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Inverse Semigroups and Extensions of Groups by Semilattices

S. W. MARGOLIS

Computer Science, Ferguson Building,
University of Nebraska, Lincoln, Nebraska 68588–0115

AND

J. E. PIN

LITP, 4, Place Jussieu, Tour 55–65,
75252 Paris Cedex 05, France

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This paper is the first part of a series of three papers devoted to the study of inverse semigroups. The subject of our second paper [7] is free inverse semigroups, the third one [8] is dedicated to finite inverse semigroups and applications to language theory, while this one is concerned with general inverse semigroups.

Much of the structure theory of inverse semigroups has revolved about constructing an arbitrary inverse semigroup from groups and semilattices, and the main results of this theory can be stated as follows. An E-semigroup S (that is, a semigroup whose idempotents commute) is said to be an extension of a group by a semilattice if there is a surjective morphism ϕ from S onto a group such that $1\phi^{-1}$ is the set of idempotents of S. First, every inverse semigroup is covered by a regular extension of a group by a semilattice and the covering map is one-to-one on idempotents. Second, regular extensions of groups by semilattices are exactly E-unitary inverse semigroups [17], or P-semigroups (in the sense of McAlister [9, 10]), or regular subsemigroups of semidirect products of a semilattice by a group [16].

The aim of this paper is to develop a similar theory in the non-regular case. However, as usual in semigroup theory, many difficulties arise when passing from the regular case to the non-regular case. The first obvious problem is to find non-regular equivalents to the notions of inverse semigroups, E-unitary inverse semigroups, P-semigroups, etc. It turns out

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that the proper definitions are the following: E-dense semigroups (that is, semigroups in which idempotents form a dense subsemilattice) generalize inverse semigroups, E-unitary dense semigroups generalize E-unitary inverse semigroups, and extensions of groups by semilattices generalize regular extensions.

With this dictionary in hand, we may try to extend the main results recalled above. It is fair to say immediately that the generalization of the first result we propose is still a conjecture.

Conjecture. Every E-dense semigroup is covered by an E-unitary dense semigroup and the covering is one-to-one on idempotents.

Notice that in an E-unitary semigroup, the stabilizer of any element is a semilattice. Thus our conjecture says that any E-dense semigroup is covered by an E-dense semigroup whose stabilizers are nice, or, to paraphrase a geometric terminology, without singularities. In this sense our conjecture is reminiscent of the following result of Rhodes and Tilson [24]: every finite semigroup is covered by a finite semigroup whose stabilizers are group-free and the covering is one-to-one on groups. Our conjecture is especially interesting for finite semigroups as is shown in our third paper [8].

We are more successful with the second fundamental result and we prove the following generalization: extensions of groups by semilattices are exactly E-unitary dense semigroups. The next step consists in extending the notion of P-semigroups and seems more embarrassing at first sight. In fact, P-semigroups have a very nice interpretation in terms of algebraic topology. First, it is necessary to adopt a slightly more general point of view by replacing semigroups by categories. Notice that categories are not considered here as "abstract nonsense" but as algebraic objects similar to groups or vector spaces. This point of view has already been investigated by different authors [2, 15] in semigroup theory and analogous techniques have proved to be very useful in the theory of free groups [5, 23]. Following Higgins [5], we next define the fundamental group of a connected category. If this category is a monoid M (that is, a one-object category), the fundamental group is just the "free group over M" in the terminology of Clifford and Preston [3]. Moreover if M is E-dense, it is the quotient of M by a classical congruence: two elements u and v are equivalent if $eu = ev$ for some idempotent e.

To any monoid morphism $\phi : M \to N$ we associate a category C, the derived category of ϕ, which is a natural extension of the "derived semigroup" introduced by Tilson [25]. This derived category covers M in the topological sense. In particular, if M is E-unitary dense and ϕ is the surjective morphism from M onto its fundamental group G then C is a connected category and the endomorphisms of any object of C forms a semilattice. (We say that C is idempotent and commutative in this case.) Moreover G acts transitively without fixpoints on C and M is isomorphic to C/G. Conversely if a group G acts transitively without fixpoints on a connected idempotent and commutative category C, then the monoid C/G is E-unitary dense. In fact this theorem is the proper extension of McAlister's theorem on P-semigroups. Indeed in the regular case, C is an inverse category and C/G is isomorphic to the P-semigroup $P(G, F, E)$, where F is the partially ordered set of $\mathcal{J}$-classes of C and E is a subsemilattice of F isomorphic to the semilattice of idempotents of C/G. Furthermore, C/G is a submonoid of a semidirect product $S * G$, where S is the semilattice of ideals of C under intersection. Thus, even in the regular case, our description of E-unitary monoids is more natural then the description using P-semigroups.

The paper breaks up into four main sections. In the Section 1 we give some definitions relative to semigroups and in Section 2 we review the structure theory of inverse semigroups. In Section 3 we introduce categories, coverings, and fundamental groups and we prove our main result, Theorem 3.15. We come back to the regular case in the Section 4 and we show the connection between categories and P-semigroups.

1. Preliminaries

Let S be a semigroup. The elements x and y of S are *inverses* if $xyx = x$ and $yxy = y$. S is said to be a *regular semigroup* if every element of S has an inverse. S is an *inverse semigroup* if every element of S has a unique inverse. It is well known that S is an inverse semigroup iff S is a regular semigroup whose idempotents commute [3]. A *semilattice* is a commutative and idempotent semigroup.

Let Q be a set. We denote by $I(Q)$ the semigroup of all partial injective (that is, one-to-one) functions from Q to Q under composition of partial functions. Then $I(Q)$ is an inverse semigroup and the following representation theorem, due to Vagner and Preston [3], holds.

THEOREM 1.1. *S is an inverse semigroup iff S is isomorphic to a regular subsemigroup of $I(Q)$ for some set Q.*

Let S be a semigroup. Then S^1 denotes the monoid constructed as follows. $S^1 = S$ if S is a monoid and if S is not a monoid $S^1 = S \cup \{1\}$, where 1 is an identity.

Let T be a subsemigroup of a semigroup S. Then T is called *unitary* if for all $t, t' \in T$ and $s \in S$, $ts = t'$ implies $s \in T$ and $st = t'$ implies $s \in T$. T is called *dense* if every element $s \in S$ can be completed on the right and on the left into an element of T, that is, if there exist $s_1, s_2 \in S^1$ such that $s_1 s \in T$ and $ss_2 \in T$.

As usual $E(S)$ denotes the set of idempotents of S. An *E-semigroup* is a semigroup such that $E(S)$ is a semilattice. An E-semigroup S is called *E-unitary* (resp. *E-dense*, *E-unitary dense*) if $E(S)$ is a unitary (resp. dense, unitary and dense) subsemigroup of S. For example, every finite E-semigroup and every inverse semigroup is E-dense. E-unitary semigroups have been especially studied in the case of inverse semigroups [17]. In fact, one can give a great number of equivalent conditions defining E-unitary inverse semigroups. The next proposition selects four of them. Of course conditions (3) and (4) could be dualized.

PROPOSITION 1.2. *Let S be an inverse semigroup. The following conditions are equivalent*:

(1) *S is E-unitary.*

(2) *S is E-unitary dense.*

(3) *For all $e \in E(S)$ and $s \in S$, $es = e$ implies $s \in E(S)$.*

(4) *For all $s, t \in S$, $st = s$ implies $t \in E(S)$.*

Proof. Condition (1) implies (2). Let $s \in S$ and let $\bar{s}$ be the inverse of s. Then $s\bar{s}$, $\bar{s}s \in E(S)$ and thus S is E-dense.

Condition (2) implies (3) follows from the definition of E-unitary.

Condition (3) implies (4). If $st = s$ then $\bar{s}st = \bar{s}s \in E(S)$. Thus by (3), $t \in E(S)$.

Condition (4) implies (1). Since S is inverse, $E(S)$ is a semilattice. Moreover if e, $es \in E(S)$ then $(es)s = e(es)s = (es)es = es$ so that $s \in E(S)$ by (4). Finally, assume that e, $se \in E(S)$. Then $ses = (ses)es$ and thus $es \in E(S)$ by (4). Therefore $(es)s = e(es)s = eses = es$ so that $s \in E(S)$ by (4) again. ∎

Let S be a semigroup and let M be a monoid with 1 as an identity. To simplify notation, we will write S additively, without assuming that S is commutative. A *left action* of M on S is a mapping

$$M \times S \to S$$
$$(m, s) \to ms$$

satisfying for all $s, s_1, s_2 \in S$ and $m, m_1, m_2 \in M$

(1) $m(s_1 + s_2) = ms_1 + ms_2$,

(2) $m_1(m_2 s) = (m_1 m_2) s$,

(3) $1s = s$.

Of course, this just amounts to giving a morphism from M to the monoid of endomorphisms acting on the left of S. This action is used to form a semigroup $S * M$ on the set $S \times M$ with multiplication

$$(s, m)(s', m') = (s + ms', mm').$$

$S * M$ is called a *semidirect product* of S and M.

Note that if elements of $S \times M$ are represented by matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ s & m \end{smallmatrix}\right)$, where $s \in S$ and $m \in M$, then the previous formula can be written

$$\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s' & m' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s + ms' & mm' \end{pmatrix}.$$

Given a semigroup S, the reverse semigroup of S is denoted S^r. Given a left action of M^r on S^r, the reverse semidirect product of S and M is the semigroup $M *_r S$ defined by

$$M *_r S = (S^r * M^r)^r.$$

More directly, one can associate to the left action of M^r on S^r a right action of M on S (denoted by $(s, m) \to s \cdot m$) by setting $s \cdot m = ms$. Then one can define the product in $M *_r S$ by $(m, s)(m', s') = (mm', s \cdot m' + s')$.

If elements of $M \times S$ are represented by matrices of the form $\left(\begin{smallmatrix} m & 0 \\ s & 1 \end{smallmatrix}\right)$ the previous formula can be written

$$\begin{pmatrix} m & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} m' & 0 \\ s' & 1 \end{pmatrix} = \begin{pmatrix} mm' & 0 \\ s \cdot m' + s' & 1 \end{pmatrix}.$$

If M is a group, the following result shows that there is no difference between semidirect and reverse semidirect product.

PROPOSITION 1.3. *Let S be a semigroup and let G be a group. Then every semidirect product $S * G$ is isomorphic to a reverse semidirect product $G *_r S$ and vice versa.*

Proof. Given a left action of G on S, define a right action of G on S by setting $s \cdot g = g^{-1}s$. A simple calculation now shows that the function $\phi\colon S * G \to G *_r S$ defined by $\left(\begin{smallmatrix} 1 & 0 \\ s & g \end{smallmatrix}\right)\phi = \left(\begin{smallmatrix} g & 0 \\ g^{-1}s & 1 \end{smallmatrix}\right)$ is an isomorphism. ∎

2. INVERSE SEMIGROUPS AS EXTENSIONS OF GROUPS BY SEMILATTICES

The aim of this short section is to give a unified presentation of the structure theory of inverse semigroups, as developed, for example, by McAlister [9–11], O'Carroll [16], Munn [14], Reilly [18], and others (see [12] for an extensive bibliography).

A natural construction to obtain inverse semigroups from semilattices and groups is the semidirect product.

PROPOSITION 2.1. [11]. *Let S be a semilattice and let G be a group. Then $S * G$ is an inverse semigroup for any left action of G on S.*

Proof. Let $x=(s,g)$ and $\bar{x}=(\bar{s},\bar{g})$ be two elements of $S*G$. Then $x\bar{x}x=(s+g\bar{s}+g\bar{g}s, g\bar{g}g)$ and $\bar{x}x\bar{x}=(\bar{s}+\bar{g}s+\bar{g}g\bar{s}, \bar{g}g\bar{g})$. Therefore $\bar{x}$ is an inverse of x iff $\bar{g}=g^{-1}$, $s+g\bar{s}=s$, and $\bar{s}=\bar{s}+g^{-1}s$. It follows that $g\bar{s}=g(\bar{s}+g^{-1}s)=g\bar{s}+s=s$ and thus $\bar{x}=(g^{-1}s, g^{-1})$ is the unique inverse of x and S is inverse. ▮

A description of the Green's relations and the ideal structure of these semidirect products is given in [12]. Unfortunately, Proposition 2.1 does not characterize all inverse semigroups. However, the following result shows how to construct an arbitrary inverse semigroup from semilattices and groups. It is a consequence of the results of McAlister [9, 10] and O'Carroll [16]. A related result was given by Tilson [24].

THEOREM 2.2. *Let R be an inverse semigroup. Then there exists an inverse semigroup T such that*

(1) *there is a surjective morphism $\phi: T \to R$ which is one-to-one on idempotents,*

(2) *T is a subsemigroup of a semidirect product $S*G$, where S is a semilattice and G is a group.*

In view of this last result, inverse subsemigroups of semidirect products $S*G$, where S is a semilattice and G is a group, play a central role in the study of inverse semigroups. There are now many known characterizations of these semigroups. One of them states that these semigroups are exactly the E-unitary inverse semigroups. A more precise description was obtained by McAlister [9, 10] in terms of P-semigroups. The notion of P-semigroup arose from Scheiblich's construction of free inverse semigroups [20]. Here is the definition, from McAlister [9].

Let F be a (down) directed partially ordered set and let E be an ideal and subsemilattice of F. Let G be a group acting on the left on F by order automorphisms in such a way that $F=G\cdot E$ and set $P(G,F,E)=\{(e,g)\in E\times G \mid g^{-1}e\in E\}$. Then if (e,g) and (f,h) are in $P(G,F,E)$ it can be shown [9] that e and gf have a greatest lower bound denoted $e\wedge gf$. Moreover $e\wedge gf$ is in E. Thus we can define a multiplication on $P(G,F,E)$ by setting

$$(e,g)(f,h)=(e\wedge gf, gh).$$

Then one shows [9, 10] that $P(G,F,E)$ is an inverse semigroup, called a

P-semigroup. We may now summarize the results of [9, 10, 16] in the following theorem.

THEOREM 2.3. *Let T be an inverse semigroup. The following conditions are equivalent*:

(1) *There exist a group G and a surjective morphism $\phi: T \to G$ such that $1\phi^{-1} = E(T)$.*

(2) *T is E-unitary.*

(3) *T is isomorphic to a P-semigroup $P(G, F, E)$, where $E = E(T)$.*

(4) *T is a subsemigroup of a semidirect product $S * G$, where S is a semilattice and G is a group.*

3. EXTENSIONS OF GROUPS BY SEMILATTICES

Theorem 2.3 characterizes *regular* E-semigroups T such that there exists a surjective morphism ϕ onto a group with $1\phi^{-1} = E(T)$. It is convenient to say in this case that T is an extension of a group by a semilattice. The aim of this section is to extend Theorem 2.3 to the non-regular case. In fact we shall only extend conditions (1), (2), and (3) of this theorem but the restriction of our extended theorem to the regular case will give back Theorem 2.3 in full.

The main problem is that all the proofs of Theorem 2.3 [9, 10, 14, 22] rely heavily on the regularity of T and do not generalize immediately. In particular, a new definition of P-semigroups is needed. It appears that the proper framework to generalize Theorem 2.3 is to study groups acting on a category. Following [5], we consider here only "small" categories in the sense that the objects of a category always form a set. In fact categories are considered here as algebraic structures in their own right, on the same footing as rings, vector spaces, or groups. Therefore the "elements" of this algebra are the morphisms and their composition is an associative partial binary operation. Functors thus appear as algebra homomorphisms. In the case of a category with just one object, the resulting algebra is just a monoid. Thus we shall define for categories a number of notions borrowed either from semigroup theory (Green's relations, ideals) or from algebraic topology (coverings, fundamental groups). Now we give the details.

3.1. *Categories*

Let C be a category. $\mathrm{Ob}(C)$ denotes the objects of C and for $u, v \in \mathrm{Ob}(C)$, $\mathrm{Mor}(u, v)$ denotes the set of all morphisms from u to v. C is *connected* if, for all $u, v \in \mathrm{Ob}(C)$, $\mathrm{Mor}(u, v)$ is non-empty. $\mathrm{Mor}(u, C)$

denotes the set of all morphisms starting from u. Thus $\mathrm{Mor}(u, C) = \bigcup_{v \in \mathrm{Ob}(C)} \mathrm{Mor}(u, v)$. Similarly $\mathrm{Mor}(C) = \bigcup_{u \in \mathrm{Ob}(C)} \mathrm{Mor}(u, C)$ is the set of all morphisms of C.

A category is a *groupoid* if all morphisms are isomorphisms.

We shall adopt an additive notation for composition of morphisms, although this composition is not assumed to be commutative. Thus if $p \in \mathrm{Mor}(u, v)$ and $q \in \mathrm{Mor}(v, w)$, $p + q \in \mathrm{Mor}(u, w)$, and if $u \in \mathrm{Ob}(C)$, O_u denotes the identity on u. Although this notation may seem a little unsettling at first, it will allow us to write group actions multiplicatively.

A category C is *regular* if for all objects $u, v \in \mathrm{Ob}(C)$ and for all $p \in \mathrm{Mor}(u, v)$ there exists $q \in \mathrm{Mor}(v, u)$ such that $p + q + p = p$. q is called an *inverse* of p if, furthermore, $q + p + q = q$. Notice that if $p + q + p = p$ then $q + p + q$ is an inverse of p.

A category is *inverse* iff every morphism has a unique inverse. If C is inverse $\bar{p}$ will denote the inverse of p.

A morphism $p \in \mathrm{Mor}(u, v)$ is *idempotent* if $p + p = p$. This implies in particular that $u = v$.

A category C is *idempotent* (resp. *commutative*, *torsion*) if for all $u \in \mathrm{Ob}(C)$, the monoid $\mathrm{Mor}(u, u)$ is idempotent (resp. commutative, torsion).

Let C be a category. We define the *Green's relations* $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$, $\mathscr{H}$, and $\mathscr{J}$ and the preorders $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{H}}$, $\leqslant_{\mathscr{J}}$ on $\mathrm{Mor}(C)$ as for a semigroup. For example, $p \leqslant_{\mathscr{R}} q$ iff there exists $r \in \mathrm{Mor}(C)$ such that $p = q + r$ and $p \mathrel{\mathscr{R}} q$ iff $p \leqslant_{\mathscr{R}} q$ and $q \leqslant_{\mathscr{R}} p$.

All the standard proofs on Green's relations and regular semigroups carry over without much trouble. Therefore we just state without proof the basic facts we need on categories.

PROPOSITION 3.1. (1) *The relations $\mathscr{R}$ and $\mathscr{L}$ commute. Therefore $\mathscr{R}\mathscr{L} = \mathscr{L}\mathscr{R} = \mathscr{D}$.*

(2) *If C is a torsion category, then $\mathscr{D} = \mathscr{J}$ on C.*

PROPOSITION 3.2. (1) *A category is regular iff every $\mathscr{R}$-class and every $\mathscr{L}$-class contains an idempotent.*

(2) *A regular category C is inverse iff, for every $u \in \mathrm{Ob}(C)$, the set of idempotents of* $\mathrm{Mor}(u, u)$ *forms a semilattice.*

PROPOSITION 3.3. *Let C be an inverse category and let p, q be two idempotents such that $p \mathrel{\mathscr{D}} q$. Then there exists a unique $r \in \mathrm{Mor}(C)$ such that $p = r + \bar{r}$ and $q = \bar{r} + r$.*

We need the following slightly more technical result.

PROPOSITION 3.4. *Let C be a regular, idempotent, and commutative category and let $u, v \in \mathrm{Ob}(C)$. Then for all $p, q \in \mathrm{Mor}(u, v)$, $p \mathcal{J} q$ implies $p = q$.*

Proof. First C is inverse by Proposition 3.2. Now $p \mathcal{J} q$ and since $p = p + \bar{p} + p$ and $q = q + \bar{q} + q$ we also have $p \mathcal{J} p + \bar{p}$, $q \mathcal{J} p + \bar{p}$ and $q \mathcal{J} q + \bar{q}$. It follows that $p + \bar{p} \mathcal{J} q + \bar{q}$, that is, there exist $a, b, c, d \in \mathrm{Mor}(C)$ such that $a + (p + \bar{p}) + b = q + \bar{q}$ and $c + (q + \bar{q}) + d = p + \bar{p}$. But $p + \bar{p}$, $q + \bar{q} \in \mathrm{Mor}(u, u)$ and thus $a, b, c, d \in \mathrm{Mor}(u, u)$. Therefore $p + \bar{p}$ and $q + \bar{q}$ are $\mathcal{J}$-related in the semilattice $\mathrm{Mor}(u, u)$ and hence $p + \bar{p} = q + \bar{q}$. A similar argument shows that $\bar{p} + p = \bar{q} + q$. Now we have $(p + \bar{q}) + q + \bar{p} = p + \bar{p} + p + \bar{p} = p + \bar{p}$ and $(p + \bar{p}) + p + \bar{q} = p + \bar{q}$. It follows that $p + \bar{p} \mathcal{J} p + \bar{q}$ and hence, by the above argument, $p + \bar{p} = p + \bar{q} = q + \bar{q}$. Finally, we obtain $p = p + \bar{p} + p = p + \bar{q} + q = q + \bar{q} + q = q$. ∎

Let C be a category. An *ideal* I of C is a subset of $\mathrm{Mor}(C)$ such that for all $u, v, w, z \in \mathrm{Ob}(C)$ and for all $x \in \mathrm{Mor}(u, v)$, $p \in \mathrm{Mor}(v, w)$, and $y \in \mathrm{Mor}(w, z)$, $p \in I$ implies $x + p + y \in I$.

A *congruence* on a category C is an equivalence relation $\sim$ on $\mathrm{Mor}(C)$ satisfying the following conditions:

(1) If $p \sim q$ then p and q are coterminal, that is, there exist $u, v \in \mathrm{Ob}(C)$ such that $p, q \in \mathrm{Mor}(u, v)$.

(2) For all $u, v, w \in \mathrm{Ob}(C)$ and for all $p_1, p_2 \in \mathrm{Mor}(u, v)$ and $q_1, q_2 \in \mathrm{Mor}(v, w)$, $p_1 \sim p_2$ and $q_1 \sim q_2$ imply $p_1 + p_2 \sim q_1 + q_2$.

Let $\sim$ be a congruence on a category C. The *quotient category* $C/\sim$ is the category whose objects are the objects of C and whose morphisms are the $\sim$-classes of the morphisms of C. Condition (2) above implies that composition of morphisms is well defined on $C/\sim$.

An important example of a category is the free category on a graph. Intuitively, a graph is a set of arrows between points called vertices. More formally *a graph* consists of a set V of vertices, a set E of edges, and two maps $\alpha: E \to V$ and $\omega: E \to V$. If e is an edge, $e\alpha$ is the origin of e and $e\omega$ is the end of e. Two edges e_1 and e_2 are consecutive if the end of e_1 is equal to the origin of e_2. A path is a sequence of consecutive edges. Given a path $p = e_1 \cdots e_n$, the origin of p is the origin of e_1 and the end of p is the end of e_n. Let Γ be a graph. The *free category* on Γ is defined as follows. The objects are the vertices of Γ and the set of morphisms from u to v is the set of all paths with origin u and end v. If $u = v$ we also include a trivial path O_u. The morphism composition is the natural path composition, completed by the rules $O_u + p = p$—for each path with origin u—and $p + O_u = p$ for each path with end u.

An *automorphism* of a category C is a functor $F: C \to C$ such that:

(1) For all $u \in \mathrm{Ob}(C)$, F induces a permutation on $\mathrm{Ob}(C)$.

(2) For all $u, v \in \mathrm{Ob}(C)$, F induces a bijection $\mathrm{Mor}(u, v) \to \mathrm{Mor}(F(u), F(v))$.

We denote by $\mathrm{Aut}(C)$ the group of automorphims of C. A *group G acts* on C if there exists a group morphism $G \to \mathrm{Aut}(C)$. In this case we write gv (resp. gp), the result of the action of g on the object v (resp. on the morphism p). Since an automorphism is a functor, we have the following identities, where $u, v, w \in \mathrm{Ob}(C)$.

(1) $g(p+q) = gp + gq$ for all $g \in G$, $p \in \mathrm{Mor}(u, v)$ and $q \in \mathrm{Mor}(v, w)$.

(2) $(gh)p = g(hp)$ for all $g, h \in G$, $p \in \mathrm{Mor}(u, v)$.

(3) $gO_u = O_{gu}$ for all $g \in G$.

(4) $1p = p$ for all $p \in \mathrm{Mor}(u, v)$.

G acts *transitively* if for all $u, v \in \mathrm{Ob}(C)$ there exists $g \in G$ such that $gu = v$. G acts *without fixpoints* if the condition "$gv = v$ for some $v \in \mathrm{Ob}(C)$" implies $g = 1$. Thus if G acts transitively without fixpoints, then for all $u, v \in \mathrm{Ob}(C)$ there exists a unique $g \in G$ such that $gu = v$.

Let G be a group acting on a category C. We define a category C/G as follows. The objects are the orbits of $\mathrm{Ob}(C)$ under G. If Gu and Gv are two orbits, the set $\mathrm{Mor}(Gu, Gv)$ is the quotient of the set

$$\bigcup_{\substack{u' \in Gu \\ v' \in Gv}} \mathrm{Mor}(u', v')$$

under the equivalence defined by $p \sim q$ iff there exists $g \in G$ such that $p = gq$. Composition of morphisms needs care. If $\dot{p} \in \mathrm{Mor}(Gu, Gv)$ and $\dot{q} \in \mathrm{Mor}(Gv, Gw)$, then $\dot{p} + \dot{q} = (p+q)^{\cdot}$ where p and q are morphisms selected as follows. First $p \in \mathrm{Mor}(u', v')$ is an arbitrary element of the equivalence class $\dot{p}$ (and thus $u' \in Gu$ and $v' \in Gv$). Now since G is transitive on Gv there exists in the equivalence class $\dot{q}$ a morphism $q \in \mathrm{Mor}(v', w')$, where $w' \in Gw$. Now $p + q \in \mathrm{Mor}(u', w')$ is well defined and one can easily verify that the class of $p + q$ modulo $\sim$ depends only on $\dot{p}$ and $\dot{q}$. The fact that C/G is now really a category is left to the reader.

We conclude this subsection by an obvious, but useful, observation.

PROPOSITION 3.5. *Let G be a group acting on a category C. If $p \mathrel{\mathcal{J}} q$ then $gp \mathrel{\mathcal{J}} gq$ for all $g \in G$.*

Proof. If $p \mathscr{J} q$ there exist $a, b, c, d \in \mathrm{Mor}(C)$ such that $q = a + p + b$ and $p = c + q + d$. It follows that $gq = ga + gp + gb$ and $gp = gc + gq + gd$, that is, $gp \mathscr{J} gq$. ∎

3.2. *Fundamental Groups*

Another important concept is the *fundamental groupoid* of a category C. Consider the graph whose vertices are the objects of C and whose edges are the triples (u, p, v) or $(v, \bar{p}, u)$ such that $p \in \mathrm{Mor}(u, v)$. Let F be the free category over this graph and let $\sim$ be the congruence on F generated by the relations $(u, p, v) + (v, \bar{p}, u) \sim O_u \sim (v, \bar{p}, u) + (u, p, v)$ and $(u, p, v) + (v, q, w) \sim (u, p + q, w)$ for all $p \in \mathrm{Mor}(u, v)$ and $q \in \mathrm{Mor}(v, w)$.

The quotient $F/\sim$ is a groupoid, called the fundamental groupoid of C. Thus for every $v \in \mathrm{Ob}(F/\sim)$, $\mathrm{Mor}(v, v)$ is a group, called the *fundamental group of C at the point v* and denoted $\Pi_1(C, v)$. Moreover if C is connected, all the groups $\Pi_1(C, v)$ are isomorphic [5], and thus we will simply refer in this case to *the fundamental group* of C, denoted by $\Pi_1(C)$.

In particular, if M is a monoid (that is, a one-object category), the fundamental group $\Pi_1(M)$ can be computed as follows. Let $F(M)$ be the free group with basis M. To avoid confusion between the elements of M and the elements of $F(M)$ we shall denote by $\iota\colon M \to F(M)$ the natural embedding. Notice that ι is *not* a monoid morphism. Now $\Pi_1(M)$ is the quotient of $F(M)$ under the relations $(u\iota)(v\iota) = (uv)\iota$ for all $u, v \in M$. Thus $\Pi_1(M)$ is nothing else than the "free group over M" in the terminology of [3]. Indeed let $\pi\colon F(M) \to \Pi_1(M)$ be the surjective group morphism onto $\Pi_1(M)$. Then the map $\eta = \iota\pi\colon M \to \Pi_1(M)$ has the following universal property:

PROPOSITION 3.6. *η is a monoid morphism and for every monoid morphism $\gamma = M \to G$ into a group such that $M\gamma$ generates G as a group, there exists a unique group morphism $\phi\colon \Pi_1(M) \to G$ such that $\gamma = \eta\phi$. Moreover ϕ is surjective.*

$$\begin{tikzcd} M \arrow[rr, "\eta"] \arrow[dr, "\gamma"'] & & \Pi_1(M) \arrow[dl, "\phi"] \\ & G & \end{tikzcd}$$

Proof. First η is a monoid morphism, since for all $u, v \in M$, we have $(u\eta)(v\eta) = (u\iota\pi)(v\iota\pi) = ((u\iota)(v\iota))\pi = (uv)\iota\pi = (uv)\eta$. Next let $\gamma\colon M \to G$ be a monoid morphism into a group such that $M\gamma$ generates G as a group. Since $F(M)$ is the free group with basis M there exists a unique group morphism $\beta\colon F(M) \to G$ such that $\iota\beta = \gamma$. Thus if there exists a group morphism ϕ such that $\gamma = \eta\phi = \iota(\pi\phi)$, the uniqueness of β implies $\beta = \pi\phi$. Moreover since $M\phi$

generates G as a group, β is surjective. Let us summarize the situation with the diagram

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "\gamma"'] & F(M) \arrow[r, "\pi"] \arrow[d, "\beta"] & \Pi_1(M) \arrow[dl, "\phi"] \\ & G & \end{tikzcd}$$

Now $\operatorname{Ker} \pi \subset \operatorname{Ker} \beta$. Indeed $\operatorname{Ker} \pi$ is the normal subgroup of $F(M)$ generated by the set $\{(uv)\iota(v\iota)^{-1}(u\iota)^{-1} \mid u, v \in M\}$ and for all $u, v \in M$ we have $(uv)\,\iota\beta = (uv)\,\gamma = (u\gamma)(v\gamma) = (u\iota\beta)(v\iota\beta) = ((u\iota)(v\iota))\,\beta$. Therefore there exists a unique group morphism ϕ such that $\pi\phi = \beta$ and ϕ is surjective. ∎

The quotients of a monoid M are naturally (pre) ordered by the relation $N_1 \leqslant N_2$ iff there exists a surjective morphism $N_2 \to N_1$. Notice that if N_2 is a group then N_1 is also a group. Proposition 3.6 implies that if $\eta: M \to \Pi_1(M)$ is surjective then $\Pi_1(M)$ is the maximal quotient group of M (relative to the preorder $\leqslant$). This is the case for an important class of monoids.

PROPOSITION 3.7. *Let M be a monoid such that the semigroup generated by $E(M)$ is dense in M. Then the fundamental group $\Pi_1(M)$ is the maximal quotient group of M.*

Proof. By Proposition 3.6 we only have to show that $\Pi_1(M)$ is a quotient of M. Let $u \in M$. Since the semigroup generated by $E(M)$ is dense there exists $v \in M$ such that uv is a product of idempotents. Therefore $(uv)\,\eta = 1$ and thus $(u\eta)^{-1} = v\eta \in M\eta$. Consequently the set $\{(u\eta)^{-1} \mid u \in M\}$ is contained in $M\eta$ and since $M\eta$ generates $\Pi_1(M)$ as a group, it also generates $\Pi_1(M)$ as a monoid. Therefore η is surjective. ∎

The last proposition applies in particular in the case where M is finite. In this case the construction of the maximal quotient group is well known [1]. Let K be the minimal ideal of M. Then K is a completely simple semigroup with structure group G. Let S be the subsemigroup of K generated by $E(K)$. Then S is again a simple semigroup with structure group $H \subset G$. Let N be the smallest normal subgroup of G containing H. Then $\Pi_1(M)$ is isomorphic to G/N.

Proposition 3.7 also applies to the case of inverse monoids. Indeed it is well known that the maximal quotient group of an inverse monoid M is the quotient of M under the congruence $\sim$ defined by $u \sim v$ iff there exists $e \in E(M)$ such that $eu = ev$. In fact this result holds for a more general class of semigroups.

PROPOSITION 3.8. *Let M be an E-dense monoid. Then $\Pi_1(M)$ is the quotient of M by the congruence $\sim$ defined by $u \sim v$ iff there exists $e \in E(M)$ such that $eu = ev$.*

Proof. Clearly $\sim$ is reflexive and symmetric. Assume that $u \sim v$ and $v \sim w$. Then there exists $e, f \in E(M)$ such that $eu = ev$ and $fv = fw$. Since $E(M)$ is a semilattice, it follows that $feu = fev = efv = efw = few$ and hence $\sim$ is transitive. Now if $x \in M$ then $eux = evx$ and thus $ux \sim vx$. Moreover since M is E-dense there exists $y \in M$ such that $yx \in E(M)$. Therefore $(yx)\,eu = (yx)\,ev = e(yx)\,u = e(yx)\,v$. Now there exists z such that $z(ey) \in E(M)$ and thus $(zey)\,xu = (zey)\,xv$ and hence $xu \sim xv$. Consequently $\sim$ is a congruence.

We claim that $M/\sim$ is a group. Indeed let $u \in M$. Then there exists $v \in M$ such that $uv \in E(M)$. Therefore $(uv) \cdot 1 = (uv)(uv)$ and thus $uv \sim 1$. Similarly, for each $u \in M$ there exists $w \in M$ such that $wu \sim 1$ and this proves the claim. Thus $M/\sim$ is a quotient group of M. Moreover if $\phi\colon M \to G$ is a surjective morphism onto a group, then $u \sim v$ implies $eu = ev$ for some $e \in E(M)$ and hence $(e\phi)(u\phi) = (e\phi)(v\phi)$, that is, $u\phi = v\phi$. Therefore there exists a surjective morphism $M/\sim \to G$ and thus $M/\sim$ is the maximal quotient group of M. Thus by Proposition 3.7, $M/\sim$ is isomorphic to $\Pi_1(M)$. ∎

All the results of this subsection were stated for monoids only. Now if S is a semigroup one can define a monoid S^1 as follows: $S^1 = S$ if S is a monoid and $S^1 = S \cup \{1\}$, where 1 is an identity if S is not a monoid. The fundamental group of S can be defined by setting $\Pi_1(S) = \Pi_1(S^1)$ and then Propositions 3.6 to 3.8 can be readily extended to semigroups.

We now generalize conditions (1) and (2) of Theorem 2.3 to the non-regular case.

THEOREM 3.9. *Let T be a semigroup whose idempotents commute. Then the following conditions are equivalent*:

(1) *There exist a group G and a surjective morphism $\phi\colon T \to G$ such that $1\phi^{-1} = E(T)$.*

(2) *There exists a surjective morphism $\eta\colon T \to \Pi_1(T)$ such that $1\eta^{-1} = E(T)$.*

(3) *T is E-unitary dense.*

Proof. We prove (3) implies (2), (2) implies (1), and (1) implies (3) in this order. Let T be a E-unitary dense semigroup. Then by Proposition 3.7 there exists a surjective morphism $\eta\colon T \to \Pi_1(T)$ and by Proposition 3.8 $\Pi_1(T)$ is the quotient of T under the congruence $\sim$. Therefore

$1\eta^{-1} = \{t \in T \mid t \sim 1\}$. But if $u \sim 1$, then $eu = e$ for some $e \in E(T)$ and thus $u \in E(T)$, since T is E-unitary. Therefore $1\eta^{-1} = E(T)$, proving (2).

Condition (2) implies (1) is obvious.

Finally, assume that (1) holds. Then if $eu = f$ for some $e, f \in E(T)$ and $u \in T$, then $(u\phi) = 1(u\phi) = (e\phi)(u\phi) = (eu)\,\phi = f\phi = 1$ and thus $u \in E(T)$. Therefore T is E-unitary. Moreover for every $s \in T$ there exists $t \in T$ such that $t\phi = (s\phi)^{-1}$ and thus $st \in 1\phi^{-1} = E(T)$. It follows that T is E-dense, proving (3).

3.3. *Coverings*

A functor $F: C \to D$ is a *covering* if for all $u \in \mathrm{Ob}(C)$, F induces a bijection from $\mathrm{Mor}(u, C)$ to $\mathrm{Mor}(F(u), D)$.

The following result is an analog of a well-known result of topology.

PROPOSITION 3.10. *Let G be a group acting without fixpoints on a category C. Then $F: C \to C/G$ is a covering.*

If G acts transitively without fixpoints, the category C/G is a monoid and a direct description of this monoid is possible. Let u be any object of C and let $C_u = \{(p, g) \mid p \in \mathrm{Mor}(u, gu)\}$. Then C_u is a monoid under the multiplication

$$(p, g)(q, h) = (p + gq, gh).$$

This is well defined since if $p \in \mathrm{Mor}(u, gu)$ and $q \in \mathrm{Mor}(u, hu)$ then $gq \in \mathrm{Mor}(gu, ghu)$ and thus $p + gq \in \mathrm{Mor}(u, ghu)$. Then we have

PROPOSITION 3.11. *Let G be a group acting transitively without fixpoints on a category C. Then for all $u \in \mathrm{Ob}(C)$, the monoid C_u is isomorphic to C/G.*

Proof. Define a function $\Theta: C_u \to C/G$ by setting $(p, g)\,\Theta = \dot{p}$. Then we have $((p, g)(q, h))\,\Theta = (p + gq, gh)\,\Theta = (p + gq)^{\cdot}$. But since $p \in \mathrm{Mor}(u, gu)$ and $gq \in \mathrm{Mor}(gu, ghu)$, $(p + gq)^{\cdot} = \dot{p} + \dot{q}$ by definition of C/G. Therefore Θ is a monoid morphism. Assume that $(p_1, g_1)\,\Theta = (p_2, g_2)\,\Theta$, that is, $\dot{p}_1 = \dot{p}_2$. Then there exists $h \in G$ such that $hp_1 = p_2$. It follows that $hu = u$ and hence $h = 1$ since G acts without fixpoints. Thus $p_1 = p_2$ and Θ is injective. Finally, let $p \in \mathrm{Mor}(C)$. Since G acts transitively on C there exists $h \in G$ such that $hp \in \mathrm{Mor}(u, C)$. Therefore if $hp \in \mathrm{Mor}(u, gu)$ we have $(hp, g)\,\Theta = (hp)^{\cdot} = \dot{p}$. Thus Θ is an isomorphism.

The multiplication in C_u is somewhat reminiscent of the multiplication in a P-semigroup. Indeed, as we shall see later, the monoids C_u—or C/G, by the previous proposition—are the generalization of P-semigroups we need to extend Theorem 2.3.

We turn now to an important example of coverings, the derived covering of a morphism. Some years ago, the derived semigroup of a morphism was introduced by Tilson [25] for applications to finite semigroups and it has proved to be one of the most important tools in the study of wreath product decompositions. Rhodes [19] recently proposed an extension to arbitrary semigroups. In fact it is both more natural and more convenient to consider the point of view of coverings.

Let $\phi\colon M \to N$ be a monoid morphism. Define a category C with $\mathrm{Ob}(C) = N$ and for $u, v \in N$, $\mathrm{Mor}(u, v) = \{(u, m, v) \in N \times M \times N \mid u(m\phi) = v\}$.

Composition is given by $(u, m, v) + (v, n, w) = (u, mn, w)$. Clearly this is associative and $O_u = (u, 1, u)$. So C is a category, called the *derived category* of ϕ. Note that the derived semigroup of ϕ is simply the set $\mathrm{Mor}(C) \cup \{0\}$ with multiplication given by

$$pq = \begin{cases} p+q & \text{if } p+q \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$$

Now the *derived covering* of ϕ is the functor $F\colon C \to M$ defined by

(a) for all $u \in \mathrm{Ob}(C)$, $F(u)$ is the only object of M,

(b) for all $(u, m, v) \in \mathrm{Mor}(u, v)$, $F(u, m, v) = m$.

It is easy to verify that F is indeed a covering.

We consider now the particular case where $\phi\colon M \to G$ is a morphism into a group.

PROPOSITION 3.12. *Let $\phi\colon M \to G$ be a morphism into a group and let $F\colon C \to M$ be the derived covering of ϕ. Then G acts transitively without fixpoints on C and M is isomorphic (as a monoid) to C/G.*

Proof. Since $\mathrm{Ob}(C) = G$, the multiplication of G defines an action of G on $\mathrm{Ob}(C)$. Now if $(h, m, k) \in \mathrm{Mor}(h, k)$, we set $g(h, m, k) = (gh, m, gk)$. This defines a transitive action of G on C and obviously this action has no fixpoints. Moreover if $g \in \mathrm{Ob}(C)$ is fixed, the function $M \to Cg$ defined by $m \to ((g, m, g(m\phi)), g(m\phi)\, g^{-1})$ is an isomorphism since

$$\begin{aligned}
&((g, m, g(m\phi)), g(m\phi)\, g^{-1})((g, n, g(n\phi)), g(n\phi)\, g^{-1}) \\
&\quad = ((g, m, g(m\phi)) + g(m\phi)\, g^{-1}(g, n, g(n\phi)), g(m\phi)(n\phi)\, g^{-1}) \\
&\quad = ((g, m, g(m\phi)) + (g(m\phi), n, g(m\phi)(n\phi)), g(m\phi)(n\phi)\, g^{-1}) \\
&\quad = ((g, mn, g(mn)\,\phi), g(mn)\,\phi\, g^{-1}).
\end{aligned}$$

It follows by Proposition 3.11 that M is isomorphic to C/G. ∎

We consider now the case where M is an E-unitary dense monoid. By Proposition 3.7, there exists a surjective morphism $\eta: M \to \Pi_1(M)$. Let C be the derived category of η and let $F: C \to M$ be the derived covering of η. Then we have.

PROPOSITION 3.13. *For all* $g \in \mathrm{Ob}(C)$, $\mathrm{Mor}(g, g)$ *is isomorphic to* $E(M)$. *Moreover* C *is connected, idempotent, and commutative and* $\mathcal{D} = \mathcal{J}$ *on* C. *Finally, if* M *is regular,* C *is regular.*

Proof. By definition $(g, m, g) \in \mathrm{Mor}(g, g)$ iff $m \in 1\eta^{-1}$. But $1\eta^{-1} = E(M)$ by Theorem 3.9 and thus $\mathrm{Mor}(g, g) = \{(g, m, g) \mid m \in E(M)\}$ is isomorphic to $E(M)$. Thus C is idempotent and commutative and, by Proposition 3.1, $\mathcal{D} = \mathcal{J}$ on C. Finally, let $g, h \in \mathrm{Ob}(C)$ and let $m \in M$ be such that $m\eta = g^{-1}h$. Then by definition $(g, m, h) \in \mathrm{Mor}(g, h)$. Thus C is connected.

Finally, assume that M is regular. Let $(g, m, h) \in \mathrm{Mor}(g, h)$ and let n be an inverse of m. Then $mnm = m$ and $nmn = n$ and thus $(mn)\eta = (nm)\eta = 1$. It follows that $(h, n, g) \in \mathrm{Mor}(h, g)$ since $h(n\eta) = g(m\eta)(n\eta) = g(mn)\eta = g$ and a simple calculation shows that (h, n, g) is an inverse of (g, m, h). Thus C is regular. ∎

Propositions 3.12 and 3.13 combined show that if M is E-unitary dense, then $\Pi_1(M)$ acts transitively without fixpoints on the connected, idempotent, and commutative category C and that M is isomorphic to $C/\Pi_1(M)$. Conversely we have the following result.

PROPOSITION 3.14. *Let* G *be a group acting transitively without fixpoints on an idempotent, commutative, and connected category* C. *Then the monoid* C/G *is* E*-unitary dense. Furthermore, if* C *is regular, then* C/G *is an inverse monoid.*

Proof. By Proposition 3.11, C/G is isomorphic to C_u for each $u \in \mathrm{Ob}(C)$. Now (p, g) is an idempotent in C_u iff $p = p + gp$ and $g = g^2$, that is, iff $g = 1$ and p is idempotent. Since C is idempotent and commutative, every morphism $p \in \mathrm{Mor}(u, u)$ is idempotent and thus $E(C_u) = \{(p, 1) \mid p \in \mathrm{Mor}(u, u)\}$ is isomorphic to $\mathrm{Mor}(u, u)$. It follows that $E(C_u)$ is a semilattice.

Assume now that $(p, 1)(q, h) \in E(C_u)$. This means that $(p + q, h)$ is idempotent and, by the previous description of the idempotents, $h = 1$ and $q \in \mathrm{Mor}(u, u)$. Thus $(q, h) \in E(C_u)$ and C_u is unitary.

Next we show that C_u is E-dense. Indeed let $(p, g) \in C_u$. Since C is connected there exists a morphism $q \in \mathrm{Mor}(u, g^{-1}u)$. Therefore $(p, g)(q, g^{-1}) = (p + gq, 1) \in E(C_u)$. Thus C_u is E-dense and C/G is an E-unitary dense monoid.

Finally, assume that C is regular. Let $(p, g) \in C_u$ and let q be an inverse of p. Then $q \in \mathrm{Mor}(gu, u)$ and thus $g^{-1}q \in \mathrm{Mor}(u, g^{-1}u)$. It follows that $(p, g)(g^{-1}q, g^{-1})(p, g) = (p+q, 1)(p, g) = (p+q+p, g) = (p, g)$ and similarly $(g^{-1}q, g^{-1})(p, g)(g^{-1}q, g^{-1}) = (g^{-1}q, g^{-1})$. Thus C_u is regular and hence an inverse monoid. ▮

The result of this section can now be summarized into the following main result, which extends Theorem 2.3 to the non-regular case.

THEOREM 3.15. *Let M be a monoid whose idempotents commute. Then the following conditions are equivalent*:

(1) *There exists a surjective morphism $\phi: M \to G$ onto a group such that $1\phi^{-1} = E(M)$.*

(2) *M is E-unitary dense.*

(3) *M is isomorphic to a monoid C/G, where G is a group acting transitively without fixpoints on a connected, idempotent, and commutative category C.*

4. BACK TO THE REGULAR CASE

In this section we apply Theorem 3.15 to the regular case. This leads to a new proof of Theorem 2.3 and to a more precise statement of this theorem. Indeed Theorem 2.3 states in particular that if there exists a surjective morphism ϕ from an inverse semigroup T onto a group G, then T is isomorphic to a subsemigroup of a semidirect product $S * G$, where S is a semilattice. However, if one follows the previous proofs of [9, 10, 14, 22], the explicit construction of such a semilattice S is rather involved. Here we show that one can simply choose for S the semilattice (under intersection) of all $\mathcal{J}$-classes of C, the derived category of ϕ.

We first state the "regular" version of Theorem 3.15 which follows from the regular version of Propositions 3.13 and 3.14.

THEOREM 4.1. *Let M be an inverse monoid. Then the following conditions are equivalent*:

(1) *There exists a surjective morphism $\phi: M \to G$ onto a group such that $1\phi^{-1} = E(M)$.*

(2) *M is E-unitary.*

(3) *M is isomorphic to a monoid C/G, where G is a group acting transitively without fixpoints on a connected idempotent, commutative inverse category C.*

We now establish the connection with P-semigroups.

THEOREM 4.2. *Let G be a group acting transitively without fixpoints on a connected idempotent, commutative inverse category C. Then the inverse monoid C/G is isomorphic to a P-semigroup $P(G, F, E)$, where F is the partially ordered set of $\mathscr{J}$-classes of C and E is a subsemilattice* of *F isomorphic to $E(C/G)$.*

Proof. We simply denote by $\leqslant$ the relation $\leqslant_{\mathscr{J}}$ on C. Then $\leqslant$ is a partial order on the set $F = C/\mathscr{J}$ of $\mathscr{J}$-classes of C. Let $u \in \mathrm{Ob}(C)$ and set $E_u = \{J \in F \mid J \cap \mathrm{Mor}(u, u) \neq \varnothing\}$. By Proposition 3.11, C/G is isomorphic to C_u. We claim that C_u is isomorphic to $P(G, F, E_u)$. We first need a lemma to verify that G, E_u, and F satisfy the conditions required to form a P-semigroup.

LEMMA 4.3. (1) $F = GE_u$.

(2) *E_u is a semilattice isomorphic to* $\mathrm{Mor}(u, u)$ *and to $E(C/G)$.*

(3) *E_u is an order ideal of F.*

Proof. (1) Let $J \in F$ and let $p \in \mathrm{Mor}(v, w)$ be an element of J. Since G acts transitively on C, $u = gv$ for some $g \in G$. Let J' be the $\mathscr{J}$-class of $gp + g\bar{p}$. Since $gp + g\bar{p} \in \mathrm{Mor}(u, u)$, $J' \in E_u$. Moreover since $gp = gp + g\bar{p} + gp$, $gp \mathrel{\mathscr{R}} gp + g\bar{p}$ and hence $gp \in J'$. Now by Proposition 3.5, G respects the $\mathscr{J}$-classes and thus $g^{-1}J' = J$. Therefore $GE_u = F$.

(2) By Proposition 3.4 one can define a bijection $\delta\colon E_u \to \mathrm{Mor}(u, u)$ by setting $J\delta = J \cap \mathrm{Mor}(u, u)$. Moreover for every $J_1, J_2 \in E_u$, $J_1 \leqslant J_2$ iff $J_1\delta \leqslant J_2\delta$ in $\mathrm{Mor}(u, u)$. Thus δ is a semilattice isomorphism. Finally, it follows from Proposition 3.11 that $\mathrm{Mor}(u, u)$ is isomorphic to $E(C/G)$.

(3) Let $J \in E_u$ and let $J' \leqslant J$. Since C is regular and $\mathscr{J} = \mathscr{D}$ on C by Proposition 3.1, J' contains an idempotent p'. Then $p' \in \mathrm{Mor}(v, v)$ for some $v \in \mathrm{Ob}(C)$. Let $p \in J \cap \mathrm{Mor}(u, u)$. Since $p' \leqslant p$ there exist $x, y \in \mathrm{Mor}(C)$ such that $p' = x + p + y$. It follows that $x \in \mathrm{Mor}(v, u)$ and $y \in \mathrm{Mor}(u, v)$. Now since $x + p + (y + p' + x) + p + y = p'$, we have $p' \mathrel{\mathscr{J}} y + p' + x$. But $y + p' + x \in \mathrm{Mor}(u, u)$ and thus $J' \in E_u$. ∎

We now prove the claim. Let $\gamma\colon C_u \to P = P(G, F, E_u)$ be the function defined by $(p, g)\gamma = (J, g)$, where J is the $\mathscr{J}$-class of p. First, (J, g) is effectively in P. Indeed, since $p + \bar{p}$ and $\bar{p} + p$ are two idempotents in the same $\mathscr{D}$-class as p, we have $\bar{p} + p \in \mathrm{Mor}(gu, gu) \cap J$ and thus $g^{-1}(\bar{p} + p) \in g^{-1}J \cap \mathrm{Mor}(u, u)$. Therefore $g^{-1}J \in E_u$ and $(J, g) \in P$.

γ is surjective. Indeed if $(J, g) \in P$, there exist by definition two idempotents $p \in J \cap \mathrm{Mor}(u, u)$ and $q \in J \cap \mathrm{Mor}(gu, gu)$. Therefore by Proposition 3.3 there is a (unique) $r \in \mathrm{Mor}(u, gu)$ such that $p = r + \bar{r}$ and $q = \bar{r} + r$. It follows that $(r, g)\gamma = (J, g)$.

γ is injective. Indeed if $(p_1, g_1)\gamma = (p_2, g_2)\gamma = (J, g)$ then $g_1 = g_2 = g$ and thus p_1 and p_2 are two $\mathscr{J}$-equivalent elements of $\mathrm{Mor}(u, gu)$. Therefore $p_1 = p_2$ by Proposition 3.4.

Finally, we show that γ is a morphism. Let $(p_1, g_1)\gamma = (J_1, g_1)$, $(p_2, g_2)\gamma = (J_2, g_2)$ and let $(p_1, g_1)(p_2, g_2) = (p, g)$. Then $(p, g)\gamma = (J, g)$, where J is the $\mathscr{J}$-class of p. Since $p = p_1 + g_1 p_2$ we have $p \leqslant p_1$ and $p \leqslant g_1 p_2$ and hence $J \leqslant J_1$ and $J \leqslant g_1 J_2$. Assume that $J' \leqslant J_1$ and $J' \leqslant g_1 J_2$ for some $J' \in F$. Then since $J_2 \in E_u$ we have $g_1 J_2 \in E_{g_1} u$ and thus $J' \in E_{g_1} u$ because $E_{g_1} u$ is an order ideal of F. Therefore there exists $q \in J' \cap \mathrm{Mor}(g_1 u, g_1 u)$ such that $q \leqslant p_1$ and $q \leqslant g_1 p_2$. But $p_1 \mathscr{J} (\bar{p}_1 + p_1)$ and $g_1 p_2 \mathscr{J} (g_1 p_2 + g_1 \bar{p}_2)$. Thus $q \leqslant \bar{p}_1 + p_1$ and $q \leqslant g_1 p_2 + \bar{g}_1 \bar{p}_2$. Since $q, \bar{p}_1 + p_1$ and $g_1 p_2 + g_1 \bar{p}_2$ are elements of the semilattice $\mathrm{Mor}(g_1 u, g_1 u)$ it follows that $q \leqslant \bar{p}_1 + p_1 + g_1 p_2 + g_1 \bar{p}_2$. On the other hand, $p_1 + g_1 p_2 \mathscr{J} \bar{p}_1 + p_1 + g_1 p_2 + \bar{g}_1 \bar{p}_2$ since $p_1 + (\bar{p}_1 + p_1 + g_1 p_2 + g_1 \bar{p}_2) + g_1 p_2 = p_1 + g_1 p_2$ and thus $q \leqslant p_1 + g_1 p_2$. Therefore $J' \leqslant J$. Consequently $J = J_1 \wedge g_1 J_2$ and γ is a morphism. ∎

Part (4) of Theorem 2.3 can now be made more precise through the following result.

THEOREM 4.4. *Let G be a group acting transitively without fixpoints on a connected idempotent, commutative inverse category C and let S be the semilattice of ideals of C under intersection. Then the inverse monoid C/G is isomorphic to a subsemigroup of a semidirect product $S * G$.*

Proof. We first define an action of G on S by setting, for all ideals I of C, $gI = \{gp \mid p \in I\}$. It is not difficult to see that this action defines a semidirect product $S * G$.

Let $\beta: C_u \to S * G$ be the function defined by $(p, g)\beta = (I, g)$, where I is the ideal generated by p. Then β is injective. Indeed if $(p_1, g_1)\beta = (p_2, g_2)\beta = (I, g)$ then $g_1 = g_2 = g$ and $p_1 \mathscr{J} p_2$. Therefore $p_1 = p_2$ by Proposition 3.4.

Now if $(p_1, g_1)\beta = (I_1, g_1)$ and $(p_2, g_2)\beta = (I_2, g_2)$ we have $((p_1, g_1)(p_2, g_2))\beta = (p_1 + g_1 p_2, g_1 g_2)\beta = (I, g_1 g_2)$, where I is the ideal generated by $p_1 + g_1 p_2$. We claim that $I = I_1 \cap g_1 I_2$. Clearly $p_1 + g_1 p_2 \in I_1 \cap g_1 I_2$ and thus $I \subset I_1 \cap g_1 I_2$. Conversely let $q \in I_1 \cap g_1 I_2$. Then $q \leqslant p_1$, $q \leqslant g_1 p_2$ and it follows from the proof of Theorem 4.2 that $q \leqslant p_1 + g_1 p_2$. Thus $q \in I$ and the claim holds. Therefore β is an injective morphism and since C/G is isomorphic to C_u by Proposition 3.11, the theorem is proved. ∎

COROLLARY 4.5. *Let M be an inverse monoid and let $\phi: M \to G$ be a surjective morphism onto a group such that $1\phi^{-1} = E(M)$. Let C be the derived category of ϕ. Then M is isomorphic to a subsemigroup of a semidirect product $S * G$, where S is the semilattice of ideals of C under intersection.*

We remark here that it can be shown that an E-unitary monoid M is isomorphic to a semidirect product of a semilattice by a group iff the derived category of the morphism $\phi: M \to G$ is equivalent, in the sense of category theory [13] to a semilattice. We omit the proof.

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