

Expansions, Free Inverse Semigroups, and Schützenberger Product

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In this paper we shall present a new construction of the free inverse monoid on a set X . Contrary to the previous constructions of [9, 11], our construction is symmetric and originates from classical ideas of language theory. The ingredients of this construction are the free group on X and the relation that associates to a word w of the free monoid on X , the set of all pairs (u, v) such that $uv = w$. It follows at once from our construction that the free inverse monoid on X can be naturally embedded into the Schützenberger product of two free groups of basis X . We shall also give some connections with the theory of expansions as developed by Rhodes and Birget [2, 3]. © 1987 Academic Press, Inc

1. THE SCHÜTZENBERGER PRODUCT

The Schützenberger product of two monoids was introduced by Schützenberger in view of applications to language theory. Given two monoids M_1 and M_2 , the Schützenberger product $\diamond(M_1, M_2)$ is defined as follows. Let $M = M_1 \times M_2$. Then $\mathcal{P}(M)$, the set of finite subsets of M , is a semiring with union as addition and the usual multiplication of subsets as multiplication. Then $\diamond(M_1, M_2)$ is the set of all 2×2 matrices P over this semiring satisfying the following conditions:

- (1) $P_{21} = \emptyset$, that is, P is upper triangular.
- (2) $P_{11} = \{(m_1, 1)\}$ for some $m_1 \in M_1$.
- (3) $P_{22} = \{(1, m_2)\}$ for some $m_2 \in M_2$.

The multiplication is the usual multiplication of matrices over a semiring.

Conditions (2) and (3) above show that an element of $\diamond(M_1, M_2)$ can be represented by matrices of the form

$$\begin{pmatrix} m_1 & Q \\ \emptyset & m_2 \end{pmatrix},$$

where $m_1 \in M_1, m_2 \in M_2$, and $Q \subset M_1 \times M_2$. If we denote union by $+$ (and the empty set by \emptyset) the multiplication of two such matrices is given by

$$\begin{pmatrix} m_1 & Q \\ \emptyset & m_2 \end{pmatrix} \begin{pmatrix} n_1 & R \\ \emptyset & n_2 \end{pmatrix} = \begin{pmatrix} m_1 n_1 & m_1 R + Q n_2 \\ \emptyset & m_2 n_2 \end{pmatrix},$$

where $m_1 R = \{(m_1 x_1, x_2) | (x_1, x_2) \in R\}$ and $Q n_2 = \{(x_1, x_2 m_2) | (x_1, x_2) \in Q\}$.

Here is a first connection between Schützenberger products and inverse monoids.

PROPOSITION 1.1. *Let G and H be groups. Then $\diamond(G, H)$ is an inverse monoid.*

Proof. Let $P = \begin{pmatrix} g & X \\ \emptyset & h \end{pmatrix}$ and $\bar{P} = \begin{pmatrix} \bar{g} & \bar{X} \\ \emptyset & \bar{h} \end{pmatrix}$. Then $P\bar{P}P = P$ and $\bar{P}P\bar{P} = \bar{P}$ if and only if $\bar{g} = g^{-1}, \bar{h} = h^{-1}, X + g\bar{X}h = X$, and $\bar{X} + \bar{g}X\bar{h} = \bar{X}$. The last two equations are equivalent to $\bar{X} = g^{-1}Xh^{-1}$ since $g\bar{X}h \subset X$ and $g^{-1}Xh^{-1} \subset \bar{X}$ imply $X = g(g^{-1}Xh^{-1})h \subset g\bar{X}h \subset X$. It follows that

$$\bar{P} = \begin{pmatrix} g^{-1} & g^{-1}Xh^{-1} \\ \emptyset & h^{-1} \end{pmatrix}$$

is the unique inverse of P . Thus $\diamond(G, H)$ is an inverse monoid. ■

This last result is also a consequence of the following proposition since it is well known that a semidirect product of a group by a semilattice is inverse.

PROPOSITION 1.2. *Let G and H be groups. Then $\diamond(G, H)$ is isomorphic to a semidirect product $S * (G \times H)$, where S is the semilattice of subsets of $G \times H$ under union.*

Proof. If $X \in S$ and $(g, h) \in G \times H$, set

$$(g, h)X = gXh^{-1} = \{(gx, yh^{-1}) | (x, y) \in X\}.$$

This defines a left action of $G \times H$ on S and we can form the semidirect product $S * (G \times H)$. Define a function

$$\varphi: S * (G \times H) \rightarrow \diamond(G, H)$$

by

$$(X, (g, h)) \varphi = \begin{pmatrix} g & Xh \\ 0 & h \end{pmatrix}.$$

It is bijective since

$$(Xh^{-1}, (g, h)) \varphi = \begin{pmatrix} g & X \\ 0 & h \end{pmatrix}.$$

Now a simple verification shows that φ is a morphism. Thus $\diamond(G, H)$ is isomorphic to $S * (G \times H)$. ■

COROLLARY 1.3. *If G and H are groups, $\diamond(G, H)$ is isomorphic to a reverse semidirect product $(G \times H) *_r S$.*

Proof. Indeed, if S is a semigroup and G is a group, every semidirect product $S * G$ is isomorphic to a reverse semidirect product $G *_r S$ and vice versa. See [8], for instance. ■

2. THE FREE INVERSE MONOID

In this paper, we view an inverse monoid M as a monoid with an involution, denoted by “ $\bar{}$ ”, satisfying the axioms

- (1) $x\bar{x}x = x$,
- (2) $(\overline{\bar{x}y}) = y\bar{x}$,
- (3) $\bar{\bar{x}} = x$,
- (4) $x\bar{x}y\bar{y} = y\bar{y}x\bar{x}$.

Axioms (1), (2), and (3) say that x is an inverse of x and (4) says that idempotents commute. If S is a subset of M , the inverse submonoid of M generated by S is the smallest submonoid of M containing $S \cup \bar{S}$, where $\bar{S} = \{\bar{s} | s \in S\}$. This is also the smallest inverse monoid containing S .

Let X be a set. We denote by $F(X)$ the free group on X and by $FI(X)$ the free inverse semigroup on X . $F(X)$ can be constructed as follows. Let \bar{X} be a disjoint copy of X and let $A = X \cup \bar{X}$. Then $F(X)$ is the quotient of the free monoid A^* by the relations $x\bar{x} = \bar{x}x = 1$, for all $x \in X$. In the sequel we shall denote by $\varphi: A^* \rightarrow F(X)$ the canonical morphism. A word of A^* is reduced if it does not contain a factor of the form $x\bar{x}$ or $\bar{x}x$ with $x \in X$. Notice that a factor of a reduced word is also reduced. Since the restriction of φ to the set of all reduced words is one-to-one, we shall identify $F(X)$ with the set of all reduced words of A^* .

For the free inverse monoid, a number of constructions are known [9–11]. Let us briefly review the construction of [10, 11]. A subset S of $F(X)$ is called a Schreier system [4, 5] if for all words u and v of A^* , $uv \in S$ implies $u \in S$. (In the terminology of [10] this is a closed subset of $F(X)$.) The next proposition is a slightly modified version of Theorem 4.2 of [9].

PROPOSITION 2.1. *If S and T are two Schreier systems and if g is an element of S , then $S \cup gT$ is a Schreier system.*

It follows that the set

$$M = \{(S, g) \mid S \text{ is a finite Schreier system and } g \in S\}$$

is a monoid under the multiplication

$$(S, g)(T, h) = (S \cup gT, gh).$$

In fact M is the free inverse monoid on X [9]. The inverse of (S, g) is $(g^{-1}S, g^{-1})$. Furthermore, M is generated (as an inverse monoid) by the set

$$\{(\{1, x\}, x) \mid x \in X\}.$$

Let A be an alphabet and let $\psi: A^* \rightarrow M$ be a surjective morphism onto a monoid M . We now define the 2-expansion of M relative to ψ . It was first introduced by K. Henckell (private communication) and considered in more detail by Birget [2].

Let $\tau: A^* \rightarrow \mathcal{P}(M \times M)$ be the function defined by

$$w\tau = \{(u\psi, v\psi) \mid uv = w\}.$$

In the terminology of language theory [1], it is called a transduction from A^* to $M \times M$. Notice that $u\tau = v\tau$ implies $u\psi = v\psi$. Moreover

LEMMA 2.2. *The equivalence \sim_τ defined on A^* by $u \sim_\tau v$ iff $u\tau = v\tau$ is a congruence on A^* .*

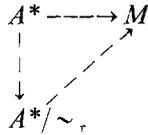
Proof. By duality it is sufficient to prove that \sim_τ is a right congruence. Let $u, v \in A^*$ be such that $u\tau = v\tau$ and let $w \in A^*$. Let $(m_1, m_2) \in (uw)\tau$. Then there exists a factorization $uw = x_1x_2$ such that $x_1\psi = m_1$ and $x_2\psi = m_2$. Now one of the following cases hold.

- (1) There exists a word $u' \in A^*$ such that $u = x_1u'$ and $u'w = x_2$.
- (2) There exists a word $w' \in A^*$ such that $x_1 = uw'$ and $w = w'x_2$.

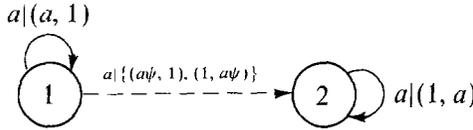
In the first case, $(x_1\psi, u'\psi) \in u\tau = v\tau$ and hence there exist $v_1, v_2 \in A^*$ such that $v_1v_2 = v$ and $v_1\psi = x_1\psi = m_1$ and $u'\psi = v_2\psi$. It follows that

$(v_1\psi, (v_2w)\psi) \in (vw)\tau$. But $v_1\psi = m_1$ and $(v_2w)\psi = (v_2\psi)(w\psi) = (u'\psi)(w\psi) = (u'w)\psi = x_2\psi = m_2$ and hence $(m_1, m_2) \in (vw)\tau$. In the second case, $((vw')\psi, x_2\psi) \in (vw)\tau$. But $(vw')\psi = (v\psi)(w'\psi) = (u\psi)(w'\psi) = x_1\psi = m_1$ and $x_2\psi = m_2$. Therefore $(m_1, m_2) \in (vw)\tau$. Thus $(uw)\tau \subset (vw)\tau$ and similarly $(vw)\tau = (uw)\tau$. It follows that \sim_τ is a right congruence. ■

By definition, A^*/\sim_τ is the 2-expansion of M relative to ψ , denoted by $M(2, \psi)$. Since $u \sim_\tau v$ implies $u\psi = v\psi$, there is a surjective morphism $A^*/\sim_\tau \rightarrow M$ and we have the commutative diagram



In fact there is a much better description of $M(2, \psi)$ using the matrix representation of τ . (The reader is referred to [1, Chap. 3] for the basic definitions on transductions.) A transducer realizing τ is represented by



for all $a \in A$.

The matrix representation of this transducer is the monoid morphism

$$\mu: A^* \rightarrow \mathcal{P}(M \times M)^{2 \times 2}$$

defined by

$$w\mu = \begin{pmatrix} \{(w\psi, 1)\} & w\tau \\ 0 & \{(1, w\psi)\} \end{pmatrix}.$$

We claim that $M(2, \psi) = A^*\mu$. Indeed if $u\mu = v\mu$ then $(u\mu)_{12} = (v\mu)_{12}$, that is, $u\tau = v\tau$. Conversely if $u\tau = v\tau$, $u\psi = v\psi$ and thus $u\tau = v\tau$.

It follows that one can identify $M(2, \psi)$ with a submonoid of $\diamond(M, M)$. Thus elements of $M(2, \psi)$ have the form

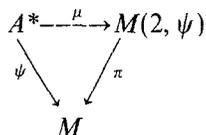
$$\begin{pmatrix} m & X \\ 0 & m \end{pmatrix},$$

where X is a finite subset of $M \times M$. However, we do not claim that all

matrices of this form are in $M(2, \psi)$. Since ψ is onto, there is a natural surjective morphism $\pi: M(2, \psi) \rightarrow M$ defined by

$$\begin{pmatrix} m & X \\ 0 & m \end{pmatrix} \pi = m.$$

Clearly the following diagram is commutative:



We come back to the case where $A = X \cup \bar{X}$ and $\psi = \varphi: A^* \rightarrow F(X)$ is the natural morphism onto the free group on X . Then we have

PROPOSITION 2.3. *The 2-expansion of $F(X)$ relative to φ is an inverse monoid.*

Proof. Let N be the 2-expansion of $F(X)$ relative to φ . Then N is a submonoid of $\diamond(F(X), F(X))$, which is inverse by Proposition 1.1. We need only show that every generator of N has an inverse in N . N is generated as a monoid by $\{a\mu | a \in A\}$. Set

$$\bar{a} = \begin{cases} \bar{x} & \text{if } a = x \in X \\ x & \text{if } a = \bar{x} \in \bar{X}. \end{cases}$$

Then we have

$$\begin{aligned} (a\tau)(\bar{a}\tau)(a\tau) &= (a\bar{a}a)\tau \\ &= \{(1\varphi, (a\bar{a}a)\varphi), (a\varphi, (\bar{a}a)\varphi), ((a\bar{a})\varphi, a\varphi), ((a\bar{a}a)\varphi, 1\varphi)\} \\ &= \{(1, a\varphi), (a\varphi, 1)\} = a\tau \end{aligned}$$

and similarly $(\bar{a}a\bar{a})\tau = \bar{a}\tau$. It follows that $\bar{a}\tau$ is an inverse of $a\tau$. ■

We are now ready to state our main result.

THEOREM 2.4. *$FI(X)$ is isomorphic to the 2-expansion of $F(X)$ relative to φ .*

Proof. Since N is an inverse monoid generated—as an inverse monoid—by $\{x\tau | x \in X\}$, it suffices to prove that there is a surjective morphism $N \rightarrow FI(X)$. We use the representation of $FI(X)$ given above.

First define a function $\gamma: A^* \rightarrow \mathcal{P}(F(X))$ by

$$u\gamma = \{v\varphi \mid v \in A^* \text{ and there exists } w \in A^* \text{ such that } vw = u\}.$$

Thus $u\gamma$ is the set of left factors of the reduced word $v\varphi$, where v is a left factor of u . It follows that $u\gamma$ is a finite Schreier system such that $u\varphi \in u\gamma$.

Let $\theta: A^* \rightarrow FI(X)$ be the function defined by

$$u\theta = (u\gamma, u\varphi).$$

Then θ is a morphism since for all $u \in A^*$ and for all $a \in A$

$$\begin{aligned} (u\theta)(a\theta) &= (u\gamma, u\varphi)(a\gamma, a\varphi) = (u\gamma \cup (u\varphi)(a\gamma), (u\varphi)(a\varphi)) \\ &= (u\gamma \cup (u\varphi)\{1, a\varphi\}, (ua)\varphi) \\ &= (u\gamma \cup \{(ua)\varphi\}, (ua)\varphi) \quad (\text{since } u\varphi \in u\gamma) \\ &= ((ua)\gamma, (ua)\varphi) \\ &= (ua)\theta. \end{aligned}$$

Moreover θ is surjective since $FI(X)$ is generated (as a monoid) by the set $\{a\theta \mid a \in A\}$. ■

COROLLARY 5.3. *$FI(X)$ is a submonoid of $\diamond(F(X), F(X))$.*

Proof. Follows immediately from Theorem 5.2. ■

Corollary 5.3 gives a two-sided construction of $FI(X)$. In fact Theorem 5.2 is just a proof that this two-sided construction is equivalent to an oriented construction based on left factors of words in A^* . Just to cover all political bases, we mention that a construction based on right factors is also possible. These correspond to embeddings of $FI(X)$ into semidirect and reverse semidirect products of $F(X)$ with an appropriate semilattice. We also remark that the notion of a left (right) factor expansion of a morphism can be refined to produce the Rhodes expansion, which plays a prominent role in finite semigroup theory and has recently been generalized to all semigroups. Finally, let us mention the existence of an n -expansion of a morphism $\varphi: A^* \rightarrow M$. It can be embedded into the n -fold Schützenberger product $\diamond(M, \dots, M)$ as defined in [12]. See [2, 3] for further details.

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