq 1$. Therefore, $p_i(gx) \leq \frac{1}{\delta}p_j(x)$ for every $(g, x) \in V \times E$ where $p_i(x) \neq 0$. Now, it is obvious that the system of pseudometrics $\{\rho_i\}_{i\in M}$, where $\rho_i(x, y) = p_i(x - y)$ for $x, y \in X$, is the desired one. $\square$

Theorem 4.3. *If $\langle G, (X, \mathcal{P}), \alpha\rangle \in \mathrm{Lip}^G$ for a certain collection $\mathcal{P}$ generating μ then $\langle G, (X, \mu), \alpha\rangle \in \mathrm{LUnif}^G$.*

Proof. Without restriction of generality we can assume that X has a fixed point. Apply the Arens-Eells [3] embedding $(X, \{\rho_i\}_{i\in M}) \hookrightarrow (E, \{\tilde{\rho}_i\}_{i\in M})$. The explicit description of $\tilde{\rho}_i - s$ shows that the associated action $\tilde{\alpha}\colon G \times E \to E$ is $\{\tilde{\rho}_i\}_{i\in M}$-Lipschitz. Since each orbit mapping $G \to E$ is continuous, then by Fact 3.10, $\tilde{\alpha}$ is continuous. $\square$

We write $\langle G, X, \alpha\rangle \in \mathrm{BTych}^G$ if X is a G-subspace of a Banach G-space. Analogously, $\langle G, (X, \mu), \alpha\rangle \in \mathrm{BUnif}^G$ means that (X, μ) is a uniform G-subspace in a Banach

G-space.

Theorem 4.4. *For a metrizable uniformity μ, the following conditions are equivalent:*

(i) $\langle G,(X,\mu),\alpha\rangle \in \mathrm{BUnif}^G$;

(ii) $\langle G,(X,d),\alpha\rangle \in \mathrm{Lip}^G$ *(or, equivalently: $\in Lip_w^G$) for a metric d;*

(iii) *There exists a countable cover $\cup\{S_n\}_{n\in\mathbb{N}} = G$ such that:*

a) Each S_n acts μ-uniformly equicontinuous.

b) A certain S_{n_0} has an interior point.

Proof. (i) $\Rightarrow$ (ii) in fact, is the same as in Theorem 4.2. For (ii) $\Rightarrow$ (iii), consider $S_n = \{g \in G \mid d(gx,gy) \le nd(x,y) \text{ for every } x,y \in X\}$.

For (iii) $\Rightarrow$ (i) take a metric $\rho \le 1$ on X which generates μ, and define inductively $A_0 = \{e\}$ and $A_{n+1} = A_n \cdot A_n \cup S_{n+1}$. Then $\{A_n\}_{n\in\mathbb{N}}$ also satisfies (iii) and, in addition, holds $A_n \cdot A_m \subset A_{n+m}$. From this inclusion easily follows that if we define

$$d(x,y) = \sup_{n\in\mathbb{N}} \left\{ \frac{1}{2^n} \sup_{g\in A_n} \{\rho(gx,gy)\} \right\}$$

then $d(gx,gy) \le 2^n d(x,y)$ for each $(g,x,y) \in A_n \times X \times X$. Using the arguments from the proof of Theorem 4.3, by the Arens-Eells construction, there exists an isometric G-embedding of (X,d) into a normed G-space E. Since $\mathrm{LUnif}^G \subset \mathrm{Unif}^G$, then Fact 1.9 completes the proof. $\square$

Theorem 4.5. *Let G be an LC σ-compact group, and X be a metrizable G-space. Then $\langle G,X,\alpha\rangle \in \mathrm{BTych}^G$. Moreover, the following statements are equivalent:*

(i) $\langle G,(X,\mu),\alpha\rangle \in \mathrm{BUnif}^G$;

(ii) *Each g-transition is μ-uniformly continuous and μ is metrizable.*

Proof. (i) $\Rightarrow$ (ii) is trivial. For (ii) $\Rightarrow$ (i) use consequently Fact 1.7(b), Lemma 3.12 and apply Theorem 4.4. In order to check $\langle G,(X,\mu),\alpha\rangle \in \mathrm{BTych}^G$, if suffices to show that there exists a metrizable saturated uniformity μ. A stronger result can be found in [18, Th. 3.8]. So our proof is completed. $\square$

Theorem 4.6. *Let X be a compact metrizable G-space and G be a closed topological subgroup of $H(X)$. Then the following statements are equivalent:*

(i) *G is LC and σ-compact;*

(ii) $\langle G, X, \alpha\rangle \in \mathrm{BTych}^G$.

Proof. (i) $\Rightarrow$ (ii) directly follows from Theorem 4.5. For (ii) $\Rightarrow$ (i) use the implication (i) $\Rightarrow$ (iii) from Theorem 4.4 and apply the Ascolli-Arzela Theorem. $\square$

Theorems 4.4 and 4.5 have been proved also in [15]. We close the paper with the following principal counterexample.

Example 4.7. There exists a G-group X which is not G-linearizable. This means that, in general, $\mathrm{ATych}^G \neq \mathrm{LTych}^G$.

Construction. Let X be the measure algebra of all equivalence classes of Lebesgue measurable subsets of the closed interval I. Under the operation Δ (the symmetric difference) and the norm $\|A\| =$ meas (A), X is a separable complete-metrizable topological Abelian group homeomorphic to the Hilbert space ℓ_2 (see Theorem 7.1 and Corollary 7.2 in [5, VI. 7]). Let G be the group of all such homeomorphisms $h\colon I \to I$ that h and h^{-1} are Lipschitz and denote by $\alpha\colon G \times I \to I$ the action $\alpha(h,x) = h(x)$. Since each α^h is a measurable function, then the associated action $\alpha^*\colon G \times X \to X$ is well defined. Moreover, each transition $(\alpha^*)^h\colon X \to X$ is $\|\cdot\|$-uniformly continuous. Endow G with the topology $\tau_{\|\cdot\|}$ of $\|\cdot\|$-uniform convergence. Then α^* is continuous. It turns out that the constructed G-group X is not G-linearizable.

Proof. In order to show that X is a desired G-group, observe that the mapping $\varphi\colon I \to X, \varphi(t) = [0,t]$ is an isometric G-embedding. Therefore, it suffices to check that $\langle G, I, \alpha\rangle$ is not G-linearizable. We will prove in fact that I, as a G-space, is not even G-mapped non-trivially into a linear G-space. Assume the contrary. Then, due to Theorem 4.2, α is weakly $\mathcal{P}$-Lipschitz for a certain family $\mathcal{P} = \{\rho_i\}_{i\in M}$ of continuous pseudometrics on I containing a non-constant ρ_{i_0}. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ in I converging in the usual topology to a point x_0 such that $\rho_{i_0}(x_n, x_0) \neq 0$ for every $n \in \mathbb{N}$. Indeed, if this is not so, then for every $x \in I$ one can choose $\delta_x > 0$ such that $\rho_{i_0}(y, x) = 0$ for every $y \in B(x, \delta_x)$, where $B(x, \delta_x) = \{y \in I \mid |y - x| < \delta_x\}$. There exists a finite cover $\{B(x_k, \delta_{x_k})\}_{k=1}^n$ of I, such that $B(x_k, \delta_{x_k}) \cap B(x_{k+1}, \delta_{x_{k+1}}) \neq \emptyset$ for each $k \in \{1, 2, \ldots, n-1\}$. Since ρ_{i_0} is a pseudometric, then by the triangle axiom, we get $\rho_{i_0}(x, y) = 0$ for every $x, y \in I$. This contradiction proves the existence of the above-mentioned $(x_n)_{n\in\mathbb{N}}$. Without any loss of generality we may assume that this sequence is monotonic. For simplicity suppose that $(x_n)_{n\in\mathbb{N}}$ is increasing and hence each x_n lies in $[0, x_0)$.

By Definition 4.1 there exists a triple (c, ρ_j, V) from $\mathbb{R} \times \mathcal{P} \times N_e(G)$ such that $V = V^{-1}$ and $\rho_(x, y) \neq 0$ implies that

$$\rho_{i_0}(h(x), h(y)) \leq c\rho_j(x, y) \quad \forall\, h \in V. \tag{*}$$

Since ρ_j is a continuous pseudometric, then $\rho_j(x_n, x_0)$ tends to zero. Thus we can choose a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_k < x_{n_k}$ and

$$2c\rho_j(x_{n_k}, x_0) \leq \rho_{i_0}(x_k, x_0) \quad \forall\, k \in \mathbb{N}. \tag{**}$$

For each $k \in \mathbb{N}$ there exists a homeomorphism $h_k \colon I \to I$ satisfying:

(1) $h_k(x_k) = x_{n_k}$;

(2) $h_k(t) = t$ for every $t \in [0, x_{k-1}] \cup [x_0, 1]$;

(3) h_k is linear on $[x_{k-1}, x_k]$ and $[x_k, x_0]$.

Clearly, each h_k belongs to G. Moreover, (2) implies that

$$\text{meas}\ (h_k(A)\Delta A) \leq |x_{k-1} - x_0| \quad \forall\, A \in X.$$

Therefore, $(h_k)_{k\in\mathbb{N}}$ converges to the identity in G. Thus, $h_m \in V$ for a certain $m \in \mathbb{N}$. Then, by (**), (*), (1), (2), we get

$$\rho_{i_0}(x_m, x_0) \geq 2c\rho_j(x_{n_m}, x_0) \geq 2\rho_{i_0}\left(h_m^{-1}(x_{n_m}), h_m^{-1}(x_0)\right) = 2\rho_{i_0}(x_m, x_0).$$

Hence, $\rho_{i_0}(x_m, x_0) \geq 2\rho_{i_0}(x_m, x_0) > 0$. This contradiction completes the proof. $\square$

Acknowledgemement. I would like to express my gratitude to Yu.M. Smirnov, J. de Vries and V.G. Pestov for valuable remarks. I am also grateful to the referee for helpful comments.

References

1. F.D. Ancel, *An alternative proof and applications of a theorem of E.G. Effros*, Michigan Math. J. **34** (1987), 39-55.
2. S.A. Antonyan, J. de Vries, *Tychonov's theorem for G-spaces*, Acta Math. Hung. **50** (1987), 253-256.
3. R.F. Arens, J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397-403.
4. D.P. Bellamy, K.F. Porter, *A homogeneous continuum that is non-Effros*, Proc. Amer. Math. Soc. **113** (1991), 593-598.
5. C. Bessaga, A. Pelczyński, *Selected Topics in Infinite-Dimensional Topology*, P.W.N., Warszawa, 1975.
6. N. Bourbaki, *Éléments de Mathématique: Intégration, Ch. VIII*, Hermann, Paris, 1966.
7. I.U. Bronstein, *Extensions of minimal transformation groups*, Sijthoff & Noordhoff, Alpher aan den Rijn, 1979.
8. R.B. Brook, *A construction of the greatest ambit*, Math. Systems Theory **4** (1970), 243-248.
9. E.G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. **(2) 81** (1965), 38-55.
10. M. Eisenberg, *Embedding a transformation group in an automorphism group*, Proc. Amer. Math. Soc. **23** (1969), 276-281.
11. R. Engelking, *General Topology*, P.W.N., Warszawa, 1977.
12. L.R. Ford, *Homeomorphism groups and coset spaces*, Trans. Amer. Math. Soc. **77** (1954), 490-497.
13. M.I. Graev, *Theory of topological groups I*, (in Russian), Uspekhi Mat. Nauk **5** (1950), 2-56.
14. I.I. Guran, *On topological groups, close to being Lindelöf*, Soviet Math. Dokl. **23** (1981), 173-175.
15. M. Megrelishvili, *Quasibounded uniform G-spaces*, (in Russian), Manuscript deposited at Gruz NIINTI (Tbilisi) on March 3, 1987, No.331-G.
16. ______, *A Tychonoff G-space not admitting a compact Hausdorff G-extension or G-linearization*, Russian Math. Surveys **43:2** (1988), 177-178.
17. ______, *Compactification and factorization in the category of G-spaces*, Categorical Topology and its Relation to Analysis, Algebra and Combinatorics (J. Adamek, S. MacLane, eds.), World Scientific, Singapore, 1989, pp. 220-237.
18. ______, *Equivariant completions*, Comment. Math. Univ. Carolinae **35:3** (1994), 539-547.
19. ______, *Group representations and construction of minimal topological groups*, Topology Appl. **62** (1995), 1-19.
20. V.G. Pestov, *Neighborhoods of unity in free topological groups*, Moscow Univ. Math. Bull. **40** (1985), 8-12.
21. ______, *To the theory of free topological groups: free groups, extensions, and compact coverability*, (in Russian), Manuscript deposited at VINITI (Moscow) on April 1, 1985, No. 2207-85.
22. ______, *On unconditionally closed sets and a conjecture of A.A. Markov*, Siberian Math. J. **29** (1988), 260-266.
23. ______, *Universal arrows to forgetful functors from categories of topological algebra*, Bull. Austral. Math. Soc. **48** (1993), 209-249.
24. D.A. Raĭkov, *Free locally convex spaces for uniform spaces*, (in Russian), Mat. Sb. (N.S.) **63** (1964), 582-590.
25. W. Roelcke, S. Dierolf, *Uniform Structures in Topological Groups and their Quotients*, McGraw-Hill, New York, 1981.
26. Yu.M. Smirnov, *On equivariant imbeddings of G-spaces*, Russian Math. Surveys **31** (1976), 198-209.
27. V.V. Uspenskiĭ, *A universal topological group with a countable base*, Functional Anal. Appl. **20** (1986), 160-161.

28. J. de Vries, *Can every Tychonoff G-space equivariantly be embedded in a compact Hausdorff G-space?*, Math. Centrum, Amsterdam, Afd. Zuivere Wisk., ZW **36** (1975).
29. ———, *On the existence of G-compactifications*, Bull. Ac. Polon. Sci. Ser. Math. **26** (1978), 275-280.
30. ———, *Topics in the theory of topological transformation groups*, Topological Structures II (Proc. Symp. Amsterdam, 1978), Math. Centre Tracts No. 116 (P.C. Baayen, J. van Mill, eds.), Math. Centrum, Amsterdam, 1979.
31. ———, *Linearization of locally compact group actions*, (in Russian), Trudy Mat. Inst. Steklov **154** (1983), 53-70.
32. J. West, *Extending certain transformation group actions in separable, infinite dimensional Frechet spaces and the Hilbert cube*, Bull. Amer. Math. Soc. **74** (1968), 1015-1019.
33. Ta-Sun Wu, *Notes on coset transformation groups*, Topological dynamics (J.Auslander, W.H. Gottschalk, eds.), W.A. Benjamin, Inc., New York, Amsterdam, 1968, pp. 507-512.

E-mail: megereli@macs.cs.biu.ac.il