ON ONE-RELATOR INVERSE MONOIDS AND ONE-RELATOR GROUPS

S.V. Ivanov, S.W. Margolis, and J.C. Meakin

Abstract. It is known that the word problem for one-relator groups and for one-relator monoids of the form $\mathrm{Mon}\langle A \parallel w = 1\rangle$ is decidable. However, the question of decidability of the word problem for general one-relation monoids of the form $M = \mathrm{Mon}\langle A \parallel u = v\rangle$ where u and v are arbitrary (positive) words in A remains open. The present paper is concerned with one-relator inverse monoids with a presentation of the form $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ where w is some word in $A \cup A^{-1}$. We show that a positive solution to the word problem for such monoids for all reduced words w would imply a positive solution to the word problem for all one-relation monoids. We prove a conjecture of Margolis, Meakin and Stephen by showing that every inverse monoid of the form $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$, where w is *cyclically* reduced, must be E-unitary. As a consequence the word problem for such an inverse monoid is reduced to the membership problem for the submonoid of the corresponding one-relator group $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ generated by the prefixes of the cyclically reduced word w. This enables us to solve the word problem for inverse monoids of this type in certain cases.

1. Introduction: Presentations of Groups and Semigroups.

We shall be concerned in this paper with presentations of groups, monoids and inverse monoids. For an alphabet (i.e. non-empty set) A we denote by A^* the free monoid on A and by $\mathcal{A}$ the (group) alphabet $\mathcal{A} = A \cup A^{-1}$, where A^{-1} is a set disjoint from A and in one-one correspondence with A in the usual way. The group presented by a set A of generators and relations of the form $w_i = 1,\ i \in I$ for some words $w_i \in \mathcal{A}^*$ will be denoted by $\mathrm{Gp}\langle A \parallel w_i = 1,\ i \in I\rangle$. It is the quotient of the free group $FG(A)$ by the normal subgroup generated by $\{w_i : i \in I\}$. The monoid presented by a set A of generators and relations of the form $u_i = v_i, i \in I$ for some words $u_i, v_i \in A^*$ is denoted by $\mathrm{Mon}\langle A \parallel u_i = v_i,\ i \in I\rangle$: it is the quotient of the free monoid A^* by the congruence generated by the corresponding relations. We refer the reader to the books by Lyndon and Schupp [LS] and Lallement [La] for standard ideas and terminology concerning presentations of groups and semigroups (monoids) respectively.

1991 *Mathematics Subject Classification.* Primary 20F05, 20F06, 20F32, 20M05, 20M18; Secondary 57M20.

The first author was supported in part by an Alfred P. Sloan Research Fellowship and NSF grants DMS 95-01056, DMS 98-01500. The second and third authors were supported in part by NSF grant DMS 96-23284

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The study of one-relator groups is by now a classical part of combinatorial group theory. We recall here that important early work on one-relator groups was done by Magnus [Ma] in the 1930's (see also [LS]). Magnus showed decidability of the word problem for a one-relator group $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$, where w is a cyclically reduced word in $\mathcal{A}^*$, and also proved the "Freiheitssatz", namely that any non-trivial relator of G must involve each letter in the word w.

The situation for one-relation monoids is considerably more complex. By using Magnus' results, Adjan [Ad] studied the word problem for one-relation monoids, i.e. monoids with a presentation of the form $M = \mathrm{Mon}\langle A \parallel u = v\rangle$, where u, v are words in A^*. He showed that the word problem for such a monoid is decidable if one of the words is empty or if both words are non-empty with different initial letters and different terminal letters. Alternative proofs of some of Adjan's results may be found in the papers of Lallement [La2] and Zhang [Zh]. There is a substantial literature devoted to the study of the word and divisibility problems for one-relation monoids. We mention here a result of Adjan and Oganessian [AO] who reduced the word problem for such a monoid to the case where u and v have different initial letters (or dually the case where u and v have different terminal letters). In other words, the problem is reduced to a consideration of one-relator monoids of the form $\mathrm{Mon}\langle A \parallel asb = atc\rangle$ where $a, b, c \in A,\ b \neq c$ and $s, t \in A^*$. Adjan [Ad] also showed that if u and v have different initial (resp. terminal) letters then the corresponding monoid is left cancellative (resp. right cancellative). In general, the word problem for one-relation monoids remains unsolved, as far as we are aware. For some recent results along these lines and for some additional references to the literature, we refer to the papers by Guba [Gu] and Watier [Wa].

We will be concerned in this paper with one-relator *inverse* monoids, more precisely with inverse monoids with a presentation of the form $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$, where w is some (not necessarily reduced) word in $\mathcal{A}^*$. We discuss some preliminary results about inverse monoids in the next section and show that the word problem for one-relator inverse monoids of the type mentioned above is at least as complex as the word problem for one-relation monoids, even in the case where w is a reduced word. We then specialize to the case where w is a *cyclically* reduced word and solve a conjecture of Margolis, Meakin and Stephen [MMS] by showing that such monoids must be E-unitary, thus reducing the word problem for such monoids to the membership problem for the submonoid of the corresponding one-relator group generated by the prefixes of the word w. In the final section of the paper we show how this may be used to solve the word problem for the one-relator inverse monoid in certain cases.

2. Inverse monoids.

An *inverse monoid* is a monoid M with the property that for each element $x \in M$ there exists a unique element denoted by $x^{-1} \in M$ such that

$$x = xx^{-1}x \text{ and } x^{-1} = x^{-1}xx^{-1}.$$

It is an easy consequence of the definition that idempotents commute in any inverse monoid M and hence that the set of idempotents of M forms a (lower) semilattice with respect to the natural partial order

$$e \leq f \text{ if and only if } ef = fe = e.$$

This may be extended to a natural partial order on M by defining

$$x \leq y \text{ for } x, y \in M \text{ if and only if there is some idempotent } e \in M \text{ such that } x = ey.$$

We shall denote the semilattice of idempotents of an inverse monoid M by $E(M)$ throughout this paper.

Inverse monoids arise naturally as monoids of partial one-one maps: in fact the first theorem in the subject (the Vagner-Preston Theorem) states that every inverse monoid may be faithfully represented as a monoid of partial one-one maps of a suitable set. We refer the reader to the book by Petrich [Pe] for this theorem and basic notation and results about inverse monoids. Such monoids are frequently referred to as "pseudogroups of transformations (or local diffeomorphisms)" in topology or differential geometry, where they play a prominent role in the theory.

We recall here that inverse monoids form a variety of algebras (in the sense of universal algebra) and hence that free inverse monoids exist. We denote the free inverse monoid on a set A by $FIM(A)$. This monoid may be viewed as a monoid of finite birooted trees whose positively oriented edges are labeled by elements of the set A, in such a way that no two edges with the same initial or terminal vertex have the same label. Such trees are referred to as *Munn trees* in the literature (see [Pe]). The Munn tree $MT(u)$ associated with a word $u \in \mathcal{A}^*$ may be identified with the finite subtree of the Cayley tree of the free group $FG(A)$ obtained by traveling along the path in this tree labeled by u, starting at 1 and ending at the reduced form $r(u)$ of u. The initial (resp. terminal) vertex of $MT(u)$ is 1 (resp. $r(u)$). A basic theorem of Munn [Mu], [Pe] asserts that two words u and v in $\mathcal{A}^*$ are equal in $FIM(A)$ if and only if they have the same Munn tree (with the same initial and terminal vertices). This provides a solution to the word problem for the free inverse monoid $FIM(A)$.

The inverse monoid presented by the set A of generators and relations of the form $u_i = v_i, \ i \in I$ for some words $u_i, v_i \in \mathcal{A}^*$ is denoted by $\text{Inv}\langle A \parallel u_i = v_i, \ i \in I\rangle$. This is the quotient of the free inverse monoid $FIM(A)$ by the corresponding congruence generated by the set of relations. Graphical and automata-theoretic methods, originally developed by Stephen [St1] have proved very useful in studying presentations of inverse monoids. We very briefly review some of these ideas here.

Let $M = \text{Inv}\langle A \parallel u_i = v_i, \ i \in I\rangle$ and identify M with the quotient $FIM(A)/\tau$ of the free inverse monoid $FIM(A)$ by the corresponding congruence τ. For each word $u \in \mathcal{A}^*$ we define the *Schützenberger* graph $S\Gamma(u)$ of u (relative to the presentation) as follows. The vertices of $S\Gamma(u)$ are the elements $v\tau$ of M that are related via Green's $\mathcal{R}$-relation to $u\tau$ in M (i.e. $(uu^{-1})\tau = (vv^{-1})\tau$ in M). For each $a \in \mathcal{A}$, there is an edge labeled by a from $v\tau$ to $(va)\tau$ in $S\Gamma(u)$ if $v\tau, (va)\tau$ are $\mathcal{R}$-related to u in M. We view $S\Gamma(u)$ as a birooted graph with initial root $(uu^{-1})\tau$ and terminal root $u\tau$. ¿From this point of view, $S\Gamma(u)$ may be regarded as an automaton with $(uu^{-1})\tau$ as initial state and $u\tau$ as terminal state. The *language* of this automaton is defined to be

$$L(u) = \{v \in \mathcal{A}^* \parallel v \text{ labels a path in } S\Gamma(u) \text{ from } (uu^{-1})\tau \text{ to } u\tau\}.$$

Note that if M is just the free inverse monoid $M = FIM(A) = \text{Inv}\langle A \parallel \emptyset \rangle$, then the Schützenberger graph of a word $u \in \mathcal{A}^*$ is identified with the Munn tree $MT(u)$ of u. The prominent role which these graphs (automata) play in the theory is illustrated in the following theorem due to Stephen [St1].

Theorem 2.1. *Let $M = \text{Inv}\langle A \parallel u_i = v_i, i \in I \rangle = FIM(A)/\tau$ and let $u, v \in \mathcal{A}^*$. Then*

(a) $L(u) = \{s \in (\mathcal{A})^* : s\tau \geq u\tau$ *in the natural partial order on* $M\}$.
(b) $u\tau = v\tau$ *in M if and only if* $L(u) = L(v)$.
(c) $u\tau = v\tau$ *in M if and only if $S\Gamma(u)$ and $S\Gamma(v)$ are isomorphic as birooted labeled graphs.*

Thus the word problem for an inverse monoid presentation is decidable if and only if the corresponding Schützenberger automata are effectively constructible. We also make note of the fact that it follows from Part (a) of Theorem 2.1 that if w is a word accepted by the Schützenberger automaton of the identity element 1 in an inverse monoid presentation, then $w\tau \geq 1\tau$ and hence $w\tau = 1\tau$ in the inverse monoid. We will use this remark explicitly in the proof of Lemma 4.9 below.

In his paper [St1], Stephen described an iterative procedure for constructing these automata. This procedure is analogous to the classical Todd-Coxeter coset enumeration procedure for constructing the Cayley graph of a group presentation and reduces to this if the inverse monoid M happens to be a group. Start with the "linear" automaton of the word $u = a_1a_2 \dots a_n$ - i.e. the automaton whose underlying graph is just a linear sequence of segments labeled by the a_i so that the entire graph is labeled from the initial vertex to the terminal vertex by the word u. Build a sequence of intermediate automata each obtained from the preceding one by application of either an "expansion" or an "edge folding". An expansion is constructed from an automaton $\mathcal{X}$ by adding to this automaton a path labeled by the word t from a vertex α to a vertex β if there is a path in $\mathcal{X}$ from α to β labeled by a word s, where $s = t$ is one of the defining relations in the monoid M. An edge folding is obtained by identifying two edges with the same label and the same initial or terminal vertex.

Stephen shows that these operations are confluent and that the (unique) automaton obtained from the linear automaton of u by closing with respect to these operations is the Schützenberger automaton of u. We refer to [St1] for details and examples of this construction. Each intermediate automaton obtained from the linear automaton of u by a sequence of expansions and edge foldings is called an *approximate automaton* of the Schützenberger automaton of u.

We now show that the word problem for one-relator inverse monoids is at least as complex as the word problem for one-relator monoids.

Theorem 2.2. *If the word problem is decidable for all inverse monoids of the form $\text{Inv}\langle A \parallel w = 1 \rangle$, where w is some reduced word in $\mathcal{A}^*$, then the word problem is also decidable for every one-relator monoid.*

Proof Assume that the word problem is decidable for all one-relator inverse monoids corresponding to any reduced word w: by the results of Adjan and Oganessian [AO] mentioned above, it suffices to show that the word problem is decidable for every one-relation monoid with a presentation of the form $M = \text{Mon}\langle A : aub = avc \rangle$ where $a, b, c \in$

A, $b \neq c$ and $u, v \in A^*$. Consider such a monoid M and the associated inverse monoid $I = \text{Inv}\langle A \parallel aubc^{-1}v^{-1}a^{-1} = 1\rangle$. We claim that M is embeddable in I.

To see this, note first that by the results of Adjan [Ad], M is right cancellative, so M has no idempotent other than 1. It follows that M embeds as the monoid of right units into its *inverse hull*, which is the inverse monoid generated by the image of M under its right regular representation into the inverse semigroup of all partial one-one maps of M (see Clifford and Preston [CP], Theorem 1.22). Denote the inverse hull of M by $IH(M)$. Now $IH(M)$ satisfies the relation $aubc^{-1}v^{-1}a^{-1} = 1$ since aub and avc are right units in $IH(M)$. Thus there are natural morphisms ν from $\mathcal{A}^*$ onto I and μ from I onto $IH(M)$. The canonical map from A^* onto $M \subseteq IH(M)$ factors as a product $\theta\phi$ where θ maps A^* onto a submonoid $M' \subseteq I$ and ϕ maps M' onto M. Now M' also satisfies $aub = avc$ and ϕ maps M' onto M. It follows from the universal property of M that ϕ is an isomorphism.

This shows that M embeds into I and since I has solvable word problem by assumption, so does M, thus completing the proof of the theorem. □

Note that the word $aubc^{-1}v^{-1}a^{-1}$ is reduced (since $b \neq c$) but not *cyclically* reduced, i.e. the last letter is the inverse of the first letter. So the situation for one-relator inverse monoids with a presentation of the form $M = \text{Inv}\langle A \parallel w = 1\rangle$ where w is a cyclically reduced word is conceivably more manageable than the general case. For this reason we restrict attention to presentations of this type in the remainder of the paper. We are able to solve the word problem for such presentations in certain cases and we are also able to study an important structural property of such monoids.

3. E-unitary Inverse Monoids.

We recall that an inverse monoid $M = \text{Inv}\langle A \parallel u_i = v_i,\ i \in I\rangle$ is called *E-unitary* if the natural morphism μ from M onto its maximal group image $G = \text{Gp}\langle A \parallel u_i = v_i, i \in I\rangle$ is *idempotent-pure*, that is the inverse image of the identity of G under the morphism μ consists precisely of the semilattice $E(M)$ of idempotents of M. Equivalently, M is E-unitary if $x \geqq e$ for $x, e \in M$ implies that x is an idempotent of M if e is an idempotent of M.

There are many alternative ways of defining this concept, which is of major importance in inverse semigroup theory. We briefly mention its connection with the classical extension problem for partial one-one maps. Given a semigroup of partial one-one maps (usually a pseudogroup of transformations of some topological space or local diffeomorphisms of some manifold) one is interested in knowing when the partial one-one maps may be extended to the action of some group on a larger space (manifold). The analogue of this in inverse semigroup theory is the concept of an *E-unitary cover* over a group. If M is an inverse monoid of partial one-one transformations on a set X, we say that M has an E-unitary cover over a group G if there is some set Y such that $X \subseteq Y$ and each partial one-one map in M is the restriction to some subset of X of a permutation in a group G of permutations of Y. This is equivalent to the existence of an inverse semigroup T and morphisms $\phi : T \to S$ and $\psi : T \to G$ such that T is E-unitary, ψ is idempotent-pure and ϕ is idempotent-separating (i.e. no two idempotents of T are identified under ϕ).

¿From the point of view of the Schützenberger graphs of M, an early observation of Meakin (see [St1]) is that M is E-unitary if and only if each Schützenberger graph of M embeds (in the natural way) into the associated Cayley graph of G. This enables us to replace the iterative procedure for approximating the Schützenberger graphs of M outlined above by an iterative procedure for building associated subgraphs of the Cayley graph of G. In certain situations, if the Cayley graph of the group G is sufficiently well understood, this may be used to solve the word problem for the inverse monoid M (see for example [MM1] for a non-trivial application of these ideas).

In general, inverse monoids with presentations of the form $M = \mathrm{Inv}\langle A \parallel w_i = 1,\ i \in I\rangle$, where the w_i are cyclically reduced, need not be E-unitary, as the following example shows.

Example Let $M = \mathrm{Inv}\langle a, b, c, d \parallel abc = 1,\ adc = 1\rangle$. We claim that M is not E-unitary. To see this note first that $bd^{-1} = a^{-1}c^{-1}ca = 1$ in the group $G = \mathrm{Gp}\langle a, b, c, d \parallel abc = 1,\ adc = 1\rangle$, so if M is E-unitary bd^{-1} must be an idempotent of M. We easily see that this is not the case by constructing the Schützenberger graph of bd^{-1} relative to the presentation defining M.

In order to construct this graph, we proceed by the iterative method outlined above. Construct first the linear automaton of the word bd^{-1} : this automaton has three vertices, the initial vertex (which is the initial vertex of an edge labeled by b), the "middle" vertex (which is the terminal vertex of the edge labeled by b and the initial vertex of the edge labeled by d^{-1}) and the terminal vertex (which is the terminal vertex of the edge labeled by d^{-1}). We may expand the graph at each of these vertices by adding loops labeled by the relators abc and adc. On each such loop the edges labeled by a fold together and the edges labeled by c fold together and the edges labeled by b and d become coterminal, but no edge folds onto the edges of the original linear automaton. The resulting graph obtained after these expansions and edge foldings has nine vertices (the three original vertices on the linear automaton and two more corresponding to each of the three expansions that were performed at these vertices). One may now repeat the process, expanding the new graph by adding loops corresponding to the relators at each of the six new vertices that were added to the original linear automaton and performing all possible edge foldings as above. The new generation of edges labeled by the letters b and d are not folded onto any previously constructed edges with these labels. Continuing by induction, one sees that the original edges of the linear automaton labeled by b and d are never identified with any new edges with these labels. Hence, in the Schützenberger automaton of the word bd^{-1}, the initial vertex and the terminal vertex remain distinct. This shows that bd^{-1} is not an idempotent in M and hence that M is not E-unitary. A sketch of the Schützenberger automaton of bd^{-1} is provided in Figure 1. □

FIGURE 1

The situation for one-relator inverse monoids corresponding to a cyclically reduced relator w is somewhat nicer however. In [MMS] the authors conjectured that an inverse monoid of the form $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$, where w is a reduced word, is E-unitary if and only if w is cyclically reduced. In one direction, this turns out to be false: Silva [Si] has given an example of an inverse monoid M presented by one reduced (but not cyclically reduced) relator w such that M is in fact a group, and hence is E-unitary of course. However, the main result of the present paper (Theorem 4.1) shows that this conjecture is true in the opposite direction, that is, a one-relator inverse monoid corresponding to a cyclically reduced relator is in fact E-unitary.

We shall prove this result in the next section. In order to provide some motivation for considering this question, we show now as a corollary that the word problem for such an inverse monoid M is reduced to the membership problem for the submonoid of the corresponding one-relator group $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ generated by the prefixes (initial segments) of the relator w. This will be exploited in Section 5 to show decidability of the word problem in certain special cases.

Let w be a cyclically reduced word over the alphabet $\mathcal{A}$. Let $\mathrm{Pre}(w) = \{v \in \mathcal{A}^* \parallel w \equiv vt$ for some $t \in \mathcal{A}^+\}$ be the set of proper prefixes of w, including the empty word. (Here we denote equality in the free monoid $\mathcal{A}^*$ by $\equiv$ in order to distinguish it from equality in other monoids or groups under consideration). Define P_w to be the submonoid of $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ generated by the image of $\mathrm{Pre}(w)$ under the natural morphism from $\mathcal{A}^*$ to G. We call P_w the *prefix monoid* of G relative to w. We say that the membership problem for P_w is decidable if there is an algorithm which on input a word $v \in \mathcal{A}^*$ outputs "yes" if the image of v is a member of P_w and "no" otherwise. We can now state the main theorem of this section.

Theorem 3.1. *If w is a cyclically reduced word then the word problem for the inverse E-unitary monoid $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ is decidable if the membership problem for P_w is decidable.*

Before proving Theorem 3.1, we need some preliminary remarks and examples. Recall that a *cyclic conjugate* of a cyclically reduced word $w \in \mathcal{A}^*$ is a word in $\mathcal{A}^*$ of the form $w' \equiv vu$ where w factors in $\mathcal{A}^*$ as $w \equiv uv$. We first note that the submonoid P_w depends not only on the group G but on the word w as well. That is, it is possible to replace w by any cyclic conjugate v of w without changing the normal closure of w and thus $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ is equal (not just isomorphic to) $H = \mathrm{Gp}\langle A \parallel v = 1\rangle$. However the monoid $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ may be very different than $N = \mathrm{Inv}\langle A \parallel v = 1\rangle$ and the submonoid P_w may be different from P_v.

Example Let $A = \{a, b\}$ and let $w = aba$. It is not difficult to see that the assignment $a \mapsto 1, b \mapsto -2$ establishes an isomorphism between $G = \mathrm{Gp}\langle \{a, b\} \parallel aba = 1\rangle$ and the integers $\mathbf{Z}$. It follows that P_{aba} is equal to $\mathbf{Z}$, but that P_{baa} is equal to the submonoid of $\mathbf{Z}$ consisting of the non-positive integers. In fact, the monoid $M = \mathrm{Inv}\langle \{a, b\} \parallel aba = 1\rangle$ is also isomorphic to the integers, since a is both a left and right divisor of 1, and thus a member of the group of units of M and thus so is $b = a^{-2}$. On the other hand, it is not difficult to prove that the monoid $N = \mathrm{Inv}\langle \{a, b\} \parallel baa = 1\rangle$ is isomorphic to the bicyclic monoid. $\square$

The proof of Theorem 3.1 will depend on some results of Stephen [St2]. It is well known that the collection of E-unitary inverse monoids forms a quasi-variety of inverse monoids, since it is defined by the implication $(e^2 = e \wedge em = e) \Rightarrow m = m^2$. As for all quasivarieties, it follows that any inverse monoid M has a maximal E-unitary image defined as the quotient of M by the intersection of all congruences whose quotients are E-unitary. In particular, given any binary relation T on the free inverse monoid $FIM(A)$ we define the E-unitary inverse monoid $G(M)$ presented by $\langle A \parallel T\rangle$ to be the maximal E-unitary image of $M = \mathrm{Inv}\langle A \parallel T\rangle$. Of course $M = G(M)$ if and only if M is E-unitary.

In [St2], Stephen implicitly considers the structure of the E-unitary monoid $G(M)$ of a monoid presented by relations all of which have the form $w = 1$. If $T = \{w_i = 1 \parallel i \in I\}$ is a collection of such relations, (where the relators w_i are not necessarily reduced words), let P_T be the submonoid of $G = \mathrm{Gp}\langle A \parallel T\rangle$ generated by the images of all $\mathrm{Pre}(w_i)$ for $i \in I$. That is, P_T is the submonoid of G generated by all proper prefixes of all relators in T. A subset X of G is said to be *connected* if $1 \in X$ and whenever $g, h \in X$ there exists a word $w = x_1 \dots x_n \in (X \cup X^{-1})^*$ such that $gw = h$ and $gx_1 \dots x_i \in X$ for $1 \leqq i \leqq n$. Equivalently, a set X containing 1 is connected if its vertices form a connected subgraph of the Cayley graph of G relative to the presentation $G = \mathrm{Gp}\langle A \parallel T\rangle$. Let $N = \{(FP_T, g) \parallel F$ is a finite connected subset of G and $g \in FP_T\}$. Define a product on N by

$$(X, g)(Y, h) = (X \cup gY, gh).$$

Let $\sigma : \mathcal{A}^* \to G$ be the natural map. If $v \in \mathcal{A}^*$, let F_v be the finite subset of G consisting of the image of all prefixes of v (including v itself). Clearly, F_v is a finite connected subset of G. Let $\theta : \mathcal{A}^* \to N$ be defined by $v\theta = (F_v P_T, v\sigma)$. The following summarizes some of the work of Stephen in [St2].

Theorem 3.2.

1. *N is an E-unitary inverse monoid with maximal group image G.*
2. *The map θ induces an isomorphism from the maximal E-unitary image $G(M)$ to N where $M = \mathrm{Inv}\langle A \parallel T\rangle$. That is, if $u, v \in \mathcal{A}^*$, then $u = v$ in $G(M)$ if and only if $u\theta = v\theta$.*

We can use Theorem 3.2 to prove the following reduction theorem for the word problem for monoids of the form $G(M)$ that will have Theorem 3.1 as an immediate corollary.

Theorem 3.3. *Let T be a subset of $\mathcal{A}^*$, let $M = \mathrm{Inv}\langle A \parallel T\rangle$ and let $G = \mathrm{gp}\langle A \parallel T\rangle$. The word problem for $G(M)$ is decidable if the word problem for G is decidable and the membership problem for the submonoid P_T of G is decidable.*

Proof Let $u, v \in \mathcal{A}^*$. By Theorem 3.2, $u = v$ in $G(M)$ if and only if $F_u P_T = F_v P_T$ and $u\sigma = v\sigma$. If the word problem for G is decidable, then we can decide the condition $u\sigma = v\sigma$. If we can decide membership in P_T, then we can also decide membership in FP_T for any effectively given finite subset $F = \{g_1, \dots , g_n\}$ of G. For $u \in FP_T$ if and only if $g_i^{-1}u \in P_T$ for some $1 \leqq i \leqq n$. Furthermore, it is clear that for this F, $FP_T \subseteq XP_T$ for another finite set X if and only if $F \subseteq XP_T$. So we can decide this last containment by checking the finitely many conditions $g_i \in XP_T, 1 \leqq i \leqq n$. It follows easily that we can algorithmically check whether $FP_T \subseteq XP_T$ and thus whether $FP_T = XP_T$ for any finite sets F and X and the result is proved. □

Proof of Theorem 3.1 Let $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ where w is a cyclically reduced word. By the main theorem of this paper (Theorem 4.1), M is E-unitary and thus $M = G(M)$.

By Magnus' Theorem [LS], the word problem for $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ is decidable and the results now follow immediately from Theorem 3.3. □

4. The E-unitary Problem.

In this section we solve the conjecture of Margolis, Meakin and Stephen [MMS] by proving the following theorem, which is the main theorem of the paper.

Theorem 4.1. *If w is a cyclically reduced word then the inverse monoid $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ is E-unitary.*

We need some preliminary ideas and results before we are able to provide a proof of this theorem. Most of the results of this section apply only to one-relator inverse monoids, but some of the concepts that we introduce are just as easily applicable to inverse monoids of the form $M = \mathrm{Inv}\langle A \parallel w_i = 1,\ i \in I\rangle$, where each word w_i is a cyclically reduced word in $\mathcal{A}^*$, so we begin by considering such presentations. We say that a cyclic conjugate w_i' of w_i is a *unit cyclic conjugate* of w_i if $w_i' = 1$ in the inverse monoid M. For example, in the bicyclic monoid $B = \mathrm{Inv}\langle a, b \parallel ab = 1\rangle$ it is clear that ba is not a unit cyclic conjugate of ab since $ba \neq 1$ in B: on the other hand, the monoid $H = \mathrm{Inv}\langle a, b \parallel aba = 1\rangle$ is easily seen to be a group (the integers), so every cyclic conjugate of the relator aba is a unit cyclic conjugate.

The unit cyclic conjugates are closely related to the group of units of the monoid M, as the following proposition shows.

Proposition 4.2. *Let $M = \mathrm{Inv}\langle A \parallel w_i = 1,\ i \in I\rangle$, where each word w_i is a cyclically reduced word in $\mathcal{A}^*$ and identify M with the quotient $\mathcal{A}^*/\tau$ where τ is the natural congruence. Then the group of units of M is the submonoid of M generated by the set of elements of the form $p_i\tau$ where q_ip_i is a unit cyclic conjugate of the defining relator $w_i, i \in I$ for some words $p_i, q_i \in \mathcal{A}^*$.*

Proof We will abuse notation slightly and denote the element $u\tau \in M$ simply by u throughout the proof: it will be clear from the context when we are referring to words in $\mathcal{A}^*$ and when we are referring to their images in M. It is clear that every element $p_i \in M$ for which q_ip_i is a unit cyclic conjugate of some relator defining M, must be a unit of M. So we need only prove that every unit of M can be written as a product of such elements.

Note that the monoid of right units of M is the set of vertices of the Schützenberger graph of 1 in M. Since this graph is built iteratively from the trivial graph (the linear automaton of 1) by repeated applications of the operations of adding loops labeled by the relators and edge foldings, it follows easily that every element of the monoid of right units of M may be written as a product of prefixes (initial segments) of the relators.

Now let s be an element of the group $U(M)$ of units of M with $s \neq 1$. By the above observation, we may write $s = p_1p_2 \ldots p_n$ in M where each p_i is a prefix of one of the relators w_j. Thus for each i there is some j (depending on i) and some word q_i such that $p_iq_i \equiv w_j$. Note that $q_i = p_i^{-1}$ in M since $p_iq_i = 1$ in M. Also, since $s \in U(M)$ we have $s^{-1}s = 1$ in M, so

$$q_n \ldots q_2q_1p_1p_2 \ldots p_n = 1 \qquad (1)$$

in M. This implies that q_n is right invertible and since we also have q_n is left invertible in M, it follows that $q_n \in U(M)$. Since $p_n = q_n^{-1}$ this implies that $p_n \in U(M)$ and also that $q_n p_n = 1$ in M, so $q_n p_n$ is a unit cyclic conjugate of the corresponding relator w_j. Now multiply equation (1) on the left by p_n and on the right by q_n: we obtain

$$q_{n-1} \dots q_2 q_1 p_1 p_2 \dots p_{n-1} = 1$$

in M. Arguing as above, we see that $q_{n-1}p_{n-1}$ is a unit cyclic conjugate of the corresponding relator w_k. Continuing this process by induction yields the desired result. □

We deduce two easy corollaries of this proposition.

Corollary 4.3. *Let $M = \mathrm{Inv}\langle A \parallel w_i = 1,\ i \in I\rangle$, where each word w_i is a cyclically reduced word in $\mathcal{A}^*$. If an element s of the group of units of M is written in any way as a product of the form $s = p_1 p_2 \dots p_n$ where each p_i is a prefix of one of the relators w_j, then in fact each p_i is a unit of M.*

Proof This is an immediate corollary of the proof of the previous proposition. □

Corollary 4.4. *Let $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ be a one-relator inverse monoid with w cyclically reduced. Then M has trivial group of units if and only if w has no unit cyclic conjugates other than w itself.*

Proof Suppose that w has a non-trivial unit cyclic conjugate of the form $w' \equiv vu$ where $w \equiv uv$ for some non-trivial words $u, v \in \mathcal{A}^*$. Clearly u and v are units of M. But u is a proper factor of w, so by a well-known result of Weinbaum ([LS], Chapter II, Proposition 5.29), $u \neq 1$ in the group $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$. Since G is the maximal group homomorphic image of M we must also have $u \neq 1$ in M. Hence the group of units of M is non-trivial. Conversely, if w has no non-trivial unit cyclic conjugates then by Proposition 4.2, the group of units of M must be trivial. □

In order to prove some structural results about one-relator inverse monoids, we shall make use of the concept of (van Kampen) diagrams over group presentations. Let a group G be given by a presentation

$$G = \langle\ A \parallel w_i = 1,\ i \in I\ \rangle\ , \qquad (2)$$

(where the w_i are cyclically reduced words over $\mathcal{A}$).

By a *map* M we mean as in [LS], [Ol] a finite planar connected (but not necessarily simply connected) simplicial 2-complex. The 0-, 1-, 2-cells of M are called the *vertices, edges, cells* of M, respectively.

A (van Kampen) diagram Δ over G given by (2) is a map that is equipped with a labeling function ϕ from the set of oriented edges of Δ to the alphabet $\mathcal{A}$ such that

(L1) If $\phi(e) = a$, then $\phi(e^{-1}) = a^{-1}$.

(L2) If Π is a cell in Δ and $\partial\Pi = e_1 ... e_k$ is the boundary cycle of Π, where $e_1, \dots, e_k$ are oriented edges, then $\phi(\partial\Pi) = \phi(e_1)\dots\phi(e_k)$ is a cyclic permutation of w_i^{ε}, where $\varepsilon = \pm 1$ and $i \in I$.

A simply connected diagram over G is called a *disk diagram*. A diagram with one hole is called an *annular diagram*.

It is convenient to fix the positive (counterclockwise) orientation for the boundary $\partial\Pi$ of a cell Π in Δ and the appropriate orientation for a component q of the boundary $\partial\Delta$ of the diagram Δ so that one gets Δ on the right hand when moving along the oriented component q. (When oriented this way, q is also termed a *contour* of Δ, see [Ol], [Iv]).

There are many ways to define the concept of a reduced diagram over a group presentation, see [LS], [Ol], [Iv]. In this paper we choose one of most straightforward definitions. Let e be an oriented edge in a diagram Δ over (2), let Π_1, Π_2 be cells in Δ and $e \in \partial\Pi_1,\ e \in \partial\Pi_2^{-1}$ (recall that $\partial\Pi_1$, $\partial\Pi_2$ are positively oriented). The cells Π_1, Π_2 are said to be a *reducible pair* provided the label $\phi(\partial\Pi_1|_{e_-})$ of the (oriented) boundary $\partial\Pi_1|_{e_-}$ starting at the initial vertex e_- of the edge e is graphically (i.e. letter-by-letter) equal to $\phi(\partial\Pi_2|_{e_-})^{-1}$. Denote by e_+ the terminal vertex of an edge e. A diagram Δ over (2) is termed *reduced* provided Δ contains no reducible pairs of cells.

The following lemma due to van Kampen is almost obvious (see [LS], [Ol]).

Lemma 4.5. *A cyclic word w equals 1 in the group G given by (2) if and only if there is a reduced disk diagram Δ over G such that $\phi(\partial\Delta) \equiv w$.*

As an immediate corollary we have the following.

Corollary 4.6. *Let $M = \mathrm{Inv}\langle A \parallel w_i = 1,\ i \in I\rangle$ where the words w_i are cyclically reduced words in $\mathcal{A}^*$, and let $G = \mathrm{Gp}\langle A \parallel w_i = 1,\ i \in I\rangle$ be the corresponding maximal group homomorphic image of M. Then M is E-unitary if and only if for every reduced disk diagram Δ over G, the word $\phi(\partial\Delta)$ is an idempotent in M.*

Proof Recall that M is E-unitary if and only if each word $s \in \mathcal{A}^*$ that is 1 in G is in fact an idempotent in M. The result follows immediately from the van Kampen lemma. □

Suppose Π is a cell in a diagram over $G = \langle A \| w_i,\ i \in I\rangle$. A vertex $o \in \partial\Pi$ is called a *distinguished vertex* (*D-vertex*) of Π if $\phi(\partial\Pi|_o) = 1$ in $M = Inv\langle A\|w_i, i \in I\rangle$. An (oriented) edge $e \in \partial\Pi$ is termed a *D-edge* of Π provided either e_- or e_+ is a D-vertex of Π.

The next result provides a sufficient condition for an inverse monoid M of the type being considered to be E-unitary. We refer to a disk diagram Δ as being *trivial* if it has no cells (i.e. if it is a tree).

Lemma 4.7. *Let $M = \mathrm{Inv}\langle A \parallel w_i = 1,\ i \in I\rangle$ where the words w_i are cyclically reduced words in $\mathcal{A}^*$, and let $G = \mathrm{Gp}\langle A \parallel w_i = 1,\ i \in I\rangle$ be the corresponding maximal group homomorphic image of M. Then M is E-unitary if for every non-trivial reduced disk diagram Δ over G there is some vertex $o \in \partial\Delta$ such that o is a D-vertex of some cell Π of Δ.*

Proof Let Δ be a disk diagram over G. If Δ has no cells then $\phi(\partial\Delta)$ is an idempotent in M since $\phi(\partial\Delta) = 1$ in the free group $FG(A)$ over A and thus $\phi(\partial\Delta)$ is an idempotent in $FIM(A)$. Proceeding by induction on the number of cells in Δ, assume Δ is a disk diagram containing cells and $o \in \partial\Delta$ is a D-vertex of a cell $\Pi \in \Delta$. Removing Π from Δ

by making a cut at o (and splitting o into o', o'', as illustrated in Fig. 2) turns Δ into a disk diagram Δ' with fewer cells. Since $\phi(\partial\Delta) = \phi(\partial\Delta')$ in M, the induction step is done and the proof is complete. □

FIGURE 2

Remark In [MMS], the authors showed that if $w = abcdacdadabbcdacd$ then there is reduced disk diagram Δ over the corresponding group $G = \mathrm{Gp}\langle A \parallel w = 1 >$ such for every cell Π of Δ, the vertex on $\partial\Pi$ at which one reads $w^{\pm 1}$ around $\partial\Pi$ is an interior vertex of the diagram Δ. They also proved that the corresponding inverse monoid $M = \mathrm{Inv}\langle A \parallel \ w = 1\rangle$ is E-unitary by showing that every cyclic conjugate of w that starts with the letter a is in fact a unit cyclic conjugate of w and thus by the Freiheitssatz, every reduced disk diagram over G must have a unit cyclic conjugate starting somewhere on its boundary. Thus it is not in general possible to prove that a one-relator inverse monoid $M = \mathrm{Inv}\langle A \parallel \ w = 1\rangle$ is E-unitary by showing that every reduced disk diagram over the corresponding group G has a boundary vertex at which the word w may be read around some cell - one must in general search for boundary vertices which are start points for unit cyclic conjugates possibly different from w.

Let w be a fixed cyclically reduced word and consider the group presentation

$$G = \mathrm{Gp}\langle A \parallel w = 1\rangle \tag{3}$$

and the corresponding inverse monoid presentation

$$M = \mathrm{Inv}\langle A \parallel w = 1\rangle \tag{4}$$

throughout the remainder of this section.

We also consider the related presentation

$$\mathrm{Inv}\langle A \parallel w_i = 1,\ i = 1, \dots, t\rangle \tag{5}$$

where $\{w_1, \dots w_t\}$ is the set of all unit cyclic conjugates of w. It is clear that (4) and (5) present the same inverse monoid M. In what follows we will also make use of the obvious fact that if Γ is an approximate graph of 1 based at a vertex α relative to the presentation (5), then any loop in Γ based at α labels a word that equals 1 in the inverse monoid M given by the presentation (4).

By Lemma 4.7, the proof of Theorem 4.1 is immediate once we prove the following result.

Lemma 4.8. *Suppose that Δ is a non-trivial reduced disk diagram over the presentation* (3)*. Then there is a cell π in Δ and a D-edge $e \in \partial\pi$ with $e^{-1} \in \partial\Delta$.*

This lemma will be proved by induction on the number of cells of Δ. The result is clearly true for diagrams with one cell. We will need some technical lemmas before we start the proof. In these lemmas we will assume that Δ is a minimal (with respect to number of cells) non-trivial reduced diagram that is a counter-example to the statement of Lemma 4.8. It is clear that such a diagram Δ must satisfy the following potentially restrictive property:

(P) For every proper non-trivial reduced disk subdiagram Δ' of Δ (with fewer cells than Δ) there is a cell π in Δ' and a D-edge $e \in \partial\pi$ with $e^{-1} \in \partial\Delta'$.

Lemma 4.9. *Let Δ be a non-trivial reduced disk diagram over (3) that satisfies property (P), let $\Pi_1, \Pi_2, \ldots, \Pi_n$ be a sequence of cells in Δ (not necessarily all distinct) and suppose that there are vertices $\alpha_1, \alpha_2, \ldots, \alpha_n$ in Δ such that α_i belongs to $\partial\Pi_i \cap \partial\Pi_{i+1}^{-1}$ for $i = 1, 2, \ldots, n$ (modulo n). If α_i is a D-vertex of Π_i for each $i = 1, 2, \ldots, n$, then α_i is a D-vertex of Π_{i+1} for each $i = 1, 2, \ldots, n$ (modulo n).*

Proof Note that $\partial\Pi_i \cap \partial\Pi_{i+1}^{-1} \neq \emptyset$ for $i = 1, 2, \ldots, n$ (modulo n). It follows that the union of the cells Π_i bounds a reduced disk subdiagram Δ' of Δ consisting of all the cells Π_i together with any other cells of Δ that are in the interior of the region of Δ enclosed by the union of these cells Π_i. The basic idea for proving the lemma is to show that the 1-skeleton of the diagram obtained from Δ' by pruning off all trees is an approximate graph for 1 in the presentation (5), based at any of the vertices α_i. This will then show that any loop in this graph based at a vertex α_i must be labeled by a word that is equal to 1 in M, by the remark following Theorem 2.1. In particular, the cyclic conjugate of w obtained by reading around $\partial\Pi_{i+1}^{\pm 1}$ starting at α_i is a unit cyclic conjugate of w, as required.

Consider Stephen's iterative procedure outlined in the introduction (Section 1 above) for constructing an approximate graph for the trivial word 1 relative to the presentation (5) for M, starting at the vertex α_i. If we start with a single vertex (that we denote by α_i), we may perform an expansion to this (trivial) graph by adding a loop labeled by an appropriate unit cyclic conjugate w' of w at this vertex, effectively building a copy of the boundary of the cell Π_i. We caution that this process does not necessarily build a copy of a cell that is homeomorphic to Π_i as it embeds in the diagram Δ - this cell may for example enclose a non-trivial van Kampen subdiagram of Δ. However, the loop labeled by w' based at α_i contains a vertex that we shall again denote by α_{i-1} (modulo n), namely the vertex that we reach along this loop by reading the segment of w' labeling the path along $\partial\Pi_i^{\pm 1}$ from α_i to α_{i-1} in Δ.

Perform another expansion by adding a loop labeled by an appropriate unit cyclic conjugate w'' of w at this vertex, effectively building a copy of the boundary of the cell Π_{i-1}. After doing as much edge folding as possible, subject to the constraint that we only fold edges of the two loops that are already identified in Δ', the resulting graph consists of two loops whose boundaries intersect in an arc that may be identified with the maximal (connected) arc of $\partial\Pi_i \cap \partial\Pi_{i+1}^{-1}$ that contains the vertex α_{i-1} in Δ. Continue this process, successively creating the vertices $\alpha_i, \alpha_{i-1}, \ldots \alpha_{i+1}$ (modulo n), expanding by adding loops labeled by appropriate unit cyclic conjugates of w at these vertices and folding as much as possible in the subdiagram already obtained, again subject to the constraint that we only fold edges that are already identified in Δ'. We denote the resulting approximate graph of 1 based at α_i (relative to the presentation (5)) by Λ.

If Δ' has no cells other than $\Pi_1, \Pi_2, \ldots, \Pi_n$ then the folding process attaches the loops labeled by the appropriate unit cyclic conjugates of w based at α_k and α_{k+1} along a common boundary that may be identified with $\partial\Pi_k \cap \partial\Pi_{k+1}^{-1}$ for each k. Thus in this case, the 1-skeleton of Δ' can be identified with Λ and thus may be viewed as an approximate graph of the empty word 1 relative to the presentation (5) based at α_i and it follows as above that the cyclic conjugate of w obtained by reading $\partial\Pi_{i+1}^{\pm 1}$ based at α_i is a unit cyclic conjugate, as desired.

Some examples of diagrams corresponding to this situation are depicted in Figures 3(a), 3(b), 3(c) and 3(d). In these figures, and in subsequent figures representing portions of van Kampen diagrams over the presentation (3), an arrow at a vertex on the boundary of a cell Π and pointing towards the interior of Π indicates that the cyclic conjugate of w obtained by reading around $\partial\Pi^{\pm 1}$ starting at this vertex is a unit cyclic conjugate of w.

Figures 3(a), 3(b), 3(c), 3(d)

In general, Δ' may contain cells that are not in $\{\Pi_1, \Pi_2, \ldots, \Pi_n\}$. We refer to such cells as *latent* cells. By the construction of Δ' no latent cell has an edge on $\partial\Delta'^{\pm 1}$, but latent cells may possibly have vertices on $\partial\Delta'$. Some examples of diagrams with latent cells are depicted in Figures 4(a), 4(b), 4(c) and 4(d). In these figures, all cells of the form L_i for some i are latent cells.

Figures 4(a), 4(b), 4(c), 4(d)

The subdiagram of Δ' consisting of the latent cells of Δ' is not necessarily connected (see Figure 4(a) for example), so it is not necessarily a reduced disk diagram. We refer to the maximal connected and simply connected components of this subdiagram as the *latent components* of Δ'. Each latent component of Δ' is a reduced disk diagram over the presentation (3) with fewer than N cells, so by Property (P) each such component has a D-vertex of some cell somewhere on its boundary. Also, the boundary of such a latent component must consist entirely of edges that are in the union of the cells Π_i. Fix a latent component Γ of Δ' and a vertex α on $\partial\Gamma$ that is a D-vertex for some latent cell Π' of Γ.

Now all of the edges of $\partial\Gamma$ are in the approximate graph Λ. It follows that we can construct the vertex α in this approximate graph and hence we can expand Λ by adding another loop labeled by an appropriate unit cyclic conjugate of w at α and then folding, again subject to the restriction that we only fold edges that get identified in Δ'. This creates a copy of the boundary of the latent cell Π' in an approximate graph of 1 based at α_i. But then the diagram obtained from Γ by removing this cell Π' either splits into several components that are reduced disk diagrams with fewer cells than Γ (see Figure 5(a)) or is a single reduced disk diagram with fewer cells than Γ (see Figure 5(b)).

Figures 5(a), 5(b)

In either case, all boundary edges of the resulting reduced disk diagram or diagrams are contained in the approximate graph of 1 based at α_i constructed so far and so we may continue the process inductively to eventually construct loops labeling the boundaries of all of the latent cells in Γ. Once all such loops have been constructed the folding process produces a graph which contains a copy of the 1-skeleton of Γ. Applying this procedure to all latent components of Δ' we eventually build the 1-skeleton of Δ' as an approximate graph of 1 based at α_i. Then it follows as above that α_i is a D-vertex of Π_{i+1} as desired.

□

Consider the following construction. Let Δ be a reduced diagram over the group G given by (3), let Π be a cell in Δ, and let e be a D-edge of Π. Clearly, e^{-1} is either an edge of $\partial\Delta$ (and then we stop), or, otherwise, $e^{-1} \in \partial\Pi_1$, where Π_1 is another cell in Δ (perhaps, $\Pi_1 = \Pi$). Denote $e_0 = e$, $\Pi_0 = \Pi$ and consider an arc u_1 of the cell Π_1 (i.e. a subpath of $\partial\Pi_1$) of the form $u_1 = e_0^{-1}v_1e_1$ such that e_1 is a D-edge of Π_1, $e_1 \neq e_0^{-1}$ and the arc v_1 has no D-edges of Π_1. Next, if $e_1^{-1} \in \partial\Delta$, then we stop. Otherwise, let $e_1^{-1} \in \partial\Pi_2$ and consider an arc u_2 of Π_2 of the form $u_2 = e_1^{-1}v_2e_2$ such that e_2 is a D-edge of Π_2, $e_2 \neq e_1^{-1}$ and the arc v_2 has no D-edges of Π_2 (as above, such an edge e_2 does exist). Analogously, defining the cells $\Pi_3, \dots, \Pi_m, \dots$, their arcs $u_3 = e_2^{-1}v_3e_3, \dots, u_m = e_{m-1}^{-1}v_me_m, \dots$, and their D-edges $e_3, \dots, e_m, \dots$, we will eventually obtain that either $e_m^{-1} \in \partial\Delta$ (see Fig. 6) or, otherwise, $e_k = e_\ell$ and $\Pi_k = \Pi_\ell$ for some $k < \ell$ (see Figs. 7-8).

Figure 6

Picking such k, ℓ so that $k, \ell - k$ are minimal, we will get the cycle $(e_k, e_{k+1}, \dots, e_{\ell-1})$ of D-edges of cells $\Pi_k, \Pi_{k+1}, \dots \Pi_{\ell-1}$ which will be called a *D-star* defined by (e_0, Π_0) and denoted by $\mathrm{St}(e_0, \Pi_0)$ (note this definition is similar to an analogous notion in [IS]). The path $v_{k+1} \dots v_{\ell-1}v_\ell)$ will be called the *boundary* of the D-star $\mathrm{St}(e_0, \Pi_0)$ and denoted by $\partial\,\mathrm{St}(e_0, \Pi_0)$. It is easy to see that the path $\partial\,\mathrm{St}(e_0, \Pi_0)$ has no self-intersections (up to arbitrarily small deformations, see [Iv]) and, therefore, one can consider a disk subdiagram $\mathrm{E}(e_0, \Pi_0)$ bounded by the cyclically reduced (possibly trivial) path obtained from $\partial\,\mathrm{St}(e_0, \Pi_0)^{\pm 1}$. (A cyclically reduced path is a path with no subpaths of the form ee^{-1} with an edge e.) In the case when the cells $\Pi_k, \Pi_{k+1}, \dots \Pi_{\ell-1}$ are not in $\mathrm{E}(e_0, \Pi_0)$ (see Fig. 7) we will say that $\mathrm{St}(e_0, \Pi_0)$ is *interior*. If the cells $\Pi_k, \Pi_{k+1}, \dots \Pi_{\ell-1}$ are in $\mathrm{E}(e_0, \Pi_0)$ (see Fig. 8) we will say that $\mathrm{St}(e_0, \Pi_0)$ is an *exterior* D-star.

Figures 7, 8

Now assume that Δ satisfies Property (P). Then it follows from Lemma 4.9 that every edge e_{k+i}^{-1} is a D-edge of the cell Π_{k+i+1} (subscripts $\mathrm{mod}(\ell - k)$). Consequently, $e_{\ell-1}$ cannot be an edge on the path v_{k-1} and thus every cell Π_j, $j < k$, will be in the disk diagram $\mathrm{E}(e_0, \Pi_0)$ provided $\mathrm{St}(e_0, \Pi_0)$ is interior and every cell Π_j, $j < k$, will not be in $\mathrm{E}(e_0, \Pi_0)$ provided $\mathrm{St}(e_0, \Pi_0)$ is exterior. The D-edges $e_0, e_1, \dots, e_{k-1}$ (if any) of $\Pi_0, \Pi_1, \dots, \Pi_{k-1}$ will be called *open* edges of $\mathrm{St}(e_0, \Pi_0)$.

Let us observe the following restrictive property of the disk subdiagram $\mathrm{E}(e_0, \Pi_0)$ of an interior D-star $\mathrm{St}(e_0, \Pi_0)$: suppose π_1 and π_{t+1} are cells in Δ so that $\pi_1 \notin \mathrm{E}(e_0, \Pi_0)$ and $\pi_{t+1} \in \mathrm{E}(e_0, \Pi_0)$. Then there are no cells $\pi_2, \dots, \pi_t$ in Δ such that there are D-edges $e_1 \in \partial\pi_1, \dots, e_t \in \partial\pi_t$ with $e_1^{-1} \in \partial\pi_2, \dots, e_t^{-1} \in \partial\pi_{t+1}$.

We will make use of this property and the terminology introduced above in the next technical lemma. Again, let Δ be a minimal counterexample to Lemma 4.8 relative to the number of cells. Let π^* be a cell that has an edge $f \in \partial\pi^*$ with $f^{-1} \in \partial\Delta$. Consider the set $\mathcal{S}(\pi^*)$ of all cells Π in Δ that have the following property: for every $\Pi \in \mathcal{S}(\pi^*)$ there exists a sequence of cells $\pi_1, \dots, \pi_\ell$ such that $\pi_1 = \pi^*$, $\pi_\ell = \Pi$, and the cell π_i, $i = 1, \dots, \ell - 1$, has a D-edge $e_i \in \partial\pi_i$ with $e_i^{-1} \in \partial\pi_{i+1}$.

Lemma 4.10. *Suppose* $\Pi \in \mathcal{S}(\pi^*)$, $e \in \partial\Pi$ *is a* D*-edge, and* Π' *is the cell in* Δ *with* $e^{-1} \in \partial\Pi'$*. Then* $\Pi' \in \mathcal{S}(\pi^*)$ *and* e^{-1} *is a* D*-edge of* Π'.

Proof. The inclusion $\Pi' \in \mathcal{S}(\pi^*)$ follows from the definition of $\mathcal{S}(\pi^*)$. Consider the D-star $\mathrm{St}(e, \Pi)$. If $\mathrm{St}(e, \Pi)$ is exterior then $\mathrm{St}(e, \Pi)$ can not have open D-edges for otherwise the disk diagram $\mathrm{E}(e, \Pi)$ would not contain Π and provide a counterexample with fewer cells (note that $\partial\,\mathrm{E}(e, \Pi)$ has no D-edges either). Suppose $\mathrm{St}(e, \Pi)$ is interior and $\mathrm{St}(e, \Pi)$ has open D-edges. Then $e \in \partial\Pi$ is an open D-edge of $\mathrm{St}(e, \Pi)$ and Π is a cell of $\mathrm{E}(e, \Pi)$. Since no cell of $\mathrm{E}(e, \Pi)$ has an edge e_1 with $e_1^{-1} \in \partial\Delta$, we have that $\pi^* \notin \mathrm{E}(e, \Pi)$. This, however, is a contradiction to the restrictive property of disk subdiagrams of interior D-stars noted above, in view of the fact that $\Pi \in \mathcal{S}(\pi^*)$ and the definition of $\mathcal{S}(\pi^*)$. Thus, in any case, $\mathrm{St}(e, \Pi)$ has no open D-edges and a reference to Lemma 4.9 shows that e^{-1} is a D-edge of Π'. $\square$

Using all of the terminology introduced above we may now proceed to the proof of Lemma 4.8, and hence of our main theorem (Theorem 4.1).

Proof of Lemma 4.8 We assume that the statement of the lemma is false and that Δ is a minimal counterexample: we may apply the results of Lemmas 4.9 and 4.10 when needed. Consider the subdiagram Γ of Δ that consists of all cells $\Pi \in \mathcal{S}(\pi^*)$. By the definition of $\mathcal{S}(\pi^*)$ and Lemma 4.10, Γ is a diagram with $k \geq 0$ holes such that if e is a D-edge of a cell $\Pi \in \Gamma$ then e^{-1} is a D-edge of a cell $\Pi' \in \Gamma$.

Let us divide each D-edge $e \in \Pi$, $\Pi \in \mathcal{S}(\pi^*)$, into two new edges e_1, e_2 so that $e = e_1 e_2$. The labels $\phi(e_1), \phi(e_2)$ are assigned to e_1, e_2 as follows: Let B be an alphabet whose letters are in bijective correspondence $\beta : A \to B$ with letters of A and $A \cap B = \emptyset$. If $\phi(e) = a \in A$ then $\phi(e_1) = a$ and $\phi(e_2) = \beta(a)$. If $\phi(e) = a^{-1} \in A^{-1}$ then $\phi(e_2) = a^{-1}$ and $\phi(e_1) = \beta(a)^{-1}$ (that is, the same rule applies to e^{-1} with $\phi(e^{-1}) = a \in A$). This results in a new disk diagram Δ' over a group H given by

$$H = \langle\, A \cup B \parallel w = 1, \bar{w} = 1 \,\rangle, \tag{6}$$

where $\phi(\partial\pi^*) = \bar{w}^{\pm 1}$, $\bar{w}$ is cyclically reduced, has occurrences of letters of $\mathcal{B}^{\pm 1}$, and erasing all letters of $\mathcal{B}^{\pm 1}$ in $\bar{w}$ results in the word w. It follows from results of [IM] that if w is not a proper power that any spherical diagram over (6) (i.e., a diagram whose underlying map is a 2-sphere) contains a reducible pair.

Note that $\phi(\partial\Delta') \equiv \phi(\partial\Delta)$. Hence attaching Δ (from above) to Δ' along $\partial\Delta$ yields a spherical diagram Δ_0 over (6). It follows from the minimality of Δ and construction of Δ'

that Δ_0 has no reducible pairs. This contradicts the result cited above on the asphericity of (6) unless w is a proper power. However, in this case our lemma is true in view of a theorem due to B. Newman [N] (see also [MKS]) that claims that if Δ is a reduced diagram with cells over (3), where $w \equiv s^n, n > 1$, then there are a cell π in Δ and an arc u of $\partial\pi$ so that $e^{-1} \in \partial\Delta$ for each edge $e \in u$ and $|u| > (n-1)|S|$. This completes the proof of Lemma 4.8 and hence of our main theorem (Theorem 4.1). □

5. The Word Problem.

Let G be a one-relator group given by the presentation, $G = \mathrm{Gp}\langle A \parallel w = 1 \rangle$ associated with a non-empty cyclically reduced relator w and let M be the corresponding inverse monoid $M = \mathrm{Inv}\langle A \parallel w = 1 \rangle$. In this section we consider some cases where we are able to solve the membership problem for the prefix submonoid P_w of G and hence, by Theorem 3.1, the word problem for M.

Let $G = \mathrm{Gp}\langle A \parallel r_i = 1,\ 1 \le i \le m\rangle$ be a presentation of a group G and let w be a (reduced) word over A. We say that this presentation for the group G is (strictly) w-positive if there is a morphism $f : G \to \mathbf{Z}$ from G onto the integers such that if $v \neq 1$ is a proper prefix of w, then $(vf > 0)$ $vf \ge 0$.

Example The presentation $G = \mathrm{Gp}\langle \{a, b\} \parallel aba = 1\rangle$ is not aba-positive. If f is any morphism from G onto $\mathbf{Z}$, then clearly $bf = -2(af)$ and thus one of the prefixes a, ab must be mapped to a positive integer while the other will be mapped to a negative integer. On the other hand, this presentation is baa-strictly positive given that the assignment $b \mapsto 2, a \mapsto -1$ is a morphism that sends both prefixes of baa to positive integers. □

We note that it is decidable given a finitely presented group $G = \mathrm{Gp}\langle A \parallel r_i = 1,\ 1 \le i \le m\rangle$ and a word w whether the presentation is (strictly) w-positive. Morphisms from G onto $\mathbf{Z}$ can be calculated by solving the integer system of m equations in $|A|$ variables arising by taking the commutative image of each relator and setting it equal to 0. The (strictly) positive condition can then be thought of as an integer programming problem by imposing the necessary inequalities to ensure that all prefixes of w map to positive or non-negative numbers.

The interest in these properties for the purpose of the current paper is the following theorem.

Theorem 5.1. *Let w be a cyclically reduced word and suppose that $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ is a w-strictly positive presentation. Then the membership problem for P_w is decidable.*

Proof Let $\theta : \mathcal{A}^* \to G$ be the natural morphism and let $f : G \to \mathbf{Z}$ be a morphism onto the integers such that if $v \neq 1$ is a proper prefix of w, then $(v\theta)f > 0$. Let $u \in \mathcal{A}^*$. It is clear that if $(u\theta)f \le 0$, then $u\theta \in P_w$ if and only if $u\theta = 1$ in G. Since the word problem for G is decidable we can decide if $u\theta \in P_w$ in this case.

So assume that $(u\theta)f > 0$. If $u\theta \in P_w$, then $u\theta = (p_1 \dots p_n)\theta$ for some prefixes p_i of w. We can assume that all the p_i are not the identity by assuming that this is the shortest representation of $u\theta$ as a member of P_w. Now $(u\theta)f = \sum_{i=1}^{n}(p_i\theta f)$. Since each $(p_i\theta)f > 0$ there are only a finite number of possible such representations of u as a member of P_w. We can effectively enumerate all of these finitely many representations and use the algorithm for the word problem for G to test whether u is equal to any of these products. $u\theta \in P_w$ if and only if we receive a positive answer to one of these finitely many tests. It follows that the membership problem for P_w is decidable. □

Corollary 5.2. *Let w be a cyclically reduced word such that $G = \mathrm{Gp}\langle A \parallel w = 1\rangle$ is a w-strictly positive presentation. Then the word problem for the inverse monoid $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ is decidable. Furthermore the group of units of M is trivial.*

Proof From Theorem 3.2 and Theorem 4.1 it follows that the word problem for $M = \mathrm{Inv}\langle A \parallel w = 1\rangle$ is decidable.

Let θ and f be as in the proof of Theorem 5.1. If $U(M) \neq 1$ then by Proposition 4.2, there is a factorization $w \equiv pq$ such that $qp = 1$ in M and $p \neq 1, p \neq w$ in A^*. Hence $(p\theta)f > 0$, so $(q\theta)f < 0$, but $qp = 1$ in M so q is right invertible in M, whence $(q\theta)f > 0$, a contradiction. □

We close the paper by considering some other partial results on the membership problem for P_w. Note the word $[a_1, b_1] \dots [a_n, b_n]$ of part (b) of the theorem below is the "standard" relator of the fundamental group of an orientable surface of genus n. Interestingly, we impose a restriction in part (c) that a word w^{-1} is not in the submonoid of the free group $FG(A)$ generated by all prefixes of the nonempty reduced word $w \in FG(A)$. However, we do not know examples of such words w and conjecture that our restriction is meaningless.

Theorem 5.3. *The membership problem for the prefix monoid P_w is decidable in the following cases:*

(a) *There is a single occurrence of a letter $a^{\pm 1} \in \mathcal{A}$ in w.*
(b) *w is a cyclic permutation of the word $[a_1, b_1] \dots [a_n, b_n]$, $n \geq 1$.*
(c) *$w \equiv w_1 w_2 \dots w_n$, where $n > 12$, each w_i is a nonempty reduced word over a subalphabet A_i so that $A = \bigcup_{1\leq i\leq n} A_i$ and A_i are disjoint, and the word w_1^{-1}, (resp. w_n) is not in the submonoid of the free group $FG(A_1)$, (resp. $FG(A_n)$) generated by all prefixes of w_1, (resp. w_n^{-1}).*

Before proving Theorem 5.3, we provide a solution for the membership problem for finitely generated submonoids of free groups.

Lemma 5.4. *Let $u_1, u_2, \dots, u_n$ be some words in a free group $FG(A)$. Then the membership problem for the submonoid $\langle\, u_1, u_2, \dots, u_n \,\rangle_S \subseteq FG(A)$ generated by $u_1, u_2, \dots, u_n$ is decidable*

Proof. We may deduce this as a consequence of a theorem of Benois [Be] characterizing the rational subsets of free groups. We provide an alternative proof here, since this proof has some independent interest. First we modify the definition of a Nielsen reduced basis (N-basis) for a subgroup $\langle\mathcal{U}\rangle$ of a free group $F = FG(A)$ (see [LS]) in order to adjust it for the submonoid $\langle\mathcal{U}\rangle_S$ generated by words in $\mathcal{U}$ as follows. An ordered set $\mathcal{U} = (U_1, U_2, \dots)$ of distinct reduced nonempty words of F will be called *N_S-reduced* provided for all triples $V_1, V_2, V_3 \in \mathcal{U}$ the following conditions are satisfied:

(N1) If $V_1 V_2 \neq 1$ then either $|V_1 V_2| \geq |V_1|, |V_2|$ or $V_1 V_2 \in \mathcal{U}$.
(N2) If $V_1 V_2 \neq 1$ and $V_2 V_3 \neq 1$ then either $|V_1 V_2 V_3| > |V_1| - |V_2| + |V_3|$ (i.e., there is at least one letter of V_2 that remains uncancelled in the product $V_1 V_2 V_3$) or $V_1 V_2 V_3 \in \mathcal{U}$.

The analogs of elementary Nielsen transformations on an ordered set $\mathcal{U} = (U_1, U_2, \dots)$ are defined as follows:

(T1) Add U_iU_j, $i \neq j$, provided $0 < |U_iU_j| < |U_i|$ or $0 < |U_iU_j| < |U_j|$ and $U_iU_j \notin \mathcal{U}$.
(T2) Delete U_i if $U_i = U_j$, $i \neq j$.

Clearly, these transformations preserve the monoid $\langle U \rangle_S$ (but are not invertible).

By analogy with the proof of Proposition 2.2 of [LS], one can prove that if $\mathcal{U}$ is finite then $\mathcal{U}$ can be carried by a sequence of elementary transformations (T1)–(T2) into a finite N_S-reduced set $\mathcal{V}$.

Now Lemma 5.4 can be proved as for subgroups of free groups on the basis of the following fact: If $\mathcal{V}$ is N_S-reduced then for every word $W \in \langle \mathcal{V} \rangle$ there are $V_1, \dots, V_k \in \mathcal{V}$ such that $W = V_1 \dots V_k$, $|V_iV_{i+1}| \geq |V_i|, |V_{i+1}|$, $|V_iV_{i+1}V_{i+2}| > |V_i| - |V_{i+1}| + |V_{i+2}|$ and so $|W| \geq k$. □

Proof of Theorem 5.3. (a) This is immediate from Lemma 5.4.

(b) If $n = 1$ then our claim is obvious since $P_w = \langle a_1, b_1 \rangle_S$. Assume $n > 1$. First notice that, up to renaming $a \to a^{-1}$, $a^{-1} \to a$, $w^{-1} \to w$ we can assume that

$$w \equiv xyx^{-1}(Ty)^{-1}, \tag{7}$$

or

$$w \equiv xyx^{-1}(yT)^{-1}, \tag{8}$$

or

$$w \equiv xyT^{-1}x^{-1}y^{-1}, \tag{9}$$

where the word T is nonempty and has no occurrences of the letters $x^{\pm 1}$, $y^{\pm 1}$,

We carry out the proof in detail in the first case: two other cases are similar and we omit the details of these cases. So assume that w is of the form (7) throughout the remainder of the proof. In this case we have

$$P_w = \langle \mathcal{P} \rangle_S, \quad \mathcal{P} = \{x, T_i, Ty, Tyx \mid 0 \leq |T_i| \leq |T|\}, \tag{10}$$

where T_i is a prefix of T of length i.

Let R be a word in $\mathcal{A}$. Without loss of generality, we may assume that R has no subwords of the form xy^k, $x^{-1}(Ty)^\ell x$ where $k, \ell \neq 0$. Assume that $R \in P_w \subseteq G$ and consider a reduced disk diagram Δ with $\partial\Delta = vr^{-1}$, where $v = v_1 \dots v_k$, $\phi(v_1), \dots, \phi(v_k) \in \mathcal{P}$, $\mathcal{P}$ is defined by (10) and $\phi(r) = R$. Note every cell Π in Δ has exactly two x-edges $e, f \in \partial\Pi$, that is, edges with $\phi(e) = x^{\pm 1}$, $\phi(f) = x^{\mp 1}$. The edge e^{-1} must be either an edge of $\partial\Delta$ or $e^{-1} \in \partial\Pi'$. Consider a chain of cells $\Pi_{-\ell_1}, \dots, \Pi_0, \Pi_1, \dots, \Pi_{\ell_2}$ in Δ, where $\Pi_0 = \Pi$, so that $e_0 = e$, $f_0 = f$, and $e_i, f_i \in \partial\Pi_i$ are x-edges such that $e_i = f_{i-1}^{-1}$, $f_i^{-1} = e_{i+1}$, $e_{-\ell_1}^{-1} \in \partial\Delta$ and $f_{\ell_2}^{-1} \in \partial\Delta$ (see Fig. 9; $e_{-\ell_1}^{-1}$ may not be f_{ℓ_2} since Δ is reduced and G is torsion-free).

Figure 9

We will refer to the disk subdiagram E consisting of the cells $\Pi_{-\ell_1}, \dots, \Pi_0, \Pi_1, \dots, \Pi_{\ell_2}$ as an *x-strip*. The factorization $\partial\,\mathrm{E} = tesf$, where $\{e, f\} = \{e_{-\ell_1}^{-1}, f_{\ell_2}^{-1}\}$ and $\phi(e) = x$, is the *standard contour* of an x-strip E. By a *trivial* x-strip we mean a subdiagram E consisting of two x-edges $e, e^{-1} \in \partial\Delta$ where $\phi(e) = x$. The standard contour of such a diagram E is $\partial\,\mathrm{E} = ee^{-1}$.

Let $\mathrm{E}_1, \dots, \mathrm{E}_m$ be all of the x-strips (including the trivial ones) in Δ with standard contours $\partial\,\mathrm{E}_1 = t_1e_1s_1f_1, \dots, \partial\,\mathrm{E}_m = t_me_ms_mf_m$. It follows from the Freiheitssatz and from the choice of R and the words $\phi(v_1), \dots, \phi(v_k)$ that $e_1, \dots, e_m \in v = v_1 \dots v_k$ and $f_1, \dots, f_m \in r^{-1}$. Changing indices, if necessary, we may assume that

$$v = u_1e_1u_2e_2 \dots u_me_mu_{m+1},$$
$$r = r_1f_1^{-1}r_2f_2^{-1} \dots r_mf_m^{-1}r_{m+1}.$$

Making use of the notation introduced above, we define disk subdiagrams $\Gamma_1, \dots, \Gamma_{m+1}$ by $\partial\Gamma_1 = u_1t_1^{-1}r_1^{-1}$, $\partial\Gamma_i = u_it_i^{-1}r_i^{-1}s_{i-1}^{-1}$, $1 < i < m+1$, and $\partial\Gamma_{m+1} = u_{m+1}r_{m+1}^{-1}s_m^{-1}$ (see Fig. 10; informally, Γ_i sits between the x-strips E_{i-1} and E_i).

Figure 10

Consider Γ_i, $1 \le i \le m+1$. Since $\phi(\partial\Gamma_i)$ has no $x^{\pm1}$, we have $\phi(\partial\Gamma_i) = 1$ in the free group $FG(A)$ and so Γ_i has no cells (recall that Δ has no closed x-strips). This means that Γ_i is over $FG(A)$ and for every edge $g \in \partial\Gamma_i$ it is true that $g^{-1} \in \partial\Gamma_i$. Since w has the form (7), each $\phi(s_i^{-1})$ is a power of y. Let $g_1, g_2 \in s_{i-1}^{-1}$ be two consecutive edges. It is easy to see from the facts that $\phi(r_i)$ does not begin with $y^{\pm1}$, $T \ne 1$ and $\phi(u_i)$ contains no $y^{\pm2}$ that g_1^{-1}, g_2^{-1} cannot both be edges of one of u_i, t_i^{-1}, r_i^{-1}. Consequently, $|s_i| \le 2$ and so the diagram Δ contains at most $2m$ cells. Now our claim is straightforward from Lemma 5.4.

(c) Consider the set $\mathcal{P}$ consisting of all prefixes of w and w^{-1}. Note it follows from the assumptions that w_1^{-1} is not in the submonoid of $FG(A_1)$ generated by all prefixes of w_1 and w_n is not in the submonoid of $FG(A_n)$ generated by all prefixes of w_n^{-1} that cancellations between prefixes of w are small. More specifically, it is not difficult to see that if V is a reduced word, $V = V_1 \dots V_k$ in $FG(A)$, where $V_1, \dots, V_k \in \mathcal{P}$, and this number k is minimal (over all such representations for V) then the following is true: If w_i^ε, $1 < i < n$, $\varepsilon = \pm1$, is a subword of V_j, $1 \le j \le k$, then no letter of w_i^ε cancels in the product $V_1 \dots V_j \dots V_k$ and any subword w_i^ε, $1 < i < n$, $\varepsilon = \pm1$, of V is a subword of one of $V_1, \dots, V_k$. In particular, this observation implies the following property that is important for this proof: suppose $V = V_1 \dots V_k$ in $FG(A)$, where $V_1, \dots, V_k \in \mathcal{P}$, is a reduced word and w_i^ε, $1 < i < n$, $\varepsilon = \pm1$, is a subword of V so that $V \equiv T_1w_i^\varepsilon T_2$. Then

$$T_1(w_{i+1} \dots w_nw_1 \dots w_{i-1})^{-\varepsilon}T_2 \in \langle\mathcal{P}\rangle_S \subseteq FG(A).$$

Now suppose R is a reduced word that has no subwords of the form (subscripts mod n)

$$(w_{j+1}w_{j+2} \dots w_{j+m})^{\pm1}, \tag{11}$$

where $m > n - m$. Note the latter property is not restrictive (as far as elements of G represented by words are concerned) because if R does not have this property we can reduce the syllable length of R (relative to the partition $A = \bigcup_{1\le i\le n} A_i$) by applying the relation $w^{\pm 1} = 1$ to R.

Now assume $R \in P_w \subseteq G$ and consider a reduced disk diagram Δ with $\partial\Delta = ur^{-1}$, where $\phi(u) \equiv U$ is a reduced word, $U \in \langle \mathcal{P} \rangle \subseteq FG(A)$, $\phi(r) \equiv R$, such that, given R, Δ has the minimal number of cells over all such words $U \in \langle \mathcal{P} \rangle \subseteq FG(A)$.

Assuming Δ has cells, we single out a subdiagram Δ_0 in Δ with $\partial\Delta_0 = u_0 r_0^{-1}$, where u_0, r_0 are subpaths of u, r, respectively, maximal relative to the the property that the first edges of u_0, r_0 are different and the last edges of u_0, r_0 are also different. Clearly, $u = u_1 u_0 u_2$ and $r = r_1 r_0 r_2$, where $u_1 = r_1$, $u_2 = r_2$ (see Fig. 11).

FIGURE 11

By Schupp's theorem [S], Δ_0 contains a cell Π such that $\partial\Pi = vt$, v^{-1} is a subpath of $\partial\Delta_0$ and $\phi(v)$ contains all the letters that occur in w. This means that $\phi(v)$ contains a subword of the form (subscripts mod n)

$$V_0 \equiv w'_j w_{j+1} \dots w_{j+n-2} w'_{j+n-1},$$

where w'_j, w'_{j+n-1} are nonempty suffix, prefix of w_j, w_{j+n-1}), respectively.

Since $n \ge 5$, we have that v is not a subpath of r (otherwise we could find a subword of the form (11) in R). Note if the subpath of v^{-1} with label $w_i^{\pm 1}$, $1 < i < n$, were a subpath of u_0 then we would have a contradiction to the minimality of Δ in view of the property of the word $\phi(u) = U \in \langle \mathcal{P} \rangle \subseteq FG(A)$ pointed out above, because taking the subpath labeled by $w_i^{\pm 1}$ along with Π out of Δ would result in a diagram Δ' with fewer number of cells and $\phi(\partial\Delta') = u' r^{-1}$, where $\phi(u') \in \langle \mathcal{P} \rangle \subseteq FG(A)$. Hence, we can assume that u_0 has no subpaths of v^{-1} labeled by $w_i^{\pm 1}$, $1 < i < n$. Consequently, we have that either v^{-1} is a subpath of $r_0^{-1} u_0$ (the case of $u_0 r_0^{-1}$ is analogous) and $(w_{j+1} w_{j+2} \dots w_{j+(n-2)-3})^{\pm 1}$ is a subword of r_0^{-1} or v^{-1} contains one of r_0^{-1}, u_0.

Since $n - 5 > 5$, in the first case we have a contradiction to the choice of R. If v^{-1} contains r_0^{-1} then $(w_{j+1} w_{j+2} \dots w_{j+(n-2)-4})^{\pm 1}$ is a subword of $\phi(r_0)^{-1}$ contrary to $n - 6 > 6$.

It remains to study the case when u_0 is a subpath of v^{-1}. Denote by Δ_1 the subdiagram of Δ_0 with $\partial\Delta_1 = t r_4^{-1}$, where $r_0 = r_3 r_4 r_5$ (see Fig. 12).

FIGURE 12

If Δ_1 has no cells, that is, $t = r_4$, then we have that $\phi(r_0)$ contains a subword of the form $(w_{j'+1} w_{j'+2} \dots w_{j'+(n-4)})^{\pm 1}$ and a contradiction to choice of R follows from $n - 4 > 4$. Assuming Δ_1 has cells we apply Schupp's theorem to Δ_1 and find a cell Π_1 with $\partial\Pi_1 = v_1 t_1$, where v_1^{-1} is a subpath of $\partial\Delta_1$ and $\phi(v_1)$ contains a subword of the form $w'_k w_{k+1} \dots w_{k+(n-2)} w'_{k+(n-1)}$, where w'_k, $(w'_{k+(n-1)})$ is nonempty prefix (suffix) of w_k (

$w_{k+(n-1)})$. If t is a subpath of v_1^{-1} then the cells Π, Π_1 form a reducible pair contrary to the minimality of Δ (which implies that Δ is reduced). Hence, a subword of $\phi(v_1^{-1})$ of the form $(w_{i'+1}w_{i'+2}\dots w_{i'+n-3})$ is a subword of $\phi(r_4)^{-1}$. Since $n-3>3$, we have a contradiction to the choice of R. Thus all possible cases have been considered and we have proved that Δ has no cells. However, in this situation, our claim becomes a corollary of Lemma 5.4. This completes the proof of the theorem. □

Remark The problem of membership in the prefix monoid P_w of a one-relator group $G = \mathrm{Gp}\langle A \parallel w = 1 >$ is a special case of the rational set problem: given a rational subset of G, is membership decidable? This last problem includes the generalized word problem for G as a special case: is the membership problem for a finitely generated subgroup of G decidable? It seems that almost nothing is known about this latter problem and it appears to be very difficult.

References

[Ad] S.I. Adjan, *Defining relations and algorithmic problems for groups and semigroups*, Trudy Math. Inst. Steklov, **85** (1966).

[AO] S.I. Adjan and Oganessian J., *Problems of equality and divisibility in semigroups with a single defining relation*, Mathem. Zametki **41** (1987), 412-421.

[Be] M. Benois, *Parties rationnelles du groups libre*, C. R. Acad. Sci. Paris, Sér A, **269** (1969), 1188-1190.

[CP] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, American Math Society, Vol. 1, 1961, Vol 2, 1967, Providence R.I.

[Gu] V. Guba, *On the relationship between the word problem and the divisibility problem in one relator semigroups*, Russian Math. Izv. **61** (1997), to appear.

[Iv] S.V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra and Comp. **4** (1994), 1-308.

[IS] S.V. Ivanov and P.E. Schupp, *On the hyperbolicity of small cancellation groups and one-relator groups*, Trans. of the Amer. Math. Soc., 350(1998), 1851-1894.

[IM] S.V. Ivanov and J.C. Meakin, *On asphericity and the Freiheitssatz for certain finitely presented groups*, preprint.

[La] G. Lallement, *Semigroups and Combinatorial Applications*, Wiley, 1979, New York.

[La2] G. Lallement, *On monoids presented by a single relation*, J. Alg. **32** (1977), 370-388.

[LS] R.C. Lyndon and P.E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[Ma] W. Magnus, *Das Identitatsproblem fur Gruppen mit einer definierenden Relation*, Math. Ann. **105**, 5274(1931).

[MKS] W. Magnus, J. Karras, and D. Solitar, *Combinatorial group theory*, Interscience Publ., John Wiley and Sons, 1966.

[MM1] S. Margolis and J. Meakin, *Inverse monoids, trees and context-free languages*, Trans. Amer. Math. Soc. **335** (1993), 259-276.

[MMS] S.W. Margolis, J. Meakin, J.B. Stephen, *Some decision problems for inverse monoid presentations*, in *Semigroups and Their Applications*, S. Goberstein and P. Higgins, editors, 99-110, Reidel, Dordrecht, 1986.

[Mu] W.D. Munn, *Free inverse semigroups*, Proc. London Math. Soc., (3) **29** (1974), 385-404.

[N] B.B. Newman, *Some results on one-relator groups*, Bull. Amer. Math. Soc. **74** (1968), 568–571.

[Ol] A.Yu. Ol'shanskii, *Geometry of defining relations in groups*, Nauka, Moscow, 1989; English translation in *Math and Its Applications* (Soviet series), **70** Kluwer Acad. Publishers, 1991.

[Pe] M. Petrich, *Inverse semigroups*, Wiley, 1984.

[S] P.E. Schupp, *A strengthened Freiheitssatz*, Math. Ann. **221**(1976), 73–80.

[Si] P.V. Silva, *Rational languages and inverse monoid presentations*, International J. of Algebra and Computation, **2** (1992), no. 2, 187-207.

[Sim] C. Sims, *Computing with Finitely Presented Groups*, Wiley, 1994.

[St1] J.B. Stephen, *Presentations of inverse monoids*, J. Pure and Appl. Alg., **63** (1990), 81-112.

[St2] J.B. Stephen, *Inverse monoids and rational subsets of related groups*, Semigroup Forum, **46** No.1 (1993), 98-108.

[Wa] G. Watier, *On the word problem for single relation monoids with an unbordered relator*, Inter. J. Algebra and Computation **7**, No. 6 (1997), 749-770.

[Zh] L. Zhang, *A short proof of a theorem of Adjan*, Proceedings of the AMS **116** (1992), 1-3.

Department of Mathematics
University of Illinois at Urbana–Champaign
1409 West Green Street, Urbana
IL 61801, U.S.A.

E-mail: ivanov@math.uiuc.edu

Department of Mathematics and Computer Science
Bar Ilan University
Ramat Gan 52900
Israel

E-mail: margolis@macs.biu.ac.il

Department of Mathematics and Statistics
University of Nebraska
810 Oldfather Hall, Lincoln
NE 68588, U.S.A.

E-mail: jmeakin@math.unl.edu