A complete rewrite system and reduced forms for $(S)_{reg}$

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1 Introduction

[Significance of $(S)_{reg}$: aperiodicity, etc.]

Notation: By $>_{\mathcal{L}}$, $\leq_{\mathcal{L}}$, $\equiv_{\mathcal{L}}$ we denote Green's well known $\mathcal{L}$-relations (and similarly for $\mathcal{R}$). We also use $\equiv_{\mathcal{D}}$ ($\mathcal{D}$-equivalence). See e.g. [4].

A less familiar notion is the $\mathcal{L}$-*incomparability* relation $\lesseqgtr\!\!\!\!/_{\mathcal{L}}$ defined as follows: $s \lesseqgtr\!\!\!\!/_{\mathcal{L}} t$ iff neither $s \leq_{\mathcal{L}} t$ nor $s \geq_{\mathcal{L}} t$. We also define $\mathcal{L}$-*comparability*: $s \lesseqgtr_{\mathcal{L}} t$ iff either $s \leq_{\mathcal{L}} t$ or $s \geq_{\mathcal{L}} t$. A similar notation is used for $\mathcal{R}$.

A semigroup S is called *unambiguous* (see [1], [2]) iff for all $s, t, u \in S - \{0\}$: $s >_{\mathcal{L}} u <_{\mathcal{L}} t$ implies $s \lesseqgtr\!\!\!\!/_{\mathcal{L}} t$ and $s >_{\mathcal{R}} u <_{\mathcal{R}} t$ implies $s \lesseqgtr\!\!\!\!/_{\mathcal{R}} t$. (Here, 0 is the zero of S if S has a zero; otherwise, $S - \{0\} = S$.)

In order to avoid confusion between products of elements in a semigroup S and strings of elements of S, we denote a *string* of length n as an n-tuple, of the form $(s_1, s_2, \ldots, s_n)$. The product of these elements in S is denoted by $s_1 s_2 \ldots s_n$ or $s_1 \cdot s_2 \cdot \ldots \cdot s_n$ ($\in S$).

2 The rewrite system

Presentation of $(S)_{reg}$ by generators and relations:

Let S be a semigroup, and let 0 be the zero of S, if S has a zero; otherwise, let 0 be a new symbol $\notin S$. Let $\overline{S - \{0\}} = \{\overline{s} : s \in S - \{0\}\}$ be a set that is disjoint from $S \cup \{0\}$, where the map $x \in (S - \{0\}) \cup \overline{S - \{0\}} \longmapsto \overline{x} \in (S - \{0\}) \cup \overline{S - \{0\}}$ is a bijection, and also an involution: $\overline{\overline{x}} = x$. (Note that we do not introduce $\overline{0}$.) Then, by definition [2], $(S)_{reg}$ has the following presentation:

Generators:

$S \ \cup \ \overline{S - \{0\}} \ \cup \{0\}$.

Relations:

*Both authors' research was supported in part by NSF grant DMS-9203981 and the Center for Communication and Information Science, University of Nebraska – Lincoln

†The second named author was supported by the Excellency Center "Group Theoretic Methods in the Study of Algebraic Varieties" of the Israeli Science Foundation and by the Binational Science Foundation of the USA and Israel, grant 1999298/1 and the support from INTAS through the Network project 99-1224.

$(s,t) = (st)$ for all $s,t \in S$
$(\overline{s},\overline{t}) = (\overline{ts})$ for all $s,t \in S - \{0\}$
$(0) = (0,0) = (0,s) = (s,0) = (0,\overline{s}) = (\overline{s},0)$ for all $s \in S$
$(s,\overline{t}) = (0)$ if $s \not\geq_{\mathcal{L}} t$, $s,t \in S - \{0\}$
$(\overline{s},t) = (0)$ if $s \not\geq_{\mathcal{R}} t$, $s,t \in S - \{0\}$
$(s,\overline{s},s) = (s)$ for all $s \in S - \{0\}$
$(\overline{s},s,\overline{s}) = (\overline{s})$ for all $s \in S - \{0\}$

Rewrite rules for $(S)_{reg}$**:**

1. *Length-reducing rules:*

The last two of the following set of rules make use of a partial function $B : S \times S \times S \to S$, that will be defined after the statement of all the rules.

(1.1) $(s,t) \to (st)$, $(\overline{s},\overline{t}) \to (\overline{ts})$ for all $s,t \in S$
(1.2) $(0,0) \to (0)$, $(0,s) \to (0)$, $(0,\overline{s}) \to (0)$, $(\overline{s},0) \to (0)$ for all $s \in S$
(1.3) $(s,\overline{t}) \to (0)$ if $s \not\geq_{\mathcal{L}} t$, $s,t \in S - \{0\}$
(1.4) $(\overline{s},t) \to (0)$ if $s \not\geq_{\mathcal{R}} t$, $s,t \in S - \{0\}$
(1.5) $(u,\overline{v},w) \to (B(u,v,w))$ if $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$, $u,v,w \in S - \{0\}$
(1.6) $(\overline{u},v,\overline{w}) \to (\overline{B(w,v,u)})$ if $u \leq_{\mathcal{R}} v \geq_{\mathcal{L}} w$, $u,v,w \in S - \{0\}$

2. *Length-preserving rules:*

For these rules we choose one representative element in every $\mathcal{R}$-class and in every $\mathcal{L}$-class; moreover, we make this choice so that $\mathcal{D}$-related representatives of $\mathcal{R}$-classes (or $\mathcal{L}$-classes) are $\mathcal{L}$-related (respectively $\mathcal{R}$-related). Such a choice can always be made. (Note that this condition on the choice of representatives was not used, and not required, in [2] and [5].)

Notation: For any $s \in S$, r_s (or ℓ_s) is the representative of the $\mathcal{R}$-class (resp. $\mathcal{L}$-class) of s.

The length-preserving rules make use of two partial functions, $B_{\mathcal{L}}$ and $B_{\mathcal{R}} : S \times S \to S$, that will be defined after the rules.

(2.1) $(s,\overline{t}) \to (r_s, \overline{B_{\mathcal{R}}(s,t)})$ if $s >_{\mathcal{L}} t$ and $s \neq r_s$
(2.2) $(\overline{s},t) \to (\overline{\ell_s}, B_{\mathcal{L}}(t,s))$ if $s >_{\mathcal{R}} t$ and $s \neq \ell_s$
(2.3) $(\overline{t},s) \to (\overline{B_{\mathcal{L}}(t,s)}, \ell_s)$ if $t \leq_{\mathcal{R}} s$ and $s \neq \ell_s$
(2.4) $(t,\overline{s}) \to (B_{\mathcal{R}}(s,t), \overline{r_s})$ if $t \leq_{\mathcal{L}} s$ and $s \neq r_s$

Definition of B**.** *If* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$*, where* $u,v,w \in S - \{0\}$*, then* $B(u,v,w) = uz$*, where* $z \in S$ *is such that* $w = vz$*.*

This operation was used in [2], but was first explicitly defined in [5]. It is easy to see that if $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ then $B(u,v,w)$ exists and is unique (i.e., it depends only on u,v,w and not on x; see Lemma 3.3 below).

Definition of $B_{\mathcal{R}}$ **and** $B_{\mathcal{L}}$**.** *If* $u \geq_{\mathcal{L}} v$*, where* $u,v \in S - \{0\}$*, then* $B_{\mathcal{R}}(u,v) = xr_u$*, where* $x \in S$ *is such that* $v = xu$*. If* $v \leq_{\mathcal{R}} u$*, where* $u,v \in S - \{0\}$*, then* $B_{\mathcal{L}}(v,u) = \ell_u y$*, where* $y \in S$ *is such that* $v = uy$*.*

This operation was implicit in [2]. Again, it is easy to see that if $u \geq_{\mathcal{L}} v$ (or $v \leq_{\mathcal{R}} u$) then $B_{\mathcal{R}}(u,v)$ (resp. $B_{\mathcal{L}}(v,u)$) exists and is unique (i.e., it depends only on u and v).

Proposition 2.1 *The rewrite system defines* $(S)_{reg}$.

Proof. The rewrite rules (when made symmetric) imply the relations of the presentation; to obtain the last two relations of the presentation, let $u = v = w$ in rules (1.5) and (1.6).

Converserly it is straightforward to show that in $(S)_{reg}$, the relations corresponding to the rules (1.5), (1.6), (2.1) – (2.4) hold (see also [2]). □

One of the main results of [2] is the following:

If S is *unambiguous* then S is a subsemigroup of $(S)_{reg}$, and (for any fixed choice of representatives of the $\mathcal{L}$- and $\mathcal{R}$-classes) every element of $(S)_{reg}$ can be written in a unique way in the *normal form* (0) or

$(r_1)\,\overline{\ell_2}\ \ldots\ r_{n-1}\,\overline{\ell_n}\ s\ \overline{r'_m}\,\ell'_{m-1}\ \ldots\ \overline{r'_2}\,(\ell'_1)$

where $(r_1 >_{\mathcal{L}})\ \ell_2 >_{\mathcal{R}}\ \ldots\ >_{\mathcal{R}} r_{n-1} >_{\mathcal{L}} \ell_n >_{\mathcal{R}} s \le_{\mathcal{L}} r'_m <_{\mathcal{R}} \ell'_{m-1} <_{\mathcal{L}}\ \ldots\ <_{\mathcal{L}} r'_2 <_{\mathcal{R}}\ (\ell'_1)$,

or in the form

$(r_1)\,\overline{\ell_2}\ \ldots\ r_{n-2}\,\overline{\ell_{n-1}}\,r_n\ \overline{s}\ \ell'_m\,\overline{r'_{m-1}}\,\ell'_{m-2}\ \ldots\ \overline{r'_2}\,(\ell'_1)$

where $(r_1 >_{\mathcal{L}})\ \ell_2 >_{\mathcal{R}}\ \ldots\ >_{\mathcal{R}} r_{n-2} >_{\mathcal{L}} \ell_{n-1} >_{\mathcal{R}} r_n >_{\mathcal{L}} s \le_{\mathcal{R}} \ell'_m <_{\mathcal{L}} r'_{m-1} <_{\mathcal{R}} \ell'_{m-2} <_{\mathcal{L}}\ \ldots\ <_{\mathcal{L}} r'_2 <_{\mathcal{R}}\ (\ell'_1)$.

Here, every $r_i, r'_j, \ell_i, \ell'_j$ is a representative of an $\mathcal{R}$- or $\mathcal{L}$-class, and s is any element of $S - \{0\}$. Elements in parentheses may be absent.

The normal form representation is the key to many structure properties of $(S)_{reg}$, e.g., the fact that S and $(S)_{reg}$ have the same $\mathcal{J}$-class structure. The main result of this paper is:

Theorem 2.1 *The above rewrite system for* $(S)_{reg}$ *is complete (i.e., confluent and terminating).*

The remainder of this paper consists of the proof of this theorem. We first give some basic properties of B, $B_{\mathcal{L}}$, and $B_{\mathcal{R}}$, then we prove termination of the rewrite system, and finally we prove local confluence.

3 Properties of the functions B, $B_{\mathcal{L}}$, and $B_{\mathcal{R}}$

In this section we collect all the basic properties of B, $B_{\mathcal{L}}$, and $B_{\mathcal{R}}$ that we will need in order to prove that the rewrite system for $(S)_{reg}$ is terminating and locally confluent. The reader may skip this section, and come back to it while reading the proofs of termination and local confluence.

Below, when we write an expression like $B_{\mathcal{R}}(x, y)$, $B_{\mathcal{L}}(x, y)$, or $B(x, y, z)$, we always implicitly assume that these expressions are defined (i.e., we assume that $x \ge_{\mathcal{L}} y$ when we use $B_{\mathcal{R}}(x, y)$, etc.).

Lemma 3.1 **(a)** *If* $u = r_u\alpha$ *then* $B_{\mathcal{R}}(u, v)\cdot\alpha = v$. *Similarly, if* $v = \beta\ell_u$ *then* $\beta\cdot B_{\mathcal{L}}(v, u) = u$. **(b)** *If* $r_u = u\alpha'$ *then* $B_{\mathcal{R}}(u, v) = v\alpha'$. *Similarly, if* $\ell_u = \beta' u$ *then* $\beta' \cdot B_{\mathcal{L}}(v, u) = \beta' u$.

The proof is trivial.

Lemma 3.2 $B_{\mathcal{R}}(r_u, v) = v$, *and* $B_{\mathcal{L}}(v, \ell_u) = v$.

The proof is trivial.

Lemma 3.3 $B_{\mathcal{R}}(u, v) \equiv_{\mathcal{R}} v$, *and* $B_{\mathcal{L}}(v, u) \equiv_{\mathcal{L}} v$.

Proof. If we multiply $r_u \equiv_{\mathcal{R}} u$ on the left by x we obtain $B_{\mathcal{R}}(u, v) = xr_u \equiv_{\mathcal{R}} xu = v$. For $\mathcal{L}$ the proof is similar. □

Lemma 3.4 *If* $s <_{\mathcal{L}} t$ *then* $B_{\mathcal{R}}(t,s) <_{\mathcal{L}} r_t$ *(and the same holds with* $<_{\mathcal{L}}$ *replaced by* $\equiv_{\mathcal{L}}$ *or* $\leq_{\mathcal{L}}$*). If* $t >_{\mathcal{R}} s$ *then* $\ell_t >_{\mathcal{R}} B_{\mathcal{L}}(s,t)$ *(and the same holds with* $<_{\mathcal{R}}$ *replaced by* $\equiv_{\mathcal{R}}$ *or* $\leq_{\mathcal{R}}$*).*

Proof. We prove the first statement, the other ones having very similar proofs. Let a be such that $ta = r_t$.

Since $B_{\mathcal{R}}(t,s) = xr_t$ for some x such that $xt = s$, we have $B_{\mathcal{R}}(t,s) = xr_t \leq_{\mathcal{L}} r_t$. Actually we have $B_{\mathcal{R}}(t,s) <_{\mathcal{L}} r_t$. Indeed, if we had $xr_t \equiv_{\mathcal{L}} r_t$, then multiplying on the right by a yields $s = xr_ta \equiv_{\mathcal{L}} r_ta = t$, i.e., $s \equiv_{\mathcal{L}} t$, which contradicts the assumption. □

Lemma 3.5 *If* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ *then* $B(u,v,w) = yw = ux = yvx$, *where* x *is such that* $w = vx$, *and* y *is such that* $u = yv$. *The value of* $B(u,v,w)$ *does not depend on the* x *or* y *chosen.*

Proof. By definition, $B(u,v,w) = ux$ where x is such that $w = vx$. Hence $B(u,v,w) = ux = yvx = yw$.

To see that $B(u,v,w)$ does not depend on the choice of x (provided that $w = vx$), let $w = vx_1 = vx_2$. Then $B(u,v,w) = yvx_1 = yvx_2$. Similarly, one sees that the choice of y does not matter (provided that $u = yv$). □

Lemma 3.6 *If* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ *and* $t \in S - \{0\}$ *then* $B(tu,v,w) = t \cdot B(u,v,w)$ *and* $B(u,v,wt) = B(u,v,w) \cdot t$.

Proof. Since $B(u,v,w) = ux$ where x is such that $w = vx$, we obtain $t \cdot B(u,v,w) = tux$ with $w = vx$. Hence by the definition of $B(tu,v,w)$ we have $B(tu,v,w) = t \cdot B(u,v,w)$.

The proof for $B(u,v,wt)$ is similar, by using Lemma 3.5. □

Lemma 3.7
(1) If $u \geq_{\mathcal{L}} su \geq_{\mathcal{L}} v$ *then* $sr_u \geq_{\mathcal{L}} B_{\mathcal{R}}(u,v)$ *and* $B_{\mathcal{R}}(su,v) = B_{\mathcal{R}}(sr_u, B_{\mathcal{R}}(u,v))$.
(2) If $su \leq_{\mathcal{L}} v \leq_{\mathcal{L}} u$ *then* $sr_u \leq_{\mathcal{L}} B_{\mathcal{R}}(u,v)$ *and* $B_{\mathcal{R}}(v,su) = B_{\mathcal{R}}(B_{\mathcal{R}}(u,v), sr_u)$.
If $su <_{\mathcal{L}} v \leq_{\mathcal{L}} u$ *then* $sr_u <_{\mathcal{L}} B_{\mathcal{R}}(u,v)$.
(3) If $su \not\lesseqgtr_{\mathcal{L}} v$ *then* $sr_u \not\lesseqgtr_{\mathcal{L}} B_{\mathcal{R}}(u,v)$.
(4) If $u \geq_{\mathcal{L}} v$ *then* $B_{\mathcal{R}}(u,sv) = s \cdot B_{\mathcal{R}}(u,v)$.
(5) Analogous properties hold for $B_{\mathcal{L}}$.

Proof. (1) By definition of $B_{\mathcal{R}}$ we have $B_{\mathcal{R}}(u,v) = xr_u$ where x is such that $v = xu$. But $v = aus$ for some a since $v \leq_{\mathcal{L}} su$, hence we can pick $x = as$. Now, $B_{\mathcal{R}}(u,v) = asr_u \leq_{\mathcal{L}} sr_u$.

By definition of $B_{\mathcal{R}}$ we have $B_{\mathcal{R}}(sr_u, B_{\mathcal{R}}(u,v)) = xr_{sr_u}$, where x is any element of S such that $B_{\mathcal{R}}(u,v) = xsr_u$.

Also, by definition of $B_{\mathcal{R}}$ we have $B_{\mathcal{R}}(su,v) = yr_{su}$, where y is such that $v = ysu$. By Lemma 3.1, multiplying $v = ysu$ by α' we obtain $B_{\mathcal{R}}(u,v) = v\alpha' = ysu\alpha' = ysr_u$. Thus, $B_{\mathcal{R}}(u,v) = ysr_u$, and since x was any element such that $B_{\mathcal{R}}(u,v) = xsr_u$, we can assume $x = y$. Now $B_{\mathcal{R}}(su,v) = xr_{su}$. Moreover, $r_{sr_u} = r_{su}$ since $u \equiv_{\mathcal{R}} r_u$. The result now follows.

(2) By definition of $B_{\mathcal{R}}$ we have $B_{\mathcal{R}}(u,v) = xr_u$ where x is such that $v = xu$. Hence $B_{\mathcal{R}}(u,v) = xr_u = v\alpha'$ where α' is such that $u\alpha' = r_u$. Moreover, $v \geq >_{\mathcal{L}} su$ (or $v >_{\mathcal{L}} su$), thus $B_{\mathcal{R}}(u,v) = v\alpha' \geq >_{\mathcal{L}} su\alpha' = sr_u$ (or $>_{\mathcal{L}} su\alpha' = sr_u$).

By definition of $B_{\mathcal{R}}$ we have $B_{\mathcal{R}}(B_{\mathcal{R}}(u,v), sr_u) = xr_{B_{\mathcal{R}}(u,v)}$, where x is such that $sr_u = x \cdot B_{\mathcal{R}}(u,v)$. By Lemma 3.1, if we multiply the last equality by α we obtain $su = xv$.

By definition we also have $B_{\mathcal{R}}(v,su) = yr_v$, where y is any element of S such that $su = yv$. But we proved that x also satisfies $su = xv$. Thus we can assume $x = y$.

So we have $B_{\mathcal{R}}(B_{\mathcal{R}}(u,v), sr_u) = yr_{B_{\mathcal{R}}(u,v)}$. Moreover, since $B_{\mathcal{R}}(u,v) \equiv_{\mathcal{R}} v$ (by Lemma 3.3), we obtain the result.

(3) This follows directly from Lemma 3.1.

(4) By definition, $B_{\mathcal{R}}(u,v) = yr_u$, where $yu = v$. Also $B_{\mathcal{R}}(u,sv) = xr_u$, where x is any element of S such that $xu = sv$. Since $yu = v$, we have $syu = sv$, hence we can pick x to be sy. The result then follows. $\Box$

Lemma 3.8 *If* $w \not\stackrel{\leq}{>}_{\mathcal{L}} s$ *then* $B(u,v,w) \not\stackrel{\leq}{>}_{\mathcal{L}} s$. *Similarly, if* $s \not\stackrel{\leq}{>}_{\mathcal{R}} u$ *then* $s \not\stackrel{\leq}{>}_{\mathcal{R}} B(u,v,w)$.

Proof. By contraposition, assume $B(u,v,w) \stackrel{\leq}{>}_{\mathcal{L}} s$. By definition, $B(u,v,w) = ux$, where x is such that $w = vx$. Since $B(u,v,w)$ exists, $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$; so $u = yv$ for some y.

Now we have $s \stackrel{\leq}{>}_{\mathcal{L}} B(u,v,w) = ux = yvx = yw \leq_{\mathcal{L}} w$.
In case $s \leq_{\mathcal{L}} ux$, the above implies $s \leq_{\mathcal{L}} w$.
In case $s \geq_{\mathcal{L}} ux$, the above implies $s \geq_{\mathcal{L}} ux \leq_{\mathcal{L}} w$, hence (by unambiguity of the $\mathcal{L}$-order): $s \stackrel{\leq}{>}_{\mathcal{L}} w$.

In either case, $s \stackrel{\leq}{>}_{\mathcal{L}} w$. $\Box$

Lemma 3.9 *(Lemma 1.1.(5) in [5].) If* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w \leq_{\mathcal{L}} s \geq_{\mathcal{R}} t$, *then* $B(u,v,w) \leq_{\mathcal{L}} s \geq_{\mathcal{R}} t$, $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} B(w,s,t)$, *and* $B(B(u,v,w),s,t) = B(u,v,B(w,s,t))$.

Proof. We have $B(u,v,w) \leq_{\mathcal{L}} w$ by the definition of B, and $w \leq_{\mathcal{L}} s \geq_{\mathcal{R}} t$, by assumption. Also, $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ by assumption, and $w \geq_{\mathcal{R}} B(w,s,t)$ by Lemma 3.5. So the claimed order relations hold.

By Lemma 3.5, $B(u,v,w) = yw$, where $u = yv$, and by definition, $B(w,s,t) = wx$, where $t = sx$. Then by definition $B(u,v,B(w,s,t)) = B(u,v,wx) = B(u,v,w) \cdot x$ (the latter equality holds by Lemma 3.6). This is equal to $yw \cdot x$. A similar reasoning shows that $B(B(u,v,w),s,t)$ is also equal to ywx. $\Box$

Lemma 3.10 *Assume that* $u \leq_{\mathcal{L}} v \equiv_{\mathcal{R}} w \geq_{\mathcal{L}} s$ *and* $c \in S - \{0\}$. *Then:*
(a) $cs = B(u,v,w)$ *iff* $u = c \cdot B(s,w,v)$,
(b) $cu = B(s,w,v)$ *iff* $s = c \cdot B(u,v,w)$.

Proof of (a). By Lemma 3.5, there exist $x, y, x', y' \in S$ such that
$B(u,v,w) = ux = yv$, $w = vx$, $u = yv$ and
$B(s,w,v) = sx' = y'v$, $v = wx'$, $s = y'w$.

If the left side of the equivalence holds then $yw = B(u,v,w) = cs = cy'w$, so if we multiply by x' we obtain $(u =)\ ywx' = cyx'$ $(= c \cdot B(s,w,v))$.

If the right side of the equivalence holds then $u = c \cdot B(s,w,v) = cy'v$, so if we multiply by x we obtain $(B(u,v,w) =)\ ux = cy'vx$; since $w = vx$, this is equal to $cy'w$,i and this equals cs (since $s = y'w$).

The proof of *(b)* is similar. $\Box$

Lemma 3.11 *Assume that* $u \leq_{\mathcal{L}} v \equiv_{\mathcal{R}} w \geq_{\mathcal{L}} s$. *Then:*
(1) $B(u,v,w) \leq_{\mathcal{L}} s$ *iff* $u \leq_{\mathcal{L}} B(s,w,v)$. *The same holds with* $\leq_{\mathcal{L}}$ *replaced by* $>_{\mathcal{L}}$ *or* $\not\stackrel{\leq}{>}_{\mathcal{L}}$.
(2. $\leq$) If $B(u,v,w) \leq_{\mathcal{L}} s$ *then* $r_s = r_{B(s,w,v)}$ *and* $B_{\mathcal{R}}(s, B(u,v,w)) = B_{\mathcal{R}}(B(s,w,v),u)$.
(2. $>$) If $B(u,v,w) >_{\mathcal{L}} s$ *then* $r_u = r_{B(u,v,w)}$ *and* $B_{\mathcal{R}}(B(u,v,w),s) = B_{\mathcal{R}}(u, B(s,w,v))$.

Analogous properties hold for $B_{\mathcal{L}}$.

Proof. (1): For $\leq_{\mathcal{L}}$ this is an immediate consequence of Lemma 3.10 *(a)*. The result (1) for $\geq_{\mathcal{L}}$ follows from Lemma 3.10 *(b)*. Since $>_{\mathcal{L}}$ holds iff we have $\geq_{\mathcal{L}}$ and not $\leq_{\mathcal{L}}$, we also obtain (1) for $>_{\mathcal{L}}$. Also, since $\lesseqgtr\!\!\!\!/_{\mathcal{L}}$ holds iff we have neither $\leq_{\mathcal{L}}$ nor $\geq_{\mathcal{L}}$, we obtain (1) for $\lesseqgtr\!\!\!\!/_{\mathcal{L}}$.

(2. $\leq$): If $B(u,v,w) \leq_{\mathcal{L}} s$ then $B(s,w,v) = sx' \leq_{\mathcal{R}} s$, and $B(s,w,v) \geq_{\mathcal{L}} B(s,w,v) \cdot x = y'vx = y'w = s$, where x', x, y' are as at the beginning of the proof of Lemma 3.10. Thus $s \equiv_{\mathcal{L}} B(s,w,v)$.

By definition, $B_{\mathcal{R}}(s, B(u,v,w)) = x''r_s$, where $x''s = B(u,v,w)$.
And $B_{\mathcal{R}}(B(s,w,v),u) = y''r_{B(s,w,v)} = y''r_s$, where $y'' \cdot B(s,w,v) = u$.
But by Lemma 3.10, $x''s = B(u,v,w)$ iff $u = x'' \cdot B(s,w,v)$. So we can choose y'' to be x''. Then the equality follows.

(2. $>$): The proof is very similar to the one of (2. $\leq$). $\square$

Lemma 3.12 *Assume that* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w \geq_{\mathcal{L}} s$, *and let* $c \in S$. *Then:*

(1) $B(u,v,w) = cs$ *iff* $B(u,v,r_w) = c \cdot B_{\mathcal{R}}(w,s)$.

(2) $c \cdot B(u,v,w) = s$ *iff* $c \cdot B(u,v,r_w) = B_{\mathcal{R}}(w,s)$.

Analogous properties hold for $B_{\mathcal{L}}$.

Proof. (1): Assume $B(u,v,w) = cs$, where (by Lemma 3.5) $B(u,v,w) = yw$ with $u = yv$. Multiplying $yw = cs$ on the right by α' (where α' is such that $w\alpha' = r_w$) we obtain:
$yr_w = cs\alpha'$.
The left side yr_w is equal to $B(u,v,r_w)$ by Lemma 3.5 (since here $u = yv$). On the other hand, by the definition of $B_{\mathcal{R}}$ we have, $B_{\mathcal{R}}(w,s) = xr_w$ with $s = xw$. Since $w\alpha' = r_w$, we have $B_{\mathcal{R}}(w,s) = xw\alpha' = s\alpha'$, which (when multiplied by c) yields the right side.

Conversely, if $B(u,v,r_w) = c \cdot B_{\mathcal{R}}(w,s)$ we will have by Lemma 3.5 and by the definition of $B_{\mathcal{R}}$, in the above notation:
$yr_w = cs\alpha'$.

Multiplying on the right by α (where α is such that $r_w\alpha = w$), we obtain: $yw = cs\alpha'\alpha = cs$ (we have $s\alpha'\alpha = s$ because we assumed $w >_{\mathcal{L}} s$). Thus $B(u,v,w)(= yw = cs\alpha'\alpha) = cs$.

The proof of (2) is quite similar to the proof of (1). $\square$

Lemma 3.13 *Assume that* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w \geq_{\mathcal{L}} s$. *Then:*

(1) $B(u,v,w) \leq_{\mathcal{L}} s$ *iff* $B(u,v,r_w) \leq_{\mathcal{L}} B_{\mathcal{R}}(w,s)$. *The same is true with* $\leq_{\mathcal{L}}$ *replaced by* $>_{\mathcal{L}}$ *or* $\lesseqgtr\!\!\!\!/_{\mathcal{L}}$.

(2. $\leq$) If $B(u,v,w) \leq_{\mathcal{L}} s$ *then* $s \equiv_{\mathcal{R}} B_{\mathcal{R}}(w,s)$ *and* $B_{\mathcal{R}}(s, B(u,v,w)) = B_{\mathcal{R}}(B_{\mathcal{R}}(w,s), B(u,v,r_w))$.

(2. $>$) If $B(u,v,w) >_{\mathcal{L}} s$ *then* $B(u,v,w) \equiv_{\mathcal{R}} B(u,v,r_w)$ *and* $B_{\mathcal{R}}(B(u,v,w),s) = B_{\mathcal{R}}(B(u,v,r_w), B_{\mathcal{R}}(w,s))$

Analogous properties hold for $B_{\mathcal{L}}$:

If $s \leq_{\mathcal{R}} u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ *then :*

(1) $s \leq_{\mathcal{R}} B(u,v,w)$ *iff* $B_{\mathcal{L}}(s,u) \leq_{\mathcal{R}} B(\ell_u,v,w)$. *The same is true with* $\leq_{\mathcal{R}}$ *replaced by* $>_{\mathcal{R}}$ *or* $\lesseqgtr\!\!\!\!/_{\mathcal{R}}$.

(2. $\leq$) If $s \leq_{\mathcal{R}} B(u,v,w)$ *then* $B(u,v,w) \equiv_{\mathcal{L}} B(\ell_u,v,w)$ *and* $B_{\mathcal{L}}(s, B(u,v,w)) = B_{\mathcal{L}}(B_{\mathcal{L}}(s,u), B(\ell_u,v,w))$.

(2. $>$) If $s >_{\mathcal{R}} B(u,v,w)$ *then* $s \equiv_{\mathcal{L}} B_{\mathcal{L}}(s,u)$ *and* $B_{\mathcal{L}}(B(u,v,w),s) = B_{\mathcal{L}}(B\ell_u,v,w), B_{\mathcal{L}}(s,u))$.

Proof. (1): The result for $\leq_{\mathcal{L}}$ follows immediately fom Lemma 3.12 (1). From Lemma 3.12 (2), we have the corresponding result for $\geq_{\mathcal{L}}$. Combining the two we obtain the result for $>_{\mathcal{L}}$ and for $\lesseqgtr\!\!\!\!/_{\mathcal{L}}$.

(2. $\leq$): By Lemma 3.3 we have $r_s = r_{B_{\mathcal{R}}(w,s)}$.

We will apply Lemma 3.7 (2), which we quote here with different parameters:

If $s_o u_o \leq_{\mathcal{L}} v_o \leq_{\mathcal{L}} u_o$ *then* $B_{\mathcal{R}}(v_o, s_o u_o) = B_{\mathcal{R}}(B_{\mathcal{R}}(u_o, v_o), s_o r_{u_o})$.
Let $v_o = s$, $u_o = y$, and $s_o = w$, where (by Lemma 3.5), $B(u, v, w) = yw$ and $B(u, v, r_w) = yr_w$ with $yv = u$. Then $s_o u_o = B(u, v, w)$ and $s_o r_{u_o} = B(u, v, r_w)$. By assumption, $B(u, v, w) \leq_{\mathcal{L}} s <_{\mathcal{L}} w$, so $s_o u_o \leq_{\mathcal{L}} u_o \leq_{\mathcal{L}} v_o$, hence Lemma 3.7 (2) is indeed applicable here. By substituting, the claimed result then follows immediately.

(2. $>$): By Lemma 3.5 we have $B(u, v, w) = yw$ and $B(u, v, r_w) = yr_w$, with $u = yv$. Since $w \equiv_{\mathcal{R}} r_w$ we obtain $B(u, v, w) \equiv_{\mathcal{R}} B(u, v, r_w)$.

We will apply Lemma 3.7 (1), which we quote here with different parameters:
If $u_o \geq_{\mathcal{L}} s_o u_o \geq_{\mathcal{L}} v_o$ *then* $B_{\mathcal{R}}(s_o u_o, v_o) = B_{\mathcal{R}}(s_o r_{u_o}, B_{\mathcal{R}}(u_o, v_o))$.
Let $s_o = y$, and $u_o = w$, where $B(u, v, w) = yw$ and $B(u, v, r_w) = yr_w$, with $yv = u$ (by Lemma 3.5). And let $v_o = s$. Since by our assumptions $w >_{\mathcal{L}} B(u, v, w) >_{\mathcal{L}} s$, Lemma 3.7 (1) can be applied. The claimed result then follows immediately by substitution. □

Lemma 3.14 *Assume that* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$. *Then* $B_{\mathcal{R}}(v, u) \leq_{\mathcal{L}} r_v \geq_{\mathcal{R}} w$ *and* $B(B_{\mathcal{R}}(v, u), r_v, w) = B(u, v, w)$.

Analogous properties hold for $B_{\mathcal{L}}$*:*
If $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ *then* $u \leq_{\mathcal{L}} \ell_v \geq_{\mathcal{R}} B_{\mathcal{L}}(w, v)$ *and* $B(u, v, w) = B(u, \ell_v, B_{\mathcal{L}}(w, v))$.

Proof. The fact that $B_{\mathcal{R}}(v, u) \leq_{\mathcal{L}} r_v \geq_{\mathcal{R}} w$ is obvious from the definition of B.

By Lemma 3.5, $B(u, v, w) = x_1 w$ for any x_1 such that $u = x_1 v$. Also, by definition, $B_{\mathcal{R}}(v, u) = x_2 r_v$ for any x_2 such that $u = x_2 v$; therefore we can choose $x_2 = x_1$.

Now $B(B_{\mathcal{R}}(v, u), r_v, w) = B_{\mathcal{R}}(v, u)\, z$ with $w = r_v z$, hence $B(B_{\mathcal{R}}(v, u), r_v, w) = x_1 r_v z = x_1 w$. This proves the result. □

Lemma 3.15 *Assume that* $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w \leq_{\mathcal{L}} s$. *Then* $B_{\mathcal{R}}(s, B(u, v, w)) = B(u, v, B_{\mathcal{R}}(s, w))$.

Analogous properties hold for $B_{\mathcal{L}}$*:*
If $s \geq_{\mathcal{R}} u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ *then* $B_{\mathcal{L}}(B(u, v, w), s) = B(B_{\mathcal{L}}(u, s), v, w)$.

Proof. By definition, $B_{\mathcal{R}}(s, B(u, v, w)) = x_1 r_s$ where $x_1 s = B(u, v, w) = uz$, with (by definition of B) $w = vz$. We also have:

$B(u, v, B_{\mathcal{R}}(s, w))$
$= y\, B_{\mathcal{R}}(s, w)$ where $u = yv$
$= y x_2 r_s$ where x_2 is such that $x_2 s = w$
$= y x_2 s \alpha'$ where α' is such that $r_s = s\alpha'$
$= y w \alpha'$ since $x_2 s = w$
$= y v z \alpha'$ since $w = vz$
$= u z \alpha'$ since $u = yv$
$= B(u, v, w)\alpha'$
$= x_1 s \alpha'$
$= x_1 r_s$
$= B_{\mathcal{R}}(s, B(u, v, w))$ as we saw in the beginning of this proof. □

Lemma 3.16 *Assume that* $u \geq_{\mathcal{L}} v \geq_{\mathcal{R}} w$. *Then*
(1) $B_{\mathcal{R}}(u, v) \equiv_{\mathcal{L}} B_{\mathcal{R}}(u, \ell_v))$,
(2) $B_{\mathcal{L}}(w, B_{\mathcal{R}}(u, v)) = B_{\mathcal{L}}(B_{\mathcal{L}}(w, u), B_{\mathcal{R}}(u, \ell_v))$.

Proof. Property (1) follows easily from Lemma 3.7 (4).

(2): Let β and β' be such that $v = \beta\ell_v$ and $\ell_v = \beta' v$. By definition, $B_{\mathcal{R}}(u,v) = xr_u$, where $xu = v$. Hence, by the definition of $B_{\mathcal{R}}$, we have $B_{\mathcal{R}}(u,\ell_v) = \beta' xr_u$ since $\beta' x$ satisfies $\beta' xu = \ell_v$.

Thus, $B_{\mathcal{L}}(w, B_{\mathcal{R}}(u,v)) = B_{\mathcal{L}}(w, xr_u) = \ell_{xr_u} y_1$, where y_1 is such that $w = xr_u y_1$.

On the other hand, $B_{\mathcal{L}}(B_{\mathcal{L}}(w,u), B_{\mathcal{R}}(u,\ell_v)) = \ell_{B_{\mathcal{R}}(u,\ell_v)} y_2 = \ell_{xr_u} y_2$, since $B_{\mathcal{R}}(u,v) \equiv_{\mathcal{L}} B_{\mathcal{R}}(u,\ell_v))$ (as we just proved in (1)). Here, by the definition of $B_{\mathcal{L}}$, y_2 is any element of S such that $B_{\mathcal{L}}(w,u) = B_{\mathcal{R}}(u,\ell_v)\, y_2$. We saw that the latter is equal to $\beta' xr_u y_2$. By the definition of $B_{\mathcal{L}}$ we also have $B_{\mathcal{L}}(w,u) = \ell_v y_3$ where y_3 is such that $w = vy_3$.

Therefore $\ell_v y_3 = \beta' xr_u y_2$. Multiplying on the left by β yields $(w =)\ vy_3 = xr_u y_2$, i.e., y_2 satisfies $w = xr_u y_2$, which is the defining property of y_1.

Hence, y_2 can be chosen above so that $y_2 = y_1$. $\square$

Lemma 3.17 *Assume that* $u' geq_{\mathcal{L}} v \le_{\mathcal{R}} w$. *Then* $B_{\mathcal{L}}(B_{\mathcal{R}}(u,v), w) = B_{\mathcal{R}}(u, B_{\mathcal{L}}(v,w))$.

Proof. By the definition of $B_{\mathcal{R}}$ and $B_{\mathcal{L}}$, $B_{\mathcal{R}}(u,v) = xr_u$, where $v = xu$, and $B_{\mathcal{L}}(v,w) = \ell_w y$, where $v = wy$. Let α, α', β and β' be such that $r_u\alpha = u$, $u\alpha' = r_u$, $\beta\ell_w = w$, and $\beta' w = \ell_w$.

Then $B_{\mathcal{L}}(B_{\mathcal{R}}(u,v), w) = \ell_w y_1$, where y_1 is such that $(xr_u =)\ B_{\mathcal{R}}(u,v) = wy_1$.

Also, $B_{\mathcal{R}}(u, B_{\mathcal{L}}(v,w)) = x_1 r_u$, where x_1 is such that $(\ell_w y =)\ B_{\mathcal{L}}(v,w) = x_1 u$. By multiplying the latter equalities by β we obtain:

(*) $$wy = \beta x_1 u.$$

We need to show that $\ell_w y_1 = x_1 r_u$.

We saw that $v = xu = xr_u\alpha = B_{\mathcal{R}}(u,v)\,\alpha$ (by the choice of x and of α, and by the definition of $B_{\mathcal{R}}$). Thus

$$B_{\mathcal{R}}(u,v)\,\alpha = v.$$

In this equation we replace v by wy (see the definition of $B_{\mathcal{L}}(v,w)$), and we replace $B_{\mathcal{R}}(u,v)$ by wy_1 (see the expression for $B_{\mathcal{L}}(B_{\mathcal{R}}(u,v), w)$). Thus,

$$wy_1\alpha = wy.$$

By (*), we can replace wy by $\beta x_1 u$. So,

$$wy_1\alpha = \beta x_1 u.$$

Multiplying this by α' (on the left) and by β' (on the right) yields $\ell_w y_1 = x_1 r_u$, which is what we wanted. $\square$

Lemma 3.18 *Assume that* $u \le_{\mathcal{L}} v \ge_{\mathcal{R}} w$. *Then* $B(B_{\mathcal{R}}(v,u), r_v, w) = B(u, \ell_v, B_{\mathcal{L}}(w,v)$.

Proof. By the definition of $B_{\mathcal{R}}$ and $B_{\mathcal{L}}$, we have:
$B(B_{\mathcal{R}}(v,u), r_v, w) = B(xr_v, vw)$ where $u = xv$,
$B(u, \ell_v, B_{\mathcal{L}}(w,v) = B(u, \ell_v, \ell_v y)$, where $w = vy$.

By the definition of B, $B(xr_v, vw) = xr_v z_1$, where $w = r_v z_1$. Hence, $B(xr_v, vw) = xw$.

Similarly, $B(u, \ell_v, \ell_v y) = uz_2$, where z_2 is any element of S satisfying $\ell_v y = \ell_v z_2$; hence we can pick z_2 to be y. Then we have $B(u, \ell_v, \ell_v y) = uy = xvy$ (since $u = xv$), and $xvy = xw$ (since $vy = w$). Thus $B(u, \ell_v, \ell_v y) = xw$, which is equal to $B(xr_v, vw)$, as we saw. $\square$

4 Termination

In this section we prove that the rewrite system for $(S)_{reg}$ is terminating.

Lemma 4.1 *If the sub-system consisting of the rules (2.1) – (2.4) terminates then the whole rewrite system terminates.*

Proof. Imagine, by contraposition, that the whole rewrite system allows an infinite rewrite chain. Since the first group of rules is strictly length-reducing, the chain contains only rules of the form (2.1) – (2.4), from some point on. Hence the rules (2.1) – (2.4) do not form a terminating system. □

The rest of this section deals with the proof that the sub-system consisting of the rules (2.1) – (2.4) terminates.

Since the rules (2.1) – (2.4) are length-preserving, the notion of *position* in a string is invariant under rewriting. More precisely, a string of length n (over the generators of $(S)_{reg}$) has positions $1, 2, \ldots, n$, and when a rule of type (2.1) – (2.4) is applied, the new string still has positions $1, 2, \ldots, n$.

Our first step is to find factorizations of strings that are preserved under rewriting. (See [3] for details about preserved factorization schemes; here we do not need exact definitions since the context will make everything clear).

Lemma 4.2 *In a string, a position occupied by 0 or by a non-0 element is invariant under rewriting. Also, a pair of positions occupied by $S \times S$ or $\overline{S} \times \overline{S}$ is invariant under rewriting.*

Proof. Since the rules (2.1) – (2.4) do not use the symbol 0, a position occupied by 0 will never change; and a non-0 symbol never turns into 0. Similarly, a pair of positions occupied by elements $(s, t) \in S \times S$ will always remain occupied by a pair $\in S \times S$ (although the value of s and t can change). Similarly for $\overline{S} \times \overline{S}$. □

Lemma 4.3 *(Preservation of $<_{\mathcal{L}}$, $\equiv_{\mathcal{L}}$, $>_{\mathcal{L}}$, and $\not\lessgtr_{\mathcal{L}}$, and similarly for $\mathcal{R}$)*
In a string, a pair of positions occupied by elements $(s, \overline{t}) \in S \times \overline{S}$ with $s <_{\mathcal{L}} t$ (or $\equiv_{\mathcal{L}}$ or $>_{\mathcal{L}}$ or $\not\lessgtr_{\mathcal{L}}$) will always remain occupied by some pair of $\in S \times \overline{S}$ related by $<_{\mathcal{L}}$ (respectively $\equiv_{\mathcal{L}}$ or $>_{\mathcal{L}}$ or $\not\lessgtr_{\mathcal{L}}$). Similarly, for a pair $\in \overline{S} \times S$ related by $<_{\mathcal{R}}$ (or $\equiv_{\mathcal{R}}$ or $>_{\mathcal{R}}$ or $\not\lessgtr_{\mathcal{R}}$), this relation is preserved between these two positions.

Let us look now at the four ways s or $\overline{t}$ could be changed when a rule is applied just to the left or right of $(s, \overline{t})$.

If the symbol to the left of $(s, \overline{t})$ is $\overline{u}$, with $u >_{\mathcal{R}} s$, then (2.2) can change $(\overline{u}, s, \overline{t})$ into $(\overline{\ell_u}, B_{\mathcal{L}}(s, u), \overline{t})$. Since $B_{\mathcal{L}}(s, u) \equiv_{\mathcal{L}} s$ (by Lemma 3.3), we still have $B_{\mathcal{L}}(s, u) <_{\mathcal{L}} t$ at this pair of positions.

If the symbol to the left of $(s, \overline{t})$ is $\overline{u}$, with $u \leq_{\mathcal{R}} s$, then (2.3) can change $(\overline{u}, s, \overline{t})$ into $(\overline{B_{\mathcal{L}}(u, s)}, \ell_s, \overline{t})$. Since $\ell_s \equiv_{\mathcal{L}} s$ we still have $\ell_s <_{\mathcal{L}} t$ at this pair of positions.

If the symbol to the left of $(s, \overline{t})$ is v with $t >_{\mathcal{R}} v$ (or $t \leq_{\mathcal{R}} v$) then the reasoning is similar. □

As a consequence of these preservation Lemmas, we can factor any string into maximal subsegments which have the following properties:

- 0 does not occur in a subsegment (unless the subsegment consists of only 0);
- neighboring positions in a subsegment are occupied by pairs in $S \times \overline{S}$ or $\overline{S} \times S$;
- the incomparability relation $\not\lessgtr$ (for $\mathcal{L}$ or $\mathcal{R}$) does not occur inside a subsegment.

We call such subsegments *continuous strings* (i.e., we view the break between two maximal such subsegments as a discontinuity).

Definition. Let x be a continuous string of length n, and let i ($1 \leq i \leq n$) be a position in x. We call this position *maximal* iff

- $i = 1$ and the relation between the elements at positions 1 and 2 is $>$;
- or $i = n$ and the relation between the elements at positions $n - 1$ and n is $\leq$;
- or $1 < i < n$ and the relations between the elements at positions $i - 1$, i, and $i + 1$ are $\leq, >$.

Lemma 4.4 *(Maximal positions)*
During the rewriting of a continuous string using rules (2.1) – (2.4), an element of $S \cup \overline{S}$ *at a maximal position is rewritten at most twice. From then on, the symbol at the maximal position never changes.*

Proof. Suppose that a maximal position is occupied by an element $s \in S$ (the case of an element of $\overline{S}$ is similar. Let $\overline{u}, \overline{v}$ be the neighboring elements in the continuous string, with $u \leq_{\mathcal{R}} s >_{\mathcal{L}} v$. If (2.3) is applied, $(\overline{u}, s)$ will be rewritten to $(\ldots, \ell_s)$. If (2.1) is applied, $(s, \overline{v})$ will be rewritten to $(r_s, \ldots)$. If (2.3) is now applied (or (2.1) is applied to the previous alternative), the element at the maximal position is rewritten to ℓ_{r_s} (respectively r_{ℓ_s}). Further rewriting with rules (2.1), (2.3) cannot change the element at the maximal position because $r_{\ell_{r_s}} = \ell_{r_s}$ and $\ell_{r_{\ell_s}} = r_{\ell_s}$, (which follows from the special choice of the representatives of the $\mathcal{L}$- and $\mathcal{R}$-classes). □

Note that the above Lemma (and the termination property itself) is not true if the representatives of the $\mathcal{L}$- and $\mathcal{R}$-classes are chosen differently than we did (except in trivial cases, e.g., when $S - \{0\}$ has no strict $>_{\mathcal{R}}$ and $>_{\mathcal{L}}$ chains).

Lemma 4.5 *(Chains >> ... and ... ≤≤ stabilize)*
If $\ldots s \ldots$ *occurs in a continuous string, with* $\ldots >_{\mathcal{L}} s >_{\mathcal{R}} \ldots$*, then after a finite number of applications of the rules (2.1) – (2.4) to the string, the symbol at the position of s will not change any more. The same is true for on occurrence of* $\ldots \overline{s} \ldots$ *with* $\ldots >_{\mathcal{R}} s >_{\mathcal{L}} \ldots$*, and for* $\ldots s \ldots$ *with* $\ldots \leq_{\mathcal{R}} s \leq_{\mathcal{L}} \ldots$*, and for* $\ldots \overline{s} \ldots$ *with* $\ldots \leq_{\mathcal{L}} s \leq_{\mathcal{R}} \ldots$*.*

Proof. Let us consider the case of $\ldots u \ldots$ with $\ldots >_{\mathcal{L}} u >_{\mathcal{R}} \ldots$. By the previous lemma, we know that the element at the maximal position towards the right of u will eventually stabilize. By induction, suppose that all elements in the descending alternatining $>_{\mathcal{L}}$ - $>_{\mathcal{R}}$ chain to the left of u have stabilized. No rule among (2.1) – (2.4) can be applied to the left of u in this chain anymore (otherwise the element just left of u would change again, since $s \neq r_s$, resp. $s \neq \ell_s$ in the rules). On the other hand, if a rule is applied to u and the element just right of u (in that case it would be rule (2.2)), then u is replaced by r_u and after this, no rule can be applied anymore at this position.

Let us also consider the case of $\ldots u \ldots$ with $\ldots \leq_{\mathcal{R}} u \leq_{\mathcal{L}} \ldots$. As before, let us assume by induction that all elements in the ascending alternatining $>_{\mathcal{L}}$ - $>_{\mathcal{R}}$ chain to the right of u have stabilized. Again, no rule will be applied to the right of u anymore. On the other hand, if a rule is applied to u and the element just left of u (in that case it will be rule (2.3), then u is replaced by ℓ_s, and after this, no rule can be applied anymore at this position.

The reasoning is similar in the other cases. □

Definition. Let x be a continuous string of length n, and let i $(1 \leq i \leq n)$ be a position in x. We call this position *minimal* iff
- $i = 1$ and the relation between the elements at positions 1 and 2 is $\leq$;
- or $i = n$ and the relation between the elements at positions $n-1$ and n is $>$;
- or $1 < i < n$ and the relations between the elements at positions $i-1$, i, and $i+1$ are $> . \leq$.

Lemma 4.6 *(Minimal positions stabilize)*
After a finite number of applications of the rules (2.1) – (2.4) to a continuous string the symbols at the minimal positions do not change anymore.

Proof. Consider the case of a minimal position occupied by an element $v \in S - \{0\}$, occurring in a context $(\ldots, \overline{u}, v, \overline{w}, \ldots)$, with $u >_{\mathcal{R}} v \leq_{\mathcal{L}} w$. By induction we assume that u and w will not change anymore. Then no rule can be applied to v (otherwise u or w would change again, since $s \neq r_s$, resp. $s \neq \ell_s$ in the rules). □

5 Local confluence

This section contains the proof that the rewrite system for $(S)_{reg}$ is locally confluent. We have to look at all the overlap cases (see [6]), which is tedious but straightforward in each case. Each case is either trivial or it is resolved by using the properties of B, $B_{\mathcal{L}}$ and $B_{\mathcal{R}}$ proved in Section 3.

Overlap 1.1 – 1.1: $(st, u) \overset{1.1}{\longleftarrow} (s, t, u) \overset{1.1}{\longrightarrow} (s, tu)$.

Then $(st, u) \overset{1.1}{\longrightarrow} (stu) \overset{1.1}{\longleftarrow} (s, tu)$, where we also use associativity of the multiplication in S.

The overlap for the $\overline{S}$-form of rule 1.1 has the form
$(\overline{ts}, \overline{u}) \overset{1.1}{\longleftarrow} (\overline{s}, \overline{t}, \overline{u}) \overset{1.1}{\longrightarrow} \overline{s}, \overline{ut})$.
Confluence follows easily as above.

Overlaps with 1.2: In all overlaps with rule 1.2 one easily shows confluence to (0).

Overlap 1.1 – 1.3:

Case 1. S-form of rule 1.1.
$(tu, \overline{v}) \overset{1.1}{\longleftarrow} (t, u, \overline{v}) \overset{1.3}{\longrightarrow} (t, 0)$ where $u \not\leq_{\mathcal{L}} v$.
Then $(t, 0) \overset{1.2}{\longrightarrow} (0) \overset{1.3}{\longleftarrow} (tu, \overline{v})$. The last application of rule 1.3 is justified by the following.

Claim: If $u \not\leq_{\mathcal{L}} v$ *then* $tu \not\leq_{\mathcal{L}} v$.

Proof of the Claim: By contraposition, if $t \geq_{\mathcal{L}} tu \geq_{\mathcal{L}} v$ then obviously $t \geq_{\mathcal{L}} v$. And if $t \geq_{\mathcal{L}} tu \leq_{\mathcal{L}} v$ then $t \mathrel{\substack{\leq \\ >}}_{\mathcal{L}} v$, by unambiguity of S. □

Case 2 $\overline{S}$-form of rule 1.1.
$(0, \overline{v}) \overset{1.3}{\longleftarrow} (t, \overline{u}, \overline{v}) \overset{1.1}{\longrightarrow} (t, \overline{vu})$, where $t \not\leq_{\mathcal{L}} u$.
Confluence is proved in the same way as above.

Overlap 1.1 – 1.4: Similar to the previous case.

Overlap 1.1 – 1.5:

Case 1. $(tu, \overline{v}, w) \overset{1.1}{\longleftarrow} (t, u, \overline{v}, w) \overset{1.5}{\longrightarrow} (t, B(u, v, w))$, where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

Then $(tu, \overline{v}, w) \overset{1.5}{\longrightarrow} (B(tu, v, w)) \overset{?}{=} (t \cdot B(u, v, w)) \overset{1.1}{\longleftarrow} (t, B(u, v, w))$.
By Lemma 3.6, $B(tu, v, w) = t \cdot B(u, v, w)$, so we have confluence.

Case 2. $(u, \overline{v}, wt) \overset{1.1}{\longleftarrow} (u, \overline{v}, w, t) \overset{1.5}{\longrightarrow} (B(u, v, w) \cdot t)$ where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

As in the previous case, we have confluence by Lemma 3.6.
Here we only considered the S-form of rule 1.1; the $\overline{S}$-form does not overlap with 1.5.

Overlap 1.1 – 1.6: Only the $\overline{S}$-form of 1.1 overlaps with 1.6. Confluence is proved in a similar way as in 1.1 – 1.5.

Overlap 1.1 (S-form) – 2.1: $(su, \overline{v}) \overset{1.1}{\longleftarrow} (s, u, \overline{v}) \overset{2.1}{\longrightarrow} (s, r_u, \overline{B_{\mathcal{R}}(u, v)})$, where $u >_{\mathcal{L}} v$.

Case 1: $su >_{\mathcal{L}} v$.

Then $(su, \overline{v}) \overset{2.1}{\longrightarrow} (r_{su}, \overline{B_{\mathcal{R}}(su, v)})$, since $su >_{\mathcal{L}} v$.
Moreover, $(s, r_u, \overline{B_{\mathcal{R}}(u, v)}) \overset{1.1}{\longrightarrow} (sr_u, \overline{B_{\mathcal{R}}(u, v)}) \overset{2.1}{\longrightarrow} (r_{sr_u}, \overline{B_{\mathcal{R}}(sr_u, B_{\mathcal{R}}(u, v))})$, where the latter application of rule 2.1 is justified since $sr_u >_{\mathcal{L}} B_{\mathcal{R}}(u, v)$ (indeed we assumed $su >_{\mathcal{L}} v$, so by Lemma 3.1, $sr_u = su\alpha' >_{\mathcal{L}} v\alpha' = B_{\mathcal{R}}(u, v)$).

To have confluence we need $r_{su} = r_{sr_u}$ (which easily follows from $u \equiv_{\mathcal{R}} r_u$), and $B_{\mathcal{R}}(su, v) = B_{\mathcal{R}}(sr_u, B_{\mathcal{R}}(u, v))$ (which is proved in Lemma 3.7 (1)).

Case 2: $su \leq_{\mathcal{L}} v$.

Then $(su, \overline{v}) \xrightarrow{2.4} (B_{\mathcal{R}}(v, su), \overline{r_v})$.

Moreover, $(s, r_u, \overline{B_{\mathcal{R}}(u, v)}) \xrightarrow{1.1} (sr_u, \overline{B_{\mathcal{R}}(u, v)}) \xrightarrow{2.4} (B_{\mathcal{R}}(B_{\mathcal{R}}(u, v), sr_u), \overline{r_{B_{\mathcal{R}}(u,v)}})$. The latter application of rule 2.4 is justified since $sr_u \leq_{\mathcal{L}} B_{\mathcal{R}}(u, v)$, which follows from the assumption $su \leq_{\mathcal{L}} v$ and from Lemma 3.1.

In order to have confluence we need $B_{\mathcal{R}}(B_{\mathcal{R}}(u, v), sr_u) = B_{\mathcal{R}}(v, su)$ (which was proved in Lemma 3.7 (2)), and $r_{B_{\mathcal{R}}(u,v)} = r_v$ (which follows from Lemma 3.3).

Case 3: $su \nleq_{\mathcal{L}} v$.

Then $(su, \overline{v}) \xrightarrow{1.3} (0)$.

Moreover, $(s, r_u, \overline{B_{\mathcal{R}}(u, v)}) \xrightarrow{1.1} (sr_u, \overline{B_{\mathcal{R}}(u, v)})$. By Lemma 3.7 (3), $sr_u \nleq_{\mathcal{L}} B_{\mathcal{R}}(u, v)$, so can now apply rule 1.3, thus obtaining confluence to (0).

Overlap 1.1 ($\overline{S}$-form) – 2.1: $(r_u, \overline{B_{\mathcal{R}}(u, v)}, \overline{s}) \xleftarrow{2.1} (u, \overline{v}, \overline{s}) \xrightarrow{1.1} (u, \overline{sv})$, where $u >_{\mathcal{L}} v$.

Then $(r_u, \overline{B_{\mathcal{R}}(u, v)}, \overline{s}) \xrightarrow{1.1} (r_u, \overline{s\, B_{\mathcal{R}}(u, v)})$, and $(u, \overline{sv}) \xrightarrow{2.1} (r_u, \overline{B_{\mathcal{R}}(u, sv)})$; 2.1 was applicable since $u >_{\mathcal{L}} v \geq_{\mathcal{L}} sv$. Confluence than follws directly from Lemma 3.7 (4).

Overlap 1.1 – 2.2: This is similar to the overlap 1.1 – 2.1.

Overlap 1.1 – 2.3: This is similar to the overlap 1.1 – 2.4, which we consider next.

Overlap S-from of 1.1 – 2.4: $(sv, \overline{u}) \xleftarrow{1.1} (s, v, \overline{u}) \xrightarrow{2.4} (s, B_{\mathcal{R}}(u, v), \overline{r_u})$, where $v \leq_{\mathcal{L}} u$.

Then $(sv, \overline{u}) \xrightarrow{2.4} (B_{\mathcal{R}}(u, sv)$.

Moreover, $(s, B_{\mathcal{R}}(u, v), \overline{r_u}) \xrightarrow{1.1} (s \cdot B_{\mathcal{R}}(u, v), \overline{r_u})$.

Confluence then follows from Lemma 3.7 (4).

Overlap $\overline{S}$–from of 1 – 2.4: $(B_{\mathcal{R}}(u, s), \overline{r_u}, \overline{v}) \xleftarrow{2.4} (s, \overline{u}, \overline{v}) \xrightarrow{1.1} (s, \overline{vu})$, where $s \leq_{\mathcal{L}} u$.

Case 1. $s \leq_{\mathcal{L}} vu \leq_{\mathcal{L}} u$.

Then $(s, \overline{vu}) \xrightarrow{2.4} (B_{\mathcal{R}}(vu, s), \overline{r_{vu}})$.

On the other hand, $(B_{\mathcal{R}}(u, s), \overline{r_u}, \overline{v}) \xrightarrow{1.1} (B_{\mathcal{R}}(u, s), \overline{vr_u}) \xrightarrow{2.4} (B_{\mathcal{R}}(vr_u, B_{\mathcal{R}}(u, s)), \overline{r_{vr_u}})$. The last application of rule 2.4 is justified by Lemma 3.7 (1).

To check confluence we observe that $vu \equiv_{\mathcal{R}} vr_u$ (obvious), and that $B_{\mathcal{R}}(vu, s) = B_{\mathcal{R}}(vr_u, B_{\mathcal{R}}(u, s))$ by Lemma 3.7 (1).

Case 2. $vu <_{\mathcal{L}} s \leq_{\mathcal{L}} u$.

Then $(s, \overline{vu}) \xrightarrow{2.1} (r_s, \overline{B_{\mathcal{R}}(s, vu)})$.

On the other hand, $(B_{\mathcal{R}}(u, s), \overline{r_u}, \overline{v}) \xrightarrow{1.1} (B_{\mathcal{R}}(u, s), \overline{vr_u}) \xrightarrow{2.1} (r_{B_{\mathcal{R}}(u,s)}, \overline{B_{\mathcal{R}}(B_{\mathcal{R}}(u, s), vr_u)})$. The last application of rule 2.4 is justified by Lemma 3.7 (2).

Confluence now follows from Lemma 3.7 (2), and from the fact that $s \equiv_{\mathcal{R}} B_{\mathcal{R}}(u, s)$ (Lemma 3.2).

Case 3. $vu \nleq_{\mathcal{L}} s$.

Then $(s, \overline{vu}) \xrightarrow{1.3} (0)$. On the other hand, $(B_{\mathcal{R}}(u, s), \overline{r_u}, \overline{v}) \xrightarrow{1.1} (B_{\mathcal{R}}(u, s), \overline{vr_u}) \xrightarrow{1.3} (0)$. We used Lemma 3.7 (3) to justify the last application of rule 1.3.

So far we have considered all overlaps involving the rule 1.1. We mentioned already that the rule 1.2 always leads to confluence to (0). Let us now look at all the overlaps that involve rule 1.3 (other than with rule 1.1, seen already).

There is no overlap of 1.3 with itself.

Overlap 1.3 – 1.4: $(0, s) \overset{1.3}{\longleftarrow} (u, \overline{v}, s) \overset{1.4}{\longrightarrow} (u, 0)$, where $u \nleqq_{\mathcal{L}} v$ and $v \nleqq_{\mathcal{R}} s$.
Then we obviously have confluence to (0).

The case of $(\overline{u}, v, \overline{s})$, where $u \nleqq_{\mathcal{R}} v$ and $v \nleqq_{\mathcal{L}} v$, is handled in a similar way.

Overlap 1.3 – 1.5: $(B(u, v, w), \overline{s}) \overset{1.5}{\longleftarrow} (u, \overline{v}, w, \overline{s}) \overset{1.3}{\longrightarrow} (u, \overline{v}, 0)$,
where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ and $w \nleqq_{\mathcal{L}} s$.

Then $(u, \overline{v}, 0) \longrightarrow (0)$ by two applications of rule 1.2. Moreover, since $B(u, v, w) \nleqq_{\mathcal{L}} s$ if $w \nleqq_{\mathcal{L}} s$ (by Lemma 3.8), we also have $(B(u, v, w), \overline{s}) \overset{1.3}{\longrightarrow} (0)$.

Overlap 1.3 – 1.6: This is similar to 1.3 – 1.5.

There are no overlaps 1.3 – 2.1, 1.3 – 2.4, nor 1.4 – 1.4, 1.4 – 2.2, 1.4 – 2.3. The overlaps 1.4 – 1.5 and 1.4 – 1.6 are similar to the case 1.3 – 1.5.

Overlaps 1.3 – 2.2, 1.3 – 2.3, or 1.4 – 2.1: This is very similar to the case considered next.

Overlap 1.4 – 2.4: $(B_{\mathcal{R}}(u, v), \overline{u}, w) \overset{2.4}{\longleftarrow} (v, \overline{u}, w) \overset{1.4}{\longrightarrow} (v, 0)$, where $v \leq_{\mathcal{L}} u \nleqq_{\mathcal{R}} w$.

Then $(v, 0) \to (0)$ by rule 1.2. Moreover, since $r_u \equiv_{\mathcal{R}} u \nleqq_{\mathcal{R}} w$ we have $(B_{\mathcal{R}}(u, v), \overline{u}, w) \longrightarrow (B_{\mathcal{R}}(u, v), 0)$ by rule 1.4; this then leads to (0) by 1.2.

Overlap 1.5 – 1.5: $(B(u, v, w), \overline{s}, t) \overset{1.5}{\longleftarrow} (u, \overline{v}, w, \overline{s}, t) \overset{1.5}{\longrightarrow} (u, \overline{v}, B(w, s, t))$,
where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w \leq_{\mathcal{L}} s \geq_{\mathcal{R}} t$.

Then $(B(u, v, w), \overline{s}, t) \overset{1.5}{\longrightarrow} (B(B(u, v, w), s, t))$; rule 1.5 was applicable here by Lemma 3.9. Also, $(u, \overline{v}, B(w, s, t)) \overset{1.5}{\longrightarrow} (B(u, v, B(w, s, t))$; rule 1.5 was applicable here by Lemma 3.9. Confluence then follows from Lemma 3.9.

Overlap 1.5 – 1.6: $(B(u, v, w), \overline{s}) \overset{1.5}{\longleftarrow} (u, \overline{v}, w, \overline{s}) \overset{1.6}{\longrightarrow} (u, \overline{B(s, w, v)})$, where $u \leq_{\mathcal{L}} v \equiv_{\mathcal{R}} w \geq_{\mathcal{L}} s$.

Case 1: $B(u, v, w) \leq_{\mathcal{L}} s$.

In this case rule 2.4 applies and $(B(u, v, w), \overline{s}) \overset{2.4}{\longrightarrow} (B_{\mathcal{R}}(s, B(u, v, w)), \overline{r_s})$. By Lemma 3.11 (1), rule 2.4 then also applies to $(u, \overline{B(s, w, v)})$, thus producing $(B_{\mathcal{R}}(B(s, w, v), u), \overline{r_{B(s,w,v)}})$. Lemma 3.11 (2.$\leq$) then shows confluence.

Case 2: $B(u, v, w) >_{\mathcal{L}} s$.

In this case $(B(u, v, w), \overline{s}) \overset{2.1}{\longrightarrow} (r_{B(u,v,w)}, \overline{B_{\mathcal{R}}(B(u, v, w), s)})$. By Lemma 3.11 (1), rule 2.1 then also applies to $(u, \overline{B(s, w, v)})$, and this yields $(r_u, \overline{B_{\mathcal{R}}(u, B(s, w, v))})$. Lemma 3.11 (2.>) then shows confluence.

Case 3: $B(u, v, w) \nleqq_{\mathcal{L}} s$.

Then $(B(u, v, w), \overline{s}) \overset{1.3}{\longrightarrow} (0)$. Moreover, by Lemma 3.11 (1), in this case we also have $u \nleqq_{\mathcal{L}} B(s, w, v)$, hence rule 1.3 also applies to $(u, \overline{B(s, w, v)})$ and produces (0).

The overlap case $\overset{1.6}{\longleftarrow} (\overline{u}, v, \overline{w}, s) \overset{1.5}{\longrightarrow}$ is similar to the case above.

Overlap 1.5 – 2.1: $(B(u,v,w),\overline{s}) \overset{1.5}{\longleftarrow} (u,\overline{v},w,\overline{s}) \overset{2.1}{\longrightarrow} (u,\overline{v},r_w,\overline{B_{\mathcal{R}}(w,s)})$, where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w >_{\mathcal{L}} s$.

Case 1: $B(u,v,w) \leq_{\mathcal{L}} s$.

Then $(B(u,v,w),\overline{s}) \overset{2.4}{\longrightarrow} (B_{\mathcal{R}}(s,B(u,v,w)),\overline{r_s})$. Moreover, $(u,\overline{v},r_w,\overline{B_{\mathcal{R}}(w,s)}) \overset{1.5}{\longrightarrow} (B(u,v,r_w),\overline{B_{\mathcal{R}}(w,s}$ $(B_{\mathcal{R}}(B_{\mathcal{R}}(w,s),B(u,v,r_w)),\overline{r_{B_{\mathcal{R}}(w,s)}})$. The last application of rule 2.4 is justified by Lemma 3.13 (1). Confluence then follows immediately form Lemma 3.13 (2. $\leq$).

Case 2: $B(u,v,w) >_{\mathcal{L}} s$.

Then $(B(u,v,w),\overline{s}) \overset{2.1}{\longrightarrow} (r_{B(u,v,w)},\overline{B_{\mathcal{R}}(B(u,v,w),s)})$. Moreover, $(u,\overline{v},r_w,\overline{B_{\mathcal{R}}(w,s)}) \overset{1.5}{\longrightarrow}$ $(B(u,v,r_w),\overline{B_{\mathcal{R}}(w,s)}) \overset{2.1}{\longrightarrow} (r_{B(u,v,r_w)},\overline{B_{\mathcal{R}}(B(u,v,r_w),B_{\mathcal{R}}(w,s))})$. The last application of rule 2.1 is justified by Lemma 3.13 (1). Confluence then follows immediately form Lemma 3.13 (2. $<$).

Case 3: $B(u,v,w) \nleq\ngtr_{\mathcal{L}} s$.

Then $(B(u,v,w),\overline{s}) \overset{1.3}{\longrightarrow} (0)$. Moreover, $(u,\overline{v},r_w,\overline{B_{\mathcal{R}}(w,s)}) \overset{1.5}{\longrightarrow} (B(u,v,r_w),\overline{B_{\mathcal{R}}(w,s)}) \overset{1.3}{\longrightarrow}$ (0). The last application of rule 1.3 is justified by Lemma 3.13 (1).

Overlap 1.5 – 2.2:

Case 1. $u \leq_{\mathcal{L}} v >_{\mathcal{R}} w$ and
$(B(u,v,w)) \overset{1.5}{\longleftarrow} (u,\overline{v},w) \overset{2.2}{\longrightarrow} (u,\overline{\ell_v},B_{\mathcal{L}}(w,v)) \overset{1.5}{\longrightarrow} (B(u,\overline{\ell_v},B_{\mathcal{L}}(w,v)))$.

Confluence then follows from the $B_{\mathcal{L}}$-version of Lemma 3.14.

Case 2. $s >_{\mathcal{R}} u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ and
$(\overline{\ell_s},B_{\mathcal{L}}(u,s),\overline{v},w) \overset{2.2}{\longleftarrow} (\overline{s},u,\overline{v},w) \overset{1.5}{\longrightarrow} (\overline{s},B(u,v,w))$.

Then $(\overline{\ell_s},B_{\mathcal{L}}(u,s),\overline{v},w) \overset{1.5}{\longrightarrow} (\overline{\ell_s},B(B_{\mathcal{L}}(u,s),v,w))$. Rule 1.5 was applicable here since by Lemma 3.3, $B_{\mathcal{L}}(u,s) \equiv_{\mathcal{L}} u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

On the other hand, $(\overline{s},B(u,v,w)) \overset{2.2}{\longrightarrow} (\overline{\ell_s},B_{\mathcal{L}}(B(u,v,w),s))$. Rule 2.2 was applicable here since $s >_{\mathcal{R}} u \geq_{\mathcal{R}} ux = B(u,v,w)$ (where the last equality holds by Lemma 3.5).

Confluence then follows from the $B_{\mathcal{L}}$-version of Lemma 3.15.

Overlap 1.5 – 2.3:

Case A. $\overset{1.5}{\longleftarrow} (u,\overline{v},w) \overset{2.3}{\longrightarrow}$, where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

This is similar to Case A of the overlap 1.5 - 2.4, treated below.

Case B. $(\overline{B_{\mathcal{L}}(s,u)},\ell_u,\overline{v},w) \overset{2.3}{\longleftarrow} (\overline{s},u,\overline{v},w) \overset{1.5}{\longrightarrow} (\overline{s},B(u,v,w))$, where $s \leq_{\mathcal{R}} u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

Then $(\overline{B_{\mathcal{L}}(s,u)},\ell_u,\overline{v},w) \overset{1.5}{\longrightarrow} (\overline{B_{\mathcal{L}}(s,u)},B(\ell_u,v,w))$.

Case B.1 $s \leq_{\mathcal{R}} B(u,v,w)$.

Then $(\overline{s},B(u,v,w)) \overset{2.3}{\longrightarrow} (\overline{B_{\mathcal{L}}(s,B(u,v,w))},\ell_{B(u,v,w)})$.

On the other hand, $(\overline{B_{\mathcal{L}}(s,u)},B(\ell_u,v,w)) \overset{2.3}{\longrightarrow} (\overline{B_{\mathcal{L}}(B_{\mathcal{L}}(s,u),B(\ell_u,v,w))},\ell_{B(\ell_u,v,w)})$. Rule 2.3 was applicable here by the $\mathcal{R}$-version of Lemma 3.13 (1).

Confluence then follows from the $\mathcal{R}$-version of Lemma 3.13 (2, $\leq$).

Case B.2 $s >_{\mathcal{R}} B(u,v,w)$.

Then $(\overline{s},B(u,v,w)) \overset{2.2}{\longrightarrow} (\overline{\ell_s},B_{\mathcal{L}}(B(u,v,w),s))$, and
$(\overline{B_{\mathcal{L}}(s,u)},B(\ell_u,v,w)) \overset{2.2}{\longrightarrow} (\overline{\ell_{B_{\mathcal{L}}(s,u)}},B_{\mathcal{L}}(B(\ell_u,v,w),B_{\mathcal{L}}(s,u)))$. Rule 2.2 was applicable here by the $\mathcal{R}$-version of Lemma 3.13 (1).

Confluence then follows from the $\mathcal{R}$-version of Lemma 3.13 (2, $<$).

Case B.3 $s \nleq\ngtr_{\mathcal{R}} B(u,v,w)$.

Then $(\overline{s}, B(u,v,w)) \stackrel{1.4}{\longrightarrow} (0)$ and
$((\overline{B_{\mathcal{L}}(s,u)}, B(\ell_u,v,w)) \stackrel{1.4}{\longrightarrow} (0)$, where the application of rule 1.4 is justified by the $\mathcal{R}$-version of Lemma 3.13 (1).

Overlap 1.5 – 2.4:

Case A: $(B_{\mathcal{R}}(v,u), \overline{r_v}, w) \stackrel{2.4}{\longleftarrow} (u, \overline{v}, w) \stackrel{1.5}{\longrightarrow} (B(u,v,w))$,
where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

Then rule 1.5 is applicable to $(B_{\mathcal{R}}(v,u), \overline{r_v}, w)$ because $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$ implies by Lemma 3.4 $B_{\mathcal{R}}(v,u) \leq_{\mathcal{L}} r_v \equiv_{\mathcal{R}} v \geq_{\mathcal{R}} w$. Applying 1.5 then yields $(B(B_{\mathcal{R}}(v,u), r_v, w))$. Thus by Lemma 3.14 we have confluence.

Case B: $(B(u,v,w), \overline{s}) \stackrel{1.5}{\longleftarrow} (u, \overline{v}, w, \overline{s}) \stackrel{2.4}{\longrightarrow} (u, \overline{v}, B_{\mathcal{R}}(s,w), \overline{r_s})$, where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w \leq_{\mathcal{L}} s$.

Then rule 2.4 is applicable to $(B(u,v,w), \overline{s})$ because by Lemma 3.5 $B(u,v,w) = yw \leq_{\mathcal{L}} w \leq_{\mathcal{L}} v$. Then 2.4 yields $(B_{\mathcal{R}}(s, B(u,v,w)), \overline{r_s})$.

On the other hand, rule 1.5 is applicable to $(u, \overline{v}, B_{\mathcal{R}}(s,w), \overline{r_s})$ because $v \geq_{\mathcal{R}} w \equiv_{\mathcal{R}} B_{\mathcal{R}}(s,w)$ (the latter by Lemma 3.3). Then 1.5 yields $(B(u,v,B_{\mathcal{R}}(s,w)), \overline{r_s})$.

By Lemma 3.15 we have confluence.

We now come to the overlaps of the rules 2.i ($i = 1, ..., 4$).

Obviously, 2.1 cannot overlap with itself nor with 2.4.

Overlap 2.1 – 2.2: $(r_u, \overline{B_{\mathcal{R}}(u,v)}, w) \stackrel{2.1}{\longleftarrow} (u, \overline{v}, w) \stackrel{2.2}{\longrightarrow} (u, \overline{\ell_v}, B_{\mathcal{L}}(w,v))$, where $u >_{\mathcal{L}} v >_{\mathcal{R}} w$.

Then $(r_u, \overline{B_{\mathcal{R}}(u,v)}, w) \stackrel{2.2}{\longrightarrow} (r_u, \overline{\ell_{B_{\mathcal{R}}(u,v)}}, B_{\mathcal{L}}(w, B_{\mathcal{R}}(u,v)))$. Rule 2.2 was applicable here since by Lemma 3.3, $B_{\mathcal{R}}(u,v) \equiv_{\mathcal{R}} v >_{\mathcal{R}} w$.

On the other hand, $(u, \overline{\ell_v}, B_{\mathcal{L}}(w,v)) \stackrel{2.1}{\longrightarrow} (r_u, \overline{B_{\mathcal{R}}(u,\ell_v)}, B_{\mathcal{L}}(w,v))$. Rule 2.1 was applicable here since $u >_{\mathcal{L}} v \equiv_{\mathcal{L}} \ell_v$.

Next, applying rule 2.2 to this yields $(r_u, \overline{\ell_{B_{\mathcal{R}}(u,\ell_v)}}, B_{\mathcal{L}}(B_{\mathcal{L}}(w,v), B_{\mathcal{R}}(u,\ell_v)))$. Rule 2.2 was indeed applicable here since by Lemma 3.3, $B_{\mathcal{R}}(u,\ell_v) \equiv_{\mathcal{R}} \ell_v \geq_{\mathcal{R}} \ell_v y = B_{\mathcal{L}}(w,v)$ where $uy = v$; moreover, the $\geq_{\mathcal{R}}$ is actually $>_{\mathcal{R}}$ (if we had $\ell_v \equiv_{\mathcal{R}} \ell_v y$, then we would also have $v \equiv_{\mathcal{R}} vy = u$, which contradicts an assumption).

Lemma 3.16 immediately shows confluence now.

The other **overlap case** for rules 2.1 and 2.2 is of the form
$(\overline{\ell_v}, B_{\mathcal{L}}(v,w), \overline{w}) \stackrel{2.2}{\longleftarrow} (\overline{u}, v, \overline{w}) \stackrel{2.1}{\longrightarrow} (\overline{u}, r_v, \overline{B_{\mathcal{R}}(v,w)})$,
where $u >_{\mathcal{R}} v >_{\mathcal{L}} w$.

This case is similar to the case above.

Overlap 2.1 – 2.3: $(r_u, \overline{B_{\mathcal{R}}(u,v)}, w) \stackrel{2.1}{\longleftarrow} (u, \overline{v}, w) \stackrel{2.3}{\longrightarrow} (u, \overline{B_{\mathcal{L}}(v,w)}, \ell_w)$,
where $u >_{\mathcal{L}} v \leq_{\mathcal{R}} w$.

Then $(r_u, \overline{B_{\mathcal{R}}(u,v)}, w) \stackrel{2.3}{\longrightarrow} (r_u, \overline{B_{\mathcal{L}}(B_{\mathcal{R}}(u,v), w)}, \ell_w)$. Rule 2.3 was applicable here since $B_{\mathcal{R}}(u,v) \equiv_{\mathcal{R}} v$.

On the other hand, $(u, \overline{B_{\mathcal{L}}(v,w)}, \ell_w) \stackrel{2.1}{\longrightarrow} (r_u, \overline{B_{\mathcal{R}}(u, B_{\mathcal{L}}(v,w))}, \ell_w)$. Rule 2.1 was applicable here since $B_{\mathcal{L}}(v,w) \equiv_{\mathcal{L}} v$.

Confluence now follows from Lemma 3.17.

The other **overlap case** for the rules 2.1 and 2.3 is of the form
$(\overline{B_{\mathcal{L}}(u,v)}, \ell_v, \overline{w}) \stackrel{2.3}{\longleftarrow} (\overline{u}, v, \overline{w}) \stackrel{2.1}{\longrightarrow} (\overline{u}, r_v, \overline{B_{\mathcal{R}}(v,w)})$,
where $u \leq_{\mathcal{R}} v >_{\mathcal{L}} w$.

This is similar to the overlap case of 2.2 – 2.4 that we will study next.

Rule 2.2 has no overlap with itself nor with 2.3.

Overlap 2.2 – 2.4: $(B_{\mathcal{R}}(v,u), \overline{r_v}, w) \stackrel{2.4}{\longleftarrow} (u, \overline{v}, w) \stackrel{2.2}{\longrightarrow} (u, \overline{\ell_v}, B_{\mathcal{L}}(w,v))$,
where $u \leq_{\mathcal{L}} v >_{\mathcal{R}} w$.

Rule 1.5 is applicable to $(B_{\mathcal{R}}(v,u), \overline{r_v}, w)$ since $B_{\mathcal{R}}(v,u) = xr_v \leq_{\mathcal{L}} r_v \equiv_{\mathcal{R}} v >_{\mathcal{R}} w$. This yields $(B(B_{\mathcal{R}}(v,u), r_v, w))$.

Rule 1.5 is also applicable to $(u, \overline{\ell_v}, B_{\mathcal{L}}(w,v))$ since $u \leq_{\mathcal{L}} v \equiv_{\mathcal{L}} \ell_v \geq_{\mathcal{R}} \ell_v y = B_{\mathcal{L}}(w,v)$. This yields $(B(u, \ell_v, B_{\mathcal{L}}(w,v))$.

Lemma 3.18 immediately implies confluence.

The other **overlap case** for the rules 2.2 and 2.4 is of the form
$(\overline{\ell_u}, B_{\mathcal{L}}(v,u), \overline{w}) \stackrel{2.2}{\longleftarrow} (\overline{u}, v, \overline{w}) \stackrel{2.4}{\longrightarrow} (\overline{u}, B_{\mathcal{R}}(w,v), \overline{r_w})$, where $u >_{\mathcal{R}} v \leq_{\mathcal{L}} w$.
This is very similar to the overlap case of 2.1 – 2.3 that we studied explicitly.

Overlap 2.3 – 2.4: $(B_{\mathcal{R}}(v,u), \overline{r_v}, w) \stackrel{2.4}{\longleftarrow} (u, \overline{v}, w) \stackrel{2.3}{\longrightarrow} (u, \overline{B_{\mathcal{L}}(v,w)}, \ell_w)$,
where $u \leq_{\mathcal{L}} v \geq_{\mathcal{R}} w$.

Then $(B_{\mathcal{R}}(v,u), \overline{r_v}, w) \stackrel{2.3}{\longrightarrow} (B_{\mathcal{R}}(v,u), \overline{B_{\mathcal{L}}(r_v,w)}, \ell_w) \stackrel{2.4}{\longrightarrow} (B_{\mathcal{R}}(B_{\mathcal{L}}(r_v,w), B_{\mathcal{R}}(v,u)), \overline{r_{B_{\mathcal{L}}(r_v,w)}}, \ell_w)$; the last application of rule 2.4 was justified since $B_{\mathcal{R}}(v,u) = xr_v \leq_{\mathcal{L}} r_v \equiv_{\mathcal{L}} B_{\mathcal{L}}(r_v,w)$ (the last $\mathcal{L}$-equivalence follows from Lemma 3.3).

On the other hand, $(u, \overline{B_{\mathcal{L}}(v,w)}, \ell_w) \stackrel{2.4}{\longrightarrow} (B_{\mathcal{R}}(B_{\mathcal{L}}(v,w), u), \overline{r_{B_{\mathcal{L}}(v,w)}}, \ell_w)$; the application of rule 2.4 was justified since $u \leq_{\mathcal{L}} v \equiv_{\mathcal{L}} B_{\mathcal{L}}(v,w)$ (where the last $\mathcal{L}$-equivalence follows from Lemma 3.3).

Confluence now follos immediately form the $\mathcal{L}--\mathcal{R}$ dual of Lemma 3.16.

The other **overlap case** for the rules 2.3 and 2.4 is of the form
$(\overline{B_{\mathcal{L}}(u,v)}, \ell_v, \overline{w}) \stackrel{2.3}{\longleftarrow} (\overline{u}, v, \overline{w}) \stackrel{2.4}{\longrightarrow} (\overline{u}, B_{\mathcal{R}}(w,v), \overline{r_v})$,
where $u \leq_{\mathcal{R}} v \geq_{\mathcal{L}} w$.
This is similar to the above case.

This completes the exhaustive analysis of overlap cases, and shows that the rewrite system for $(S)_{reg}$ is *locally confluent*.

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