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RESEARCH ARTICLE

ON M-VARIETIES GENERATED BY POWER MONOIDS

by

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I. INTRODUCTION

In this paper all semigroups considered will be finite. Let M be a monoid. Then P(M), the power set of M, forms a monoid under the usual multiplication of subsets. Power monoids have recently been studied from the algebraic point of view [7],[8] and for their connection with the theory of languages [4],[6],[10],[12].

Here we study M-varieties which are generated by power monoids. Recall that an M-variety is a collection of monoids closed under division and finite direct product. If $\underline{V}$ is an M-variety let $\underline{PV}$ be the M-variety generated by $\{P(M) | M \in \underline{V}\}$. The operation $\underline{V} \to \underline{PV}$ has been studied in [6],[10],[12].

An M-variety $\underline{V}$ is proper if $\underline{V}$ is not equal to $\underline{M}$ the M-variety of all finite monoids. The main theorem of this paper shows that $\underline{PV}$ is proper if and only if $\underline{V}$ is contained in $\underline{DS}$ the M-variety of monoids whose regular $\mathcal{D}$-classes are subsemigroups. Equivalently we will see that $\underline{PV} = \underline{M}$ if and only if BA_2, the 2x2 aperiodic Brandt monoid, is in $\underline{V}$. This answers a question raised by Pin in [6].

Let M and N be monoids. Our main technique is to study various properties of morphisms $\theta: M \to N$ which are inherited by the natural extension $\bar{\theta}: P(M) \to P(N)$. We will especially be interested in the case when N is a semilattice.

As an application of these methods we will show that if M is a union of groups, then the complexity of M is equal to the complexity of P(M). On the other hand we will give an example of an aperiodic monoid M_n such that $P(M_n)$ has complexity n, for each $n \geq 0$. We will

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also show that if M is in $\underline{DS}$, then the maximal subgroups of P(M) are in the M-variety generated by the maximal subgroups of M. See [15] for an expostion of complexity theory.

II. PRELIMINARIES

Our terminology and notation will follow [1],[3], and [15]. We refer the reader to these texts for any details not included in this paper.

If M and N are monoids and $\phi:M \to N$ is a (functional) morphism then $\bar{\phi}:P(M) \to P(N)$ will denote the natural extension. The proof of the following useful lemma is elementary and is left to the reader.

LEMMA 1. Let M and N be monoids and let $\phi:M \to N$ be a functional morphism. If X and Y are contained in M, then $X\bar{\phi} = Y\bar{\phi}$ if and only if X and Y intersect the same classes of (mod ϕ) nontrivially.

For $n \geq 1$ let $\underset{\sim}{n} = \{0, \ldots, n-1\}$ and let BA_n be the monoid consisting of the identity transformation together will all partial functions $f:\underset{\sim}{n} \to \underset{\sim}{n}$ with the property that $\text{card}(\underset{\sim}{n}f^{-1}) \leq 1$. BA_n is called the aperiodic Brandt monoid of size n. The following was proved in [6] using language theoretic methods. We present a direct algebraic proof. See [2, Ch. 7].

LEMMA 2. Let V be an M-variety. If $BA_2 \in V$ then $PV = M$, the M-variety of all finite monoids.

Proof. The following two facts are easy to establish:

1) If $m,n \geq 1$ then $BA_{mn} \prec BA_m \times BA_n$.
2) If $m \leq n$ then $BA_m \prec BA_n$.

In particular, if follows by 1) that $BA_2 \in \underline{V}$ implies $BA_{2^k} \in \underline{V}$ for all $k \geq 1$. Therefore, by 2) $BA_n \in \underline{V}$ for all $n \geq 1$.

Let R_n denote the monoid of relations on n. The function $\phi:P(BA_n) \to R_n$ given by

$$X\phi = \bigcup_{f \in X} f$$

for $X \in P(BA_n)$ is a surjective functional morphism. Therefore $R_n \in \underline{\underline{PV}}$ for all $n \geq 1$, and thus $\underline{\underline{PV}} = \underline{M}$. ∎

This proves one part of the main theorem. In order to prove the converse we will need to study M-varieties defined by certain classes of relational morphisms. We introduce the necessary terminology.

Let S and T be semigroups. Recall that a relation $\phi: S \to T$ is a relational morphism if

1) $s\phi \neq \emptyset$ for all $s \in S$.
2) $(s_1\phi)(s_2\phi) \subseteq (s_1s_2)\phi$ for all $s_1, s_2 \in S$.

If S and T are monoids we also require

3) $1 \in 1\phi$.

Let $\underline{V}$ and $\underline{W}$ be S-varieties. That is $\underline{V}$ and $\underline{W}$ are collections of semigroups closed under division and finite direct product. A relational morphism $\phi: S \to T$ is a $\underline{V}$-$\underline{W}$ morphism if for every subsemigroup T' of T

$$T' \in \underline{W} \text{ implies } T'\phi^{-1} \in \underline{V}.$$

We shall be particularly interested in the cases $\underline{W} = \underline{V}$ and $\underline{W} = \{1\}$, the variety consisting of the trivial semigroup 1. In the first case we call $\phi: S \to T$ a $\underline{V}$-morphism [15]. Notice that $\phi: S \to T$ is a $\underline{V}$-$\{\underline{1}\}$ morphism if and only if $\{e\phi^{-1} \mid e = e^2 \in T\} \subseteq \underline{V}$.

Clearly every $\underline{V}$-morphism is a $\underline{V}$-$\{\underline{1}\}$ morphism but the converse is not true. Furthermore the collection of $\underline{V}$-morphisms is easily seen to be closed under composition whereas this need not be true of the collection of $\underline{V}$-$\{\underline{1}\}$ morphisms.

EXAMPLE 1. Let U_n denote the monoid consisting of n right zeroes and an identity. It is well known that the exclusion $\langle U_2 \rangle$ of U_2 defined by $\langle U_2 \rangle = \{S \mid U_2 \not\prec S\}$ is an S-variety.

The unique surjective functional morphism $\phi: U_2 \to U_1$ is a $\langle U_2 \rangle$ - $\{\underline{1}\}$ morphism which is not a $\langle U_2 \rangle$-morphism. Furthermore the morphism $\gamma: U_1 \to \{\underline{1}\}$ is a U_2 -$\{\underline{1}\}$ morphism but $\phi\gamma: U_2 \to \{\underline{1}\}$ is not.

If Φ is the collection of all $\underline{V}$-$\underline{W}$ morphisms and $\underline{V}'$ is an M-variety let

$$\Phi^{-1}\underline{V}' = \{M \mid \text{there exists } N \in \underline{V}' \text{ and } \phi: M \to N \in \Phi\}.$$

It is easy to check that $\Phi^{-1}\underline{V}'$ is an M-variety. Varieties of the form $\Phi^{-1}\underline{V}'$ arise naturally in language theory. For example, let Φ be the collection of all aperiodic morphisms. In [13] Straubing shows that a *-variety of languages (see [13]) is closed under concatenation for each alphabet A if and only if the corresponding M-variety $\underline{V}$ is closed under the operation $\underline{V} \to \Phi^{-1}\underline{V}$.

III. THE MAIN RESULT

In this section we state the main theorem and prove it modulo a technical lemma. Recall that $\underline{DS}$ is the M-variety of monoids whose regular $\mathcal{D}$ classes are subsemigroups.

THEOREM 3. Let V be an M-variety. The following are equivalent:

1) $\underline{V}$ is not contained in $\underline{DS}$,
2) $BA_2 \in \underline{V}$,
3) $\underline{\underline{P}}\underline{V} = \underline{M}$ the variety of all finite monoids.

The hardest part of theorem 3 is 3) implies 1). That is we must show that if $\underline{V}$ is contained in $\underline{DS}$ then there exists a monoid $M \notin \underline{\underline{P}}\underline{V}$. The following lemma, whose proof is postponed until the next section, will allow us to construct such an M. Recall that $\langle U_2 \rangle$ is the S-variety of U_2-free semigroups. See example 1 above.

LEMMA 4. Let Φ be the collection of $\langle U_2 \rangle$-{1} relational morphisms and let W be the M-variety of commutative aperiodic monoids. Then

$$\mathcal{P}(\mathrm{DS}) \subseteq \Phi^{-1}\underline{W}.$$

In other words if $M \in \underline{\mathcal{P}}(\underline{DS})$, then there exists a commutative aperiodic monoid N and a relational morphism $\phi: M \rightarrow N$ such that $\{e\phi^{-1} \mid e = e^2 \in N\} \subseteq \langle U_2 \rangle$.

We now construct a monoid which is not in $\underline{\mathcal{P}}(\underline{DS})$.

EXAMPLE 2. Let U_2 be the monoid consisting of an identity and two right zeroes a and b. Form the Rees matrix semigroup

$$S = \mathcal{M}(U_2, \{a_1, a_2\}, \{b_1, b_2\}, [\begin{smallmatrix} 1 & a \\ b & 1 \end{smallmatrix}])$$

over U_2 and let $M = S^1$.

M is regular and has 3 $\mathcal{D}$ classes:

$D_1 = \{1\}$

$D_2 = \{(a_i, 1, b_j) \mid i, j \in \{1,2\}\}$

$D_3 = \{(a_i, x, b_j) \mid x \in \{a,b\},\ i,\ j \in \{1,2\}\}$.

$D_3 < D_2 < D_1$ in the usual $\mathcal{D}$ class ordering.

LEMMA 5. M is not in $\mathcal{P}(\mathrm{DS})$.

Proof. Let N be a commutative aperiodic monoid and let $\phi: M \rightarrow N$ be a relational morphism. It suffices by lemma 4 to show that ϕ is not a $\langle U_2 \rangle$-{1} morphism.

The set $R = \{(m,n) \mid n \in m\phi\}$ is a submonoid of MxN. Let

$$\pi_1 : R \to M$$
$$\pi_2 : R \to N$$

be the restriction of the projections $M \times N \to M$ and $M \times N \to N$, respectively. Note that $\phi = \pi_1^{-1}\pi_2$.

Let $D \subseteq R$ be a regular $\mathcal{D}$ class such that $D\pi_1 = D_2 \subsetneq M$. Then $D\pi_2$ is contained in a regular $\mathcal{D}$ class of N and thus $D\pi_2 = e$ for some idempotent $e \in N$ (since N is commutative and aperiodic).

Therefore $e\phi^{-1} = e\pi_2^{-1}\pi_1$ contains the subsemigroup S of M generated by D_2.

But

$$(a_2,1,b_1)^2 = (a_2,a,b_1) \in S$$
$$\text{and } (a_1,1,b_2)^2 = (a_1,b,b_2) \in S$$

Therefore

$$(a_2,a,b_2) = (a_2,a,b_1)(a_1,1,b_2) \in S$$
$$\text{and } (a_2,b,b_2) = (a_2,a,b_1)(a_1,b,b_2) \in S$$

Thus $U_2 \simeq \{(a_2,1,b_2),\ (a_2,a,b_2),\ (a_2,b,b_2)\} \subseteq S \subsetneq e\phi^{-1}$ and ϕ is not a $\langle U_2\rangle - \{\underline{1}\}$ morphism.∎

We can now prove theorem 3. By lemma 2 and lemma 5 it suffices to prove 1) implies 2).

Let $\underline{V}$ be a variety which is not contained in $\underline{DS}$. Then there is a monoid $M \in \underline{V}$ and a regular $\mathcal{D}$ class D of M which is not a subsemigroup. It is easy to see that a monoid of the form

$$N = M^0(\{1\}, 2\ , 2\ , [\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}])^1 \quad x\in\{0,1\}$$

divides M. If $x = 0$, then $N \simeq BA_2$ and we are done since $N \in \underline{V}$. If $x = 1$, then a simple calculation shows that

$$BA_2 < N \times N$$

and therefore $BA_2 \in \underline{V}$ as desired.∎

We remark that theorem 3 remains true for semigroups and S-varieties.

The following result of Putcha [7] will allow us to state a theorem for M-varieties of aperiodic monoids analogous to theorem 3.

<u>THEOREM 6</u>. <u>Let M be a aperiodic monoid. Then P(M) is aperiodic if and only if BA_2 does not divide M</u>.

Let $\underline{DA}$ be the M-variety of monoids whose regular $\mathcal{D}$-classes are aperiodic semigroups.

COROLLARY. Let V be an M-variety of aperiodic monoids. The following are equivalent:

1) $\underline{V}$ is contained in $\underline{DA}$.
2) $BA_2 \notin \underline{V}$.
3) $\underline{\underline{P}}\underline{V}$ is an aperiodic M-variety.

Proof. The equivalence of 1) and 2) is proved as in theorem 3. The equivalence of 2) and 3) follows from theorem 6.∎

More generally, we have:

THEOREM 7. Let S be a semigroup in DS and let G be a subgroup in P(S). Then G is in the M-variety generated by the maximal subgroups of S.

Proof. Let D be a regular $\mathcal{D}$-class of S. Define a map $f_D:G \to P(D)$ by $Xf_D = X \cap D$. Then f_D is a morphism. For clearly,

$$(X \cap D)(Y \cap D) \subseteq (XY \cap D)$$

for all $X,Y \in G$. On the other hand, let $z = xy \in (XY \cap D)$. Let $e = e^2 \mathcal{H} z$. Then $e \in T$ where $T = T^2 \in G$. But $ex \in TX = X$ and $ye \in YT = Y$. Therefore, $z = e(xy)e = (ex)(ye) \in (X \cap D)(Y \cap D)$.

Let $D_1, \ldots, D_n$ be the regular $\mathcal{D}$ classes of S which intersect the maximal $\mathcal{D}$ classes of T nontrivially. Then the morphism $f:G \to P(D_1) x \ldots x P(D_n)$ is injective where

$$gf = (gf_{D_1}, \ldots, gf_{D_n})$$

Indeed, suppose $Xf = Tf$ for some $X \in G$. Let $t \in T$. Then $t = us_i v$ for some $u,v \in T$ and $s_i \in D_i \cap T$ and some $1 \leq i \leq n$. But $D_i \cap T = D_i \cap X$ and thus $t \in TXT = X$. Therefore $T \subseteq X$. It follows that $X = TX \subseteq X^2$ and by induction $X \subseteq X^k$ for all $k \geq 1$. But $X^n = T$ for some $n \geq 1$ and thus $X \subseteq T$ also.

To prove theorem 7, it suffices then to prove the following lemma.

LEMMA 8. Let S be a completely simple semigroup and let G be a subgroup of P(S). Then G divides a maximal subgroup of S.

Proof. Let $G \subseteq P(S)$ and let $T = T^2 \in G$. Let H be a maximal subgroup of S such that $T \cap H \neq \emptyset$. A proof that the map $f:G \to P(H)$ sending $X \to X \cap H$ is an injective morphism is similar to the proof above and is omitted. Therefore G is isomorphic to a subgroup of P(H). It is well known that every subgroup of P(H) divides H. See [6] for example. ∎

We close by using theorem 7 to prove a theorem which generalizes theorem 6. Let G be any M-variety of groups not containing all finite groups. Define $\overline{G}$ to be the M-variety consisting of monoids all of whose subgroups are in G. Let $\underline{DG} = DS \cap \overline{G}$.

THEOREM 9. Let $\underline{V}$ be an M-variety contained in $\overline{G}$. The following conditions are equivalent.

1) $\underline{V}$ is contained in $\underline{DG}$.
2) $\underline{PV}$ is contained in $\overline{G}$.
3) $\underline{PV}$ is proper.
4) $BA_2 \notin \underline{V}$.

Proof. 1) =>2) Follows from theorem 7.

2) =>3) Trivial since G is not the M-variety of all finite groups.

3) =>4) Follows from theorem 3.

4) =>1) Since $\underline{V}$ is contained in $\overline{G}$, this follows as in theorem 3. ∎

Compare theorem 7 with the following result of Putcha [7]. Recall that an M-variety $\underline{V}$ is closed if the wreath product of two members of $\underline{V}$ is also in $\underline{V}$.

THEOREM. Let S be a finite semigroup and let G be a subgroup in P(S). Then G is in the closed M-variety generated by the maximal subgroups of $P(S_i)$ where S_i $i = 1, \ldots, n$ are the principal factors of S.

If BA_n is the aperiodic Brandt monoid of size n, then we have seen in lemma 2 that the monoid of relations on n divides $P(BA_n)$. Thus the subgroups in P(S) are in general much more complicated than the subgroups in S.

We close this section with an application to language theory. It is well known that every theorem on M-varieties leads, via the Eilenberg variety theorem ([1],[3]), to a theorem on *-varieties of recognizable languages. We assume the reader is familiar with the basic definitions and ideas in the theory of varieties of languages.

The operation $V \to \underline{PV}$ on M-varieties corresponds to the following operation on *-varieties.

Let $\mathcal{V}$ be a *-variety and let A be a finite alphabet. Define $A^*(\pi\mathcal{V})$ to be the Boolean algebra generated by sets of the form $L\phi$, where $L \in B^*\mathcal{V}$ for some finite alphabet B and $\phi: B^* \to A^*$ is a morphism such that $B\phi \subseteq A$.

THEOREM 10. *If $\mathcal{V}$ corresponds to the M-variety $\underline{V}$, then $\pi\mathcal{V}$ corresponds to $\underline{P}\underline{V}$.*

Proof. See [6], [10], or [12]. ∎

THEOREM 11. *Let $\mathcal{V}$ be a *-variety and let $A = \{a,b\}$. The following are equivalent:*

1) $(ab)^* \in A^*\mathcal{V}$.
2) $\pi\mathcal{V} = \underline{RAT}$ the variety of all rational languages.

Proof. Follows from theorem 3, theorem 10, the Eilenberg variety theorem and the fact that the syntactic monoid of $(ab)^*$ is BA_2. ∎

IV. THE M-VARIETIES $\underline{DS}$ AND $\underline{P}(\underline{DS})$

In this section we complete the proof of theorem 3 by proving lemma 4. Recall that U_1 is the 2 element semilattice and that $\langle U_1 \rangle$ is the S-variety of U_1-free semigroups. Thus $S \in \langle U_1 \rangle$ if and only if S is a nilpotent ideal extension of its minimal ideal.

The proof of lemma 4 will proceed in 2 steps:

1) If $M \in \underline{DS}$, then there exists a functional $\langle U_1 \rangle$-morphism $\phi: M \to N$ onto a semilattice N.

2) The extension $\overline{\phi}: P(M) \to P(N)$ is a $\langle U_2 \rangle$-$\{\underline{1}\}$ morphism. Since P(N) is commutative and aperiodic the result follows.

The morphism $\phi: M \to N$ in 1) will be nothing more than the Clifford map in case M is union of groups. The existence of N and the morphism $\phi: M \to N$ follows from the theory of semilattice decompositions developed by Tamura, Putcha, Petrich, etc. ([5], [9], [14]). However, we prefer, for the sake of completeness, to give a direct proof suited to our present purposes.

LEMMA 12. *Let M be any monoid and let D be a regular $\mathcal{D}$ class of M which is a subsemigroup of M. Then $T_D = \{x \in M \mid MxM \cap D \neq \emptyset\}$ is a subsemigroup of M and D is the minimal ideal of T_D.*

Proof. Let $x,y \in T_D$. Then

$uxv \in D$ and

$syt \in D$

for some $u,v,s,t \in M$.

Since D is a subsemigroup of M, D is regular. Choose idempotents $e,f \in D$ such that

$e\mathcal{R}uxv$ and

$f\mathcal{L}syt$.

Then $uxv = euxv$ and $syt = sytf$ and it follows that $eux \in D$ and $ytf \in D$. Therefore,

$$(eux)(ytf) = eu(xy)tf \in D$$

since D is a subsemigroup of M. Thus $xy \in T_D$ and T_D is a subsemigroup. Clearly D is the minimal ideal of T_D. ∎

COROLLARY. $M-T_D$ is an ideal of M.

Therefore the characteristic function $\chi_D : M \to U_1$ of T_D is a functional morphism. Here

$$m\chi_D = \begin{cases} 1 & \text{if } m \in T_D \\ 0 & \text{if } m \in M-T_D \end{cases}$$

Let $D_1, \ldots, D_n$ be the regular $\mathcal{D}$-classes of M which are subsemigroups. Then the morphism

$$(*) \quad \chi : M \to \prod_{i=1}^{n} U_1$$

where

$$m\chi = (m\chi_{D_1}, m\chi_{D_2}, \ldots, m\chi_{D_n})$$

separates $D_1, \ldots, D_n$. That is, if $s \in D_i$ and $t \in D_j$, then $s\chi = t\chi$ implies that $i = j$. In particular if every regular $\mathcal{D}$-class of M is a subsemigroup, then $e\chi^{-1}$ contains exactly one regular $\mathcal{D}$ class for each $e \in M\chi$. Thus $e\chi^{-1}$ is U_1 free and χ is a $\langle U_1 \rangle$ morphism.

LEMMA 13. Let $M \in \underline{DS}$. Then there exists a semilattice N and a $\langle U_1 \rangle$-free morphism $\chi : M \to N$. Furthermore, if D is a regular $\mathcal{D}$-class of M, then $D\chi = e$ for some $e \in N$ and $e\chi^{-1} = \{m \mid m^n \in D \text{ for some } n \in \mathbb{N}\}$.

Proof. Let $N = M\chi$ where χ is as in (*). The discussion preceding the lemma shows that $\chi : M \to N$ satisfies the requirements.

Let D be a regular $\mathcal{D}$-class of M. Then DX is contained in a regular $\mathcal{D}$-class of N and thus $DX = e$ for some $e \in N$. Since $mX = m^nX$ for all $n \geq 1$ it follows that $eX^{-1} = \{m | m^n \in D\}$. Conversely, suppose $mX = e$. Choose $n \geq 1$ such that m^n is regular. Since X separates regular $\mathcal{D}$-classes and $(m^n)X = e$ it follows that $m^n \in D$. ∎

See also [9] theorem 2.13.

We now study the induced morphism $\bar{X}:P(M) \to P(N)$. Recall that $\langle U_2 \rangle$ is the S-variety of U_2-free semigroups.

LEMMA 14. Let $M \in \underline{DS}$. Let N and $X:M \to N$ be as in lemma 13. Then $\bar{X}:P(M) \to P(N)$ is a $\langle U_2 \rangle$-$\{1\}$ morphism.

Proof. Let $E = E^2 \in P(N)$. We must show that $E\bar{X}^{-1}$ is in $\langle U_2 \rangle$. Assume that $U_2 \prec E\bar{X}^{-1}$. By a well known result $U_2 \subseteq E\bar{X}^{-1}$.

Let $\{S_1, S_2, T\} \simeq U_2 \subseteq E\bar{X}^{-1}$ with

$$S_iS_j = S_j \quad i,j = 1,2$$

and

$$S_iT = TS_i = S_i, \; T^2 = T \quad i = 1,2.$$

It suffices to prove that $T \subseteq S_1 \cap S_2$ for then

$$S_1 = S_1T \subseteq S_1S_2 = S_2$$

and dually $S_2 \subseteq S_1$.

Since T, S_1, S_2 are idempotents of P(M) they are subsemigroups of M. Furthermore the maximal $\mathcal{D}$ classes of T, S_1, S_2 are all regular. Let $D_1, \ldots, D_k$ be the $\mathcal{D}$ classes of M containing the maximal $\mathcal{D}$ classes of T.

If $t \in T$ there exists $u, v \in T$ and an idempotent $e_i \in D_i$ for some i, $1 \leq i \leq k$ such that

(1) $t = ue_iv$.

Since $T\bar{X} = S_j\bar{X}$, j = 1,2, it follows by lemma 1 and lemma 13 that there exists $y_{ij} \in D_i \cap S_j$. But D_i is a completely simple semigroup so there exists an $n \geq 1$ such that

(2) $e_i = (e_iy_{ij}e_i)^n \in (TS_jT)^n = S_j$.

By (1) we then have

$$t \in TS_jT = S_j$$

and thus $T \subseteq S_1 \cap S_2$. ∎

We can now prove lemma 4. We wish to prove that $\underline{P}(\underline{DS}) \subseteq \Phi^{-1}\underline{W}$ where $\underline{W}$ is the M-variety of commutative aperiodic semigroups and Φ is the collection of $\langle U_2 \rangle$-$\underline{1}$ morphisms.

Recall that a relational morphism $\phi:S \to T$ is injective (or elementary [15]) if

$$s_1\phi \cap s_2\phi \neq \emptyset \Rightarrow s_1 = s_2$$

for all $s_1, s_2 \in S$. It is easy to see that $S \prec T$ iff there is an injective relational morphism $\phi:S \to T$. Furthermore $\phi:S \to T$ is injective iff $\phi^{-1}:T \to S$ is a surjective partial function.

Now let $M \in P(\underline{DS})$. Then

(3) $M \prec P(M_1)x \ldots P(M_k)$

for some $M_i \in \underline{DS}$, $1 \leq i \leq k$. Let $X_i:M_i \to N_i$ be as in lemma 9 and consider $\bar{X}_i:P(M_i) \to P(N_i)$. Let

$$\theta = \phi(\bar{X}_1 x\bar{X}_2 x \ldots x\bar{X}_n):M \to P(N_1)x \ldots xP(N_k)$$

where $\phi:M \to P(M_i)x \ldots xP(M_k)$ is an injective relational morphism. It follows from lemma 14 that θ is a $<U_2>-\{\underline{1}\}$ morphism. Furthermore $P(N_i)$ is certainly commutative and is also aperiodic by theorem 6. Therefore $M \in \Phi^{-1}\underline{W}$. ▮

<u>COROLLARY 1. Let $<U_1>$ be the S-variety of U_1 free semigroups. Then $\underline{P}(<U_1>) \subseteq <U_2>$.</u>

<u>Proof</u>. If $S \in <U_1>$ then the morphism

$$\gamma_S:S \to \{1\}$$

is a $<U_1>$ morphism. Therefore by lemma 14 applied to semigroups

$$\bar{\gamma}_S:P(S) \to P(\{1\})$$

is a $<U_2> - \{\underline{1}\}$ morphism. Since $(\emptyset)\bar{\gamma}_S^{-1} = \emptyset$ it follows that in fact $\bar{\gamma}_S$ is a $<U_2>$ morphism. Therefore $P(S)$ is U_2-free and $\underline{P}(<U_1>) \subseteq <U_2>$. ▮

We recall that a basic fact about $<U_2>$ is that every member has complexity ≤ 1. (See [15]). We therefore have:

<u>COROLLARY 2. If S is U_1 free then $P(S)c \leq 1$. Moreover,</u>

$$P(S)c = Sc = \begin{cases} 0 & \text{if S is aperiodic} \\ 1 & \text{if S is not aperiodic} \end{cases}$$

<u>Proof</u>. If S is aperiodic, then so is P(S) by theorem 4. If S is not aperiodic, then Sc = 1 since $S \in <U_1>$. But $Sc \leq P(S)c \leq 1$. ▮

<u>COROLLARY 3. If S is a simple semigroup, then $Sc = P(S)c \leq 1$.</u>

<u>Proof</u>. S is U_1 free. ▮

On the other hand if $S = M^0(\{1\}, n, n, I_n)$, a completely 0-simple semigroup, then we have seen that R_n, the monoid of relations on n, divides P(S). This can be used to show that $P(S)c = n - 1$. Thus if S is completely 0-simple, the complexity of P(S) depends on the scarcity of idempotents in the egg box picture of S-{0}.

V. UNION OF GROUPS, POWER MONOIDS, AND COMPLEXITY

In this section we generalize corollary 3 above, by showing that if M is a union of groups, then the complexity of M is equal to the complexity of P(M).

We assume the reader has some familiarity with the basic definitions and theorems of complexity theory. See [15]. In particular let S be a semigroup and let $\gamma_S : S \to \{1\}$ be the collapsing morphism. Then the complexity of S is equal to the least number n such that:

$$(*) \quad \gamma_S = \alpha_0\beta_1\alpha_1 \cdots \beta_n\alpha_n$$

where each α_i is an aperiodic relational morphism and each β_j is a U_2-free relational morphism.

An important fact about unions of groups is that the α_i and β_j in (*) above can all be chosen to be functional morphisms. In fact even more is true.

Let K be any of Green's relation. A functional morphism $f:S \to T$ is a K-morphism if $s_1f = s_2f$ implies $s_1 K s_2$. Notice that an L morphism is a U_2-free morphism (but not conversely).

The following theorem appears in [2] chapter 9:

THEOREM 15. Let S be a union of groups. Then the complexity of S is equal to the least n such that

$$\gamma_S = f_0g_1f_1 \cdots g_nf_n$$

where each f_i is an aperiodic and $\mathcal{D}$ functional morphism and each g_j is a functional L morphism.

COROLLARY. Let S be a union of groups with $Sc = n > 0$. Then there exist unions of group T_1, T such that:

1) there is an aperiodic and $\mathcal{D}$ functional morphism $f:S \twoheadrightarrow T_1$,
2) there is an L-morphism $g:T_1 \to T$,
3) $Tc = n - 1$.

LEMMA 16. Let S be a union of groups. If $f:S \to T$ is a functional aperiodic and $\mathcal{D}$ morphism, then $\overline{f}:P(S) \to P(T)$ is an aperiodic functional morphism.

Proof. Let S be a union of groups and let $f:S \to T$ be an aperiodic and $\mathcal{D}$ functional morphism. We show that $\overline{f}$ is one to one on subgroups of P(S).

Let G be a subgroup of P(S) and let $T = T^2 \in G$. Let $X \in G$. Then $TXT = X$ and there is an $n > 0$ such that $X^n = T$. Assume $X\overline{f} = T\overline{f}$. Since S is a union of groups, it easily follows that $\text{card}(X) \leq \text{card}(X^k)$ for all $k > 0$. In particular $\text{card}(X) \leq \text{card}(X^n) = \text{card}(T)$. Therefore it suffices to show that $T \subseteq X$.

Let $t \in T$. Then there exists $x \in X$ such that $xf = tf$. Since f is a $\mathcal{D}$ morphism it follows that $x\mathcal{D}t_2$. Let e be an idempotent $\mathcal{H}$ related to t. Since $T = T^2$, T is a subsemigroup of S and thus $e \in T$. Furthermore $(exe)\mathcal{H}t$ and

$$(exe)f = (ete)f = tf.$$

Since f is aperiodic it follows that $t = exe \in TXT = X$.∎

LEMMA 17. Let S be a union of groups. If $f:S \to T$ is a functional $\mathcal{L}$ morphism then $\overline{f}:P(S) \to P(T)$ is a U_2-free morphism.

Proof. We must show that $\overline{f}$ is 1-1 on every copy of $U_2 \subseteq P(S)$. Let $U_2 \simeq \{T,S_1,S_2\} \subseteq P(S)$. Then

$$(**) \quad \begin{array}{l} TS_i = S_iT \\ S_iS_j = S_j \quad i,j = 1,2 \\ T = T^2 \end{array}$$

If $T\overline{f} = S_i\overline{f}$ $i = 1$ or 2, then (**) clearly implies $S_1\overline{f} = S_2\overline{f}$. Therefore it suffices to show that $S_1\overline{f} = S_2\overline{f}$ implies $S_1 = S_2$.

Suppose $S_1\overline{f} = S_2\overline{f}$. If $s_1 \in S_1$ there is $s_2 \in S_2$ such that $s_1f = s_2f$. Let e be an idempotent $\mathcal{H}$ related to s_2. Since S_2 is a subsemigroup of S, $e \in S_2$. Furthermore, $s_1\mathcal{L}s_2$ and thus:

$$s_1 = s_1e \in S_1S_2 = S_2$$

Therefore $S_1 \subseteq S_2$ and by symmetry $S_2 \subseteq S_1$.∎

THEOREM 18. Let S be a union of groups. Then $Sc = P(S)c$.

Proof. Since S divides P(S) if suffices to show that $P(S)c \leq Sc$. We prove this by induction on Sc.

If Sc = 0, then S is a band. Therefore P(S) is aperiodic by theorem 6.

Assume $Sc = n > 0$. Let T_1,T and $f:S \twoheadrightarrow T_1$, $g:T_1 \twoheadrightarrow T$ be as in the corollary to theorem 15. By lemma 16 and lemma 17

$$\overline{f}:P(S) \twoheadrightarrow P(T_1) \text{ is aperiodic}$$
$$\text{and } \overline{g}:P(T_1) \twoheadrightarrow P(T) \text{ is } U_2\text{-free}.$$

Therefore,

$$P(S)c \leq P(T_1)c \leq 1 + P(T)c \leq 1 + (n-1) = n$$

by induction and the fact that if $\phi:S \to T$ is an aperiodic (U_2-free) morphism, then $Sc \leq Tc$ ($Sc \leq 1 + Tc$). See [15].∎

VI. SOME OPEN PROBLEMS

1) Let DG be as in theorem 9. Give necessary and sufficient conditions for a monoid to be a member of P(DG).
2) If $M \in$ DS, does Mc = (PM)c?

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