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An Upper Bound for the Complexity of Transformation Semigroups

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INTRODUCTION

A transformation semigroup (ts) $X = (Q_X, S_X)$ in this paper consists of a finite set of states Q_X and a subsemigroup of transformations S_X of $\mathrm{PF}(Q_X)$, the monoid of all *partial* functions on Q_X with composition as multiplication.

For $n \geqslant 1$, $\bar{\mathbf{n}}^{\cdot}$ denotes the transformation monoid (tm) with n states and with the n constant maps on those n states (along with the identity function) as transformations.

The main theorem (Theorem 2.2) of this paper shows that if a ts X does not contain a copy of $\bar{\mathbf{n}}^{\cdot}$ as a sub-ts, then the complexity of X is less than n.

Recall that a ts X has complexity less than or equal to n (written $Xc \leqslant n$) iff

$$X \prec A_0 \circ G_1 \circ A_1 \circ \cdots \circ A_{n-1} \circ G_n \circ A_n, \tag{0.1}$$

where the A_i are aperiodic ts's, the G_i are groups, and $\prec$ and $\circ$ denote division and wreath product, respectively. The reader is refered to [1, 4] for a complete exposition of this subject. We will follow the notation of these references.

Let $X = (Q, S)$ be a ts. Then $X^{\cdot}$ denotes the transformation monoid (tm) obtained from X by adjoining to S (if necessary) the identity transformation on Q.

For each $q \in Q$, we denote by $\tilde{q}\colon Q \to Q$ the constant function with value q

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(i.e., $q'\tilde{q} = q$ for all $q' \in Q$). By $\bar{X}$ we mean the ts obtained from X by adjoining to S all the constant maps on Q.

Let n be a non-negative integer. By $\mathbf{n}$ we mean the ts with states $\{0, 1, \ldots, n-1\}$ and with no transformations. We then obtain the ts's

$$\mathbf{n}^{\boldsymbol{\cdot}}, \ \bar{\mathbf{n}}, \text{ and } \bar{\mathbf{n}}^{\boldsymbol{\cdot}}.$$

Let χ be a collection of ts's. Then $[\chi]$ is the smallest collection of ts's containing χ that is closed under division and wreath products, and $\langle\chi\rangle = \{X \mid Y \nmid X \text{ for all } Y \in \chi\}$.

Let X and Y be ts's. Then we say X is Y-free if $Y \nmid X$, i.e., $X \in \langle Y \rangle$.

Then, for example, $\langle \bar{\mathbf{2}}^{\boldsymbol{\cdot}} \rangle$ denotes the collection of $\bar{\mathbf{2}}^{\boldsymbol{\cdot}}$-free ts's.

The importance of the tm $\bar{\mathbf{2}}^{\boldsymbol{\cdot}}$ is exhibited by the following theorems:

$$[\bar{\mathbf{2}}^{\boldsymbol{\cdot}}, \langle \bar{\mathbf{2}}^{\boldsymbol{\cdot}} \rangle] = \mathbf{TS} \text{ (all ts's).} \tag{0.2}$$

$$[\bar{\mathbf{2}}^{\boldsymbol{\cdot}}] = \text{all aperiodic (group-free) ts's.} \tag{0.3}$$

Also, it is shown in [4] that the groups in expression (0.1) can be replaced with $\bar{\mathbf{2}}^{\boldsymbol{\cdot}}$-free ts's with no resulting change in the complexity function. Consequently, if X is $\bar{\mathbf{2}}^{\boldsymbol{\cdot}}$-free, then $Xc < 2$. The main theorem, then, generalizes this fact to all $n \geqslant 1$.

Section I introduces the notions of inseparable ts's and semilocal classes of ts's, and shows their connection. Section III gives necessary and sufficient conditions for a ts to be inseparable. The results of Section III are needed in Section II, but for clarity of presentation we have chosen this arrangement.

The main theorem is proved in Section II. The proof uses the following result, due to Margolis [2].

A partial function $s: Q \to Q$ is called a *k-map* if

$$\operatorname{card} qs^{-1} \leqslant k \qquad \text{for all} \quad q \in Q.$$

A ts X is a k-ts if every transformation of X is a k-map. Define the function $\tau: \mathbf{TS} \to \mathbf{N}$ by

$$X\tau = \inf\{k \mid X \text{ is a } k\text{-ts}\}.$$

Thus $X\tau \leqslant k$ iff at most k states can be mapped to one state under the action of any transformation of X.

THEOREM 0.1. *For any* ts X,

$$Xc \leqslant X\tau.$$

In other words, τ is an upper bound to complexity.

Section IV presents a corresponding theorem for relational morphisms of semigroups.

This paper assumes a working knowledge of the derived ts of a relational cover. For the convenience of the reader, an appendix has been provided expositing this theory. Complete details can be found in [1].

I. Semilocal Classes

A family $\mathbf{X}$ of ts's is called a *class* if it is closed under division, i.e., $Y \prec X$ and $X \in \mathbf{X}$ imply $Y \in \mathbf{X}$. If χ is a collection of ts's, then $[\chi]$ and (χ) are classes.

If $\mathbf{X}$ and $\mathbf{Y}$ are classes, then we define the class

$$\mathbf{X} \circ \mathbf{Y} = \{Z \mid Z \prec X \circ Y, X \in \mathbf{X}, Y \in \mathbf{Y}\}.$$

If $\mathbf{1}^{\bullet} \in \mathbf{X}$, then $\mathbf{Y} \subset \mathbf{X} \circ \mathbf{Y}$; if $\mathbf{1}^{\bullet} \in \mathbf{Y}$, then $\mathbf{X} \subset \mathbf{X} \circ \mathbf{Y}$.

It is useful to extend the notion of complexity to classes. If $\mathbf{X}$ is a class of ts's, then

$$\mathbf{X}c = \sup\{Xc \mid X \in \mathbf{X}\}.$$

Of course, if there is no bound on the complexity of members of $\mathbf{X}$, then $\mathbf{X}c = \infty$.

If $\mathbf{X}$ and $\mathbf{Y}$ are classes, it readily follows that

$$\mathbf{X} \subset \mathbf{Y} \Rightarrow \mathbf{X}c \leqslant \mathbf{Y}c \tag{1.1}$$

and

$$(\mathbf{X} \circ \mathbf{Y})c \leqslant \mathbf{X}c + \mathbf{Y}c. \tag{1.2}$$

Let $\mathbf{X}$ be a class of ts's. The *localization* of $\mathbf{X}$, denoted $\mathbf{LX}$, is the largest class $\mathbf{Y}$ such that

$$Z \in \mathbf{Y}, \qquad Z \text{ a tm} \Rightarrow Z \in \mathbf{X},$$

i.e.,

$$\mathbf{Y} \cap \mathbf{TM} \subset \mathbf{X}$$

Equivalently,

$$\mathbf{LX} = \{X \mid X_e \in \mathbf{X} \text{ for all idempotents } e \text{ in } X\},$$

where X_e is the tm (Qe, eSe) in X. $\mathbf{LX}$ is a class, and $\mathbf{X} \subset \mathbf{LX} = \mathbf{LLX}$. A class $\mathbf{X}$ is *local* if $\mathbf{LX} = \mathbf{X}$.

It is shown in [1, Exercise IV, 7.2] that

$$\mathbf{X} \circ [\bar{\mathbf{2}}] \subseteq \mathbf{LX} \tag{1.3}$$

for any class $\mathbf{X}$ containing $\mathbf{1}^{\cdot}$. The question of what conditions on $\mathbf{X}$ force (1.3) to be an equality is unresolved; there are examples both ways.

Consider the main series of prime ts's (see Chapter IV of [1])

$$\mathbf{0}, \quad \mathbf{1}, \quad \mathbf{2}, \quad E, \quad C, \quad F, \quad \bar{\mathbf{2}}.$$

If $\mathbf{X} = \langle \mathbf{2} \rangle, \langle C \rangle, \langle F \rangle$ or $\langle \bar{\mathbf{2}} \rangle$, then equality holds in (1.3). If, however, $\mathbf{X} = \langle E \rangle$, the inequality (1.3) is strict. $\mathbf{1}^{\cdot} \notin \langle \mathbf{0} \rangle$ and $\langle \mathbf{1} \rangle$, so (1.3) does not apply.

Another case where the inequality (1.3) is strict is when $\mathbf{X}$ is the class of all ts's with complexity less than or equal to one (see [3]).

We shall call a class $\mathbf{X}$ of ts's *semilocal* if

$$\mathbf{LX} \subseteq \mathbf{X} \circ [\bar{\mathbf{2}}].$$

A local class is semilocal, but the reverse, of course, is not true.

Let $\mathbf{X}$ be a semilocal class. Then

$$\mathbf{X} \subset \mathbf{LX} \subset \mathbf{X} \circ [\bar{\mathbf{2}}].$$

Since $[\bar{\mathbf{2}}]c = 0$, it follows from (1.1) and (1.2) that

$$\mathbf{LX}c = \mathbf{X}c. \tag{1.4}$$

We now proceed to identify a large number of semilocal classes. In what follows we assume familiarity with relational covers, traces, and derived ts's. The reader is refered to the Appendix or to Chapter III of [1] for details.

A ts X is *inseparable* if whenever X divides a derived ts Φ, then X divides one of the traces of Φ.

In Section III, we will completely classify all inseparable ts's. It will be shown that among the inseparable ts's are all tm's and all complete ts's X such that $S_X \neq \varnothing$. The next theorem motivates the definition of inseparable ts's.

THEOREM 1.1. *Let* χ *be a collection of inseparable* ts's. *If* $\mathbf{1}^{\cdot} \in \langle \chi \rangle$, *then* $\langle \chi \rangle$ *is semilocal.*

Proof. Since $\mathbf{1}^{\cdot} \in \langle \chi \rangle$, $[\bar{\mathbf{2}}] \subseteq \langle \chi \rangle \circ [\bar{\mathbf{2}}]$. Let $X \in \mathbf{L}\langle \chi \rangle$. If $S_X = \varnothing$, then $X \approx \mathbf{n}$, where $n = \text{card } Q_X$. Therefore $X \approx \mathbf{n} \in [\bar{\mathbf{2}}] \subseteq \langle \chi \rangle \circ [\bar{\mathbf{2}}]$.

Now suppose that $S_X \neq \varnothing$. Then the Trace–Delay theorem (Theorem A2 in the Appendix) guarantees the existence of a relational cover

$$X \lhd_{\varphi} Y$$

and a parametrization of φ such that

(i) $Y \in [\bar{\mathbf{2}}]$,

(ii) for each $p \in Q_Y$, there is an idempotent e in X such that $\mathrm{Tr}_p \prec X_e$.

Since $X \in \mathbf{L}(\chi)$, condition (ii) implies

$$\mathrm{Tr}_p \in (\chi) \qquad \text{for each} \quad p \in Q_Y.$$

Theorem A1 yields

$$X \prec \Phi \circ Y,$$

where Φ is the derived ts of φ. Since $Y \in [\bar{\mathbf{2}}]$, it suffices to show that $\Phi \in (\chi)$.

If $Z \in \chi$ and $Z \prec \Phi$, then Z must divide Tr_p for some $p \in Q_Y$, for Z is inseparable. But $\mathrm{Tr}_p \in (\chi)$, so this is impossible. Therefore $\Phi \in (\chi)$ and $X \in (\chi) \circ [\bar{\mathbf{2}}]$. Therefore

$$\mathbf{L}(\chi) \subset (\chi) \circ [\bar{\mathbf{2}}]$$

and (χ) is semilocal. ∎

It should be noted that if χ is a collection of ts's satisfying the hypothesis of Theorem 1.1 then

$$\mathbf{L}(\chi) = (\chi) \circ [\bar{\mathbf{2}}].$$

II. The Class $(\bar{\mathbf{n}}^{\boldsymbol{\cdot}})$

We now present the main theorem of this paper. Its proof depends upon

Proposition 2.1. *For each $n \geqslant 1$, $\bar{\mathbf{n}}$ is inseparable.*

The proof of Proposition 2.1 is fairly easy, but rather than writing it out here we prefer to present a complete classification of inseparable ts's in the next section from which Proposition 2.1 follows. The reader may prefer to develop his own proof of Proposition 2.1 and skip Section III.

Theorem 2.2. *Let $n \geqslant 1$. If $X \in (\bar{\mathbf{n}}^{\boldsymbol{\cdot}})$, then $Xc < n$. In other words*

$$(\bar{\mathbf{n}}^{\boldsymbol{\cdot}})c < n.$$

Proof. Since $\bar{\mathbf{n}}$ is inseparable, (1.4) and Theorem 1.1 imply

$$\mathbf{L}(\bar{\mathbf{n}})c = (\bar{\mathbf{n}})c.$$

It is well known that $\mathbf{L}\langle\bar{\mathbf{n}}\rangle = \langle\bar{\mathbf{n}}^{\bullet}\rangle$ (e.g., see Proposition IV, 1.1 of [1]). Thus we obtain

$$\langle\bar{\mathbf{n}}^{\bullet}\rangle c = \langle\bar{\mathbf{n}}\rangle c.$$

Therefore it suffices to prove $\langle\bar{\mathbf{n}}\rangle c < n$. This we will do in the following series of propositions.

Let $X = (Q, S)$ be a ts. A non-empty set R of Q is *transitive* if for any two distinct states $q_1, q_2 \in R$ we have $q_1 \in q_2 S$. The maximal transitive subsets of Q are called the *transitivity* (or $\mathscr{R}$) *classes* of X.

If Q itself is transitive, we say X is a *transitive* ts.

PROPOSITION 2.3. *Let X be a* ts. *Then there exists a transitive* ts Y *satisfying*

(i) $Y \subset X$,

(ii) $Yc = Xc$.

Proof. Let R be a transitivity class of X. Define the function $\theta_R : S \to PF(R)$ by

$$\begin{aligned} q(s\theta_R) &= qs \qquad \text{if} \quad qs \in R \\ &= \varnothing \qquad \text{otherwise.} \end{aligned}$$

Since q and $qs_1 s_2$ being in R implies $qs_1 \in R$, it easily follows that θ_R is a homomorphism. There results the transitive ts $X_R = (R, S_R)$, where $S_R = S\theta_R$. Clearly $X_R \subset X$, and hence $X_R c \leqslant Xc$.

It thus suffices to show that $X_R c \geqslant Xc$ for some transitivity class R of X. Since the complexity of a ts (Q, S) is the same as that of S, we shall show that $S_R c \geqslant Sc$ for some R.

Let $R_1, \ldots, R_n$ be the transitivity classes of X and define the product morphism

$$\theta : S \to \prod_{i=1}^{n} S_{R_i},$$

$$s\theta = (s\theta_1, \ldots, s\theta_n) \qquad \text{where} \quad \theta_i = \theta_{R_i}.$$

Let $T = S\theta$. Then T is a subsemigroup of the direct product of the S_{R_i}. Consequently (applying basic properties of complexity),

$$Tc \leqslant \max\{S_{R_i} c : i = 1, \ldots, n\}.$$

Thus for some R we have $Tc \leqslant S_R c$.

Now we will show that θ is an aperiodic morphism. The Fundamental Lemma of Complexity then yields $Sc \leqslant Tc$ and subsequently, $Sc \leqslant S_R c$ for some R.

To show that θ is aperiodic, we need to prove that θ is injective on groups in S. Let G be a group in S with identity e and let $g \in G$ with $g \neq e$. Then there is a state $q \in Q$ such that $qe \neq qg$. But qe and qg are in the same transitivity class, say R, because $(qe)g = qg$ and $(qg)g^{-1} = qe$. Since $(qe)e \neq (qe)g$, it follows that $e\theta_R \neq g\theta_R$ and, of necessity, $e\theta \neq g\theta$. θ is therefore injective on groups in S, and the assertion is established. ∎

Recall that if $X = (Q, S)$ is a ts, then

$$X\tau \leqslant k \qquad \text{iff each transformation of } X \text{ is a } k\text{-map.}$$

A transformation $s \in S$ is a k-map if

$$\operatorname{card} qs^{-1} \leqslant k \qquad \text{for all} \quad q \in Q.$$

If $\bar{\mathbf{n}} \prec X$, then $\bar{\mathbf{n}} \subset X$, and if $\bar{\mathbf{n}}$ is contained in X, then clearly $X\tau \geqslant n$. The converse does not hold, however, unless X is transitive.

Proposition 2.4. *Let X be transitive. Then $X\tau \geqslant n$ iff $\bar{\mathbf{n}} \subset X$.*

Proof. Let $X = (Q, S)$. If $X\tau \geqslant n$, then there are n distinct states $q_1, \ldots, q_n \in Q$, an element $s \in S$ and state $q \in Q$ such that $q_i s = q$ for each $i = 1, \ldots, n$. Since X is transitive, there exist elements $t_i \in S$, such that $qt_i = q_i$, $i = 1, \ldots, n$. It follows that

$$\bar{\mathbf{n}} \approx (\{q_i\colon i = 1, \ldots, n\}, \{st_i\colon i = 1, \ldots, n\})$$

so we have $\bar{\mathbf{n}} \subset X$. ∎

Proposition 2.5. $(\bar{\mathbf{n}})c < n$.

Proof. Let $X \in (\bar{\mathbf{n}})$. By Proposition 2.3, there is a transitive ts Y such that $Y \subset X$ and $Yc = Xc$. Since $Y \subset X$, we have $Y \in (\bar{\mathbf{n}})$. Since $\bar{\mathbf{n}}$ does not divide Y, Proposition 2.4 implies that $Y\tau < n$. Now Theorem 0.1 gives us the desired result

$$Xc = Yc \leqslant Y\tau < n. \quad ∎$$

This completes the proof of Theorem 2.2.

For each $n \geqslant 1$, U_n is the monoid $\{1, u_1, \ldots, u_n\}$ with $u_i u_j = u_j$ for all $1 \leqslant i, j \leqslant n$. Except in the case $n = 1$, the semigroup of $\bar{\mathbf{n}}^{\bullet}$ is U_n. $S_{\bar{1}^{\bullet}} = 1$, not U_1.

Since $\bar{\mathbf{n}}^{\bullet}$ is a complete ts (all its transformations are functions), it follows that $\bar{\mathbf{n}}^{\bullet} \prec X = (Q, S)$ implies $S_{\bar{\mathbf{n}}^{\bullet}} \prec S$. This leads to

Corollary 2.6. *Let S be a semigroup and let $n \geqslant 2$. If S is U_n-free, then $Sc < n$.*

It may be that a ts $X = (Q, S)$ is $\bar{\mathbf{n}}^{\bullet}$-free while $U_n < S$. For example let $X = (\mathbf{5}, \mathrm{PF}(\mathbf{5}))$. There are six distinct $\mathscr{R}$-equivalent idempotents $e_i \in \mathrm{PF}(5)$ with $|5e_i| = 2$, $i = 1,\ldots, 6$. It follows that there is a copy of U_6 in PF(5). But clearly $X \in (\bar{\mathbf{6}}^{\bullet})$. Thus the condition $\bar{\mathbf{n}}^{\bullet} < (Q, S)$ is more restrictive than the condition $U_n < S$.

Let $Y_n = Z_2 \circ (\mathbf{n}, \mathrm{PF}(\mathbf{n}))$, where Z_2 is the cyclic group of order 2. It is easy to check that Y_n is transitive and that $Y_n\tau = n$. By Proposition 2.4, $Y_n \in (\overline{\mathbf{n + 1}}) \subset (\overline{\mathbf{n + 1}}^{\bullet})$. It is also known that $Y_n c = n$ so that the bound on complexity in Theorem 2.2 is the best possible. That is, $(\bar{\mathbf{n}}^{\bullet})c = n - 1$.

III. Inseparable ts's

In this section we give necessary and sufficient conditions for a ts to be inseparable. We will show that all tm's and all complete ts's with transformations are inseparable, among others.

Recall that a ts X is inseparable if whenever X divides a derived ts, then X divides one of the traces of the derived ts.

In this section we make use of the following notation; let $X = (Q, S)$ and let $s \in S$. Then set

$$\begin{aligned} \operatorname{dom} s &\equiv Qs^{-1} = \{q \mid qs \neq \varnothing\}, \\ \operatorname{rg} s &\equiv Qs = (\operatorname{dom} s)s, \\ \operatorname{dom} S &\equiv QS^{-1} = \bigcup \{\operatorname{dom} s \mid s \in S\}, \\ \operatorname{rg} S &\equiv QS = (\operatorname{dom} S)S. \end{aligned}$$

The proof of Theorem 2.2 depends upon results in this section. Therefore, of course, no result of Section II is assumed here.

We first present a lemma which will prove useful in the sequel.

LEMMA 3.1. *Let X be an inseparable* ts *and suppose there exists a relational cover*

$$X \lhd_{\varphi} Y$$

with the property that for each $s \in S_X$, there exists a unique $p \in Q_Y$ such that $\operatorname{dom} s \cap p\varphi \neq \varnothing$. *Then there exists a $p \in Q_Y$ such that $p\varphi = Q_X$. Furthermore, with respect to any parametrization of φ,*

$$X \subset \mathrm{Tr}_p.$$

Proof. Let $X = (Q, S)$, $Y = (P, T)$ and let Φ be the derived ts of φ with respect to any parametrization (Ω, α, β). We will first establish the cover

$$X <_{\psi} \Phi, \tag{3.1}$$

where ψ is defined by $(q, p)\psi = q$.

Cover $s \in S$ by (p_1, ω, p_2), where p_1 is the unique member of P such that $\operatorname{dom} s \cap p_1\varphi \neq \varnothing$, and where $\omega\alpha = s$ and $p_1\omega \subset p_2$. Now if

$$(q, p)\,\psi s = qs \neq \varnothing,$$

then $q \in \operatorname{dom} s$. Since p_1 is the only state of P satisfying $\operatorname{dom} s \cap p_1\varphi \neq \varnothing$ and since $\operatorname{dom} s \subset p_1\varphi$, it follows that $p = p_1$. Now, on the other hand

$$(q, p_1)(p_1, \omega, p_2)\psi = (q\omega, p_2)\psi = qs,$$

so we have shown $\psi s \subset (p_1, \omega, p_2)\psi$ and established (3.1).

Since X is inseparable, (3.1) implies that

$$X <_{\eta} \mathrm{Tr}_p$$

for some $p \in P$. Recall that

$$\mathrm{Tr}_p = (p\varphi, S_1),$$

where S_1 is the subsemigroup of S generated by

$$\{\omega\alpha \mid \omega \in \Omega^+, p\omega \subset p\}.$$

Since $p\varphi \subset Q$ and η is a surjective partial function, it follows that $p\varphi = Q$ and η is a permutation on Q. Furthermore, $(Q, S_1) \subset (Q, S)$, so by the transitivity of division we obtain

$$(Q, S) <_{\eta^k} (Q, S_1)$$

for any $k \geqslant 1$. Choosing k so that η^k is the identity on Q yields

$$X \subset \mathrm{Tr}_p. \quad \blacksquare$$

A state q in $X = (Q, S)$ is called *isolated* if $q \notin \operatorname{dom} S \cup \operatorname{rg} S$. Thus, in a diagram for X, no arrow will enter or leave an isolated state.

PROPOSITION 3.2. *Let X be an inseparable* ts *with* card $Q_X \geqslant 2$. *Then X has no isolated states.*

Proof. Let $X = (Q, S)$ be inseparable with $\operatorname{card} Q \geqslant 2$, and suppose $q_0 \in Q$ is isolated. Let Y be the ts given by the diagram

$$\sigma \circlearrowright 1 \qquad 0.$$

Then clearly

$$X \lhd_\varphi Y,$$

where $1\varphi = Q - \{q_0\}$ and $0\varphi = q_0$, and where each transformation of X is covered by σ.

This relational cover satisfies the hypothesis of Lemma 3.1, so either 0φ or 1φ must equal Q. But since card $Q \geqslant 2$, this is impossible. Therefore X is not inseparable. ∎

We now introduce a series of definition needed for the next proposition.

Let $X = (Q, S)$ be a ts. Define a relation $\sim$ on dom S by

$$\begin{array}{llll} q_1 \sim q_2 & \text{iff either} & q_1, q_2 \in \operatorname{dom} s, & \text{for some } s \in S \\ & \text{or} & q_1, q_2 \in \operatorname{rg} s, & \text{for some } s \in S. \end{array}$$

Let $\equiv$ be the smallest equivalence relation on dom S that contains $\sim$. We will call $\equiv$ the *separation relation* of X.

Let $P = \{p_1, \ldots, p_n\}$ be the equivalence classes of $\equiv$. It follows directly from the definition of $\sim$ that for each $s \in S$ there exist $p, p' \in P$ such that

$$\operatorname{dom} s \subset p, \qquad \operatorname{rg} s \cap \operatorname{dom} S \subset p'.$$

Let $X = (Q, S)$ be a ts and let Γ be a subset of S. We say $s \in S$ is *weakly generated* by Γ if $s \subset t_1 \cdots t_n$ for some $t_1, \ldots, t_n \in \Gamma$. If every element of S is weakly generated by Γ, we say X is *weakly generated* by Γ.

Let S_1 be the subsemigroup of S generated by Γ. Then saying Γ weakly generates X is equivalent to saying

$$X \subset (Q, S_1).$$

A transformation $s \in S$ is called *terminal* if $\operatorname{rg} s \cap \operatorname{dom} S = \varnothing$, i.e., $QsS = \varnothing$. Otherwise, s is called *non-terminal.*

PROPOSITION 3.3.. *Let X be an inseparable* ts *with* card $Q_X \geqslant 2$. *Then*

(i) *The separation relation of X is trivial, and*

(ii) *X is weakly generated by its non-terminal elements.*

Proof. Let $X = (Q, S)$ and let $P = \{p_1, \ldots, p_n\}$ be the equivalence classes of the separation relation of X. We will establish

$$X \lhd_\varphi \overline{\mathbf{n+1}}, \tag{3.2}$$

where φ: $\{0,..., n\} \to Q$ is defined by

$$0\varphi = Q - \text{dom } S,$$
$$i\varphi = p_i \cup (Q - \text{dom } S), \qquad i \geqslant 1.$$

Since $\bigcup \{p_i \mid i = 1,..., n\} = \text{dom } S$, φ is surjective.

Define the parametrization (Ω, α, β) by $\Omega = S$, $s\alpha = s$, and

$$\begin{aligned} s\beta &= \tilde{j} && \text{if } s \text{ is non-terminal and rg } s \cap \text{dom } S \subset p_j \\ &= \tilde{0} && \text{if } s \text{ is terminal.} \end{aligned}$$

To establish (3.2), we must show $\varphi s \subset (s\beta)\varphi$ for all $s \in S$.

Let $q \in i\varphi$, $s \in S$ with $qs \neq \varnothing$. If s is terminal, then $qs \in Q - \text{dom } S$. But

$$i(s\beta)\varphi = i\tilde{0}\varphi = 0\varphi = Q\text{-dom } S.$$

Therefore $\varphi s \subset (s\beta)\varphi$ when s is terminal.

If s is non-terminal, then $qs \in p_j \cup (Q\text{-dom } S)$ where $\text{rg } s \cap \text{dom } S \subset p_j$. But in this case

$$i(s\beta)\varphi = i\tilde{j}\varphi = j\varphi = p_j \cup (Q\text{-dom } S).$$

Therefore (3.2) is established.

Now (3.2) satisfies the hypothesis of Lemma 3.1, so there exists an $i \in \{0,..., n\}$ such that

$$i\varphi = Q$$

and

$$X \subset \text{Tr}_i.$$

If $i = 0$, then $\text{dom } S = \varnothing$, and all the states of X are isolated. Since X is inseparable and card $Q \geqslant 2$, Proposition 3.2 rules out $i = 0$.

Therefore $Q = p_i \cup (Q\text{-dom } S)$ for some i. Necessarily, then, $i = 1$ and $p_1 = \text{dom } S$. In other words, the separation relation of X is trivial.

Let $\text{Tr}_1 = (Q, S_1)$. S_1 is the subsemigroup generated by the set

$$\{s \in S \mid 1(s\beta) \subset 1\}.$$

But this is exactly the set of non-terminal elements of S. Since

$$X \subset (Q, S_1),$$

we can conclude that X is weakly generated by its non-terminal elements. ∎

THEOREM 3.4. *X is an inseparable* ts *iff either $X \prec \mathbf{1}^{\cdot}$ or*

(i) *X has no isolated states,*

(ii) *the separation relation of X is trivial, and*

(iii) *X is weakly generated by its non-terminal elements.*

Proof. Propositions 3.2 and 3.3 have shown that if X is inseparable, then either $X \prec \mathbf{1}^{\cdot}$ or X satisfies conditions (i)–(iii). Conversely, it is an easy exercise to show that all divisors of $\mathbf{1}^{\cdot}$ are inseparable. Thus it remains to show that a ts with two or more states that satisfies conditions (i)–(iii) is inseparable.

Let $X = (Q, S)$ satisfy (i)–(iii) with card $Q \geqslant 2$, and let

$$X \prec_{\psi} \Phi,$$

where Φ is the derived ts of

$$(R, W) \lhd_{\varphi} (P, T)$$

with respect to some parametrization (Ω, α, β).

Recall that for $p \in P$,

$$\mathrm{Tr}_p = (p\varphi, \{\omega \in \Omega^+ \mid p\omega \subset p\}),$$

$$\Phi_p = (\{(r, p) \mid r \in p\varphi\}, \{(p, \omega, p) \mid p\omega \subset p\}) \subset \Phi,$$

and $\mathrm{Tr}_p \approx \Phi_p$. In the following discussion we will identify $p\varphi$ with $\{(r, p) \mid r \in p\varphi\}$.

We first prove

$$\text{There exists } p \in P \text{ such that } (\mathrm{dom}\, S)\, \psi^{-1} \subset p\varphi. \tag{3.3}$$

Let $s \in S$ be covered by (p, ω, p') in Φ, and let $q \in \mathrm{dom}\, s$. If $(r, p'')\psi = q$, then

$$(r, p'')(p, \omega, p')\psi = qs \neq \varnothing.$$

Therefore p'' must be p. This shows:

(a) For each $q \in \mathrm{dom}\, S$, there exists $p_q \in P$ such that

$$q\psi^{-1} \subset p_q \varphi.$$

(b) If $q_1, q_2 \in \mathrm{dom}\, s$ for some $s \in S$, then $p_{q_1} = p_{q_2}$. Now if q_1, $q_2 \in \mathrm{rg}\, s \cap \mathrm{dom}\, S$ for some $s \in S$ and s is covered by (p, ω, p'), then there exists r_i, $i = 1, 2$ such that

$$\begin{aligned} q_i &= (r_i, p)(p, \omega, p')\psi \\ &= (r_i \omega, p')\psi. \end{aligned}$$

Therefore:

(c) If $q_1, q_2 \in \operatorname{rg} s \cap \operatorname{dom} S$ for some $s \in S$, then $p_{q_1} = p_{q_2}$.

Statements (b) and (c) imply

$$q_1 \sim q_2 \Rightarrow p_{q_1} = p_{q_2}$$

which in turn implies

$$q_1 \equiv q_2 \Rightarrow p_{q_1} = p_{q_2}.$$

Since the separation relation of X is assumed to be trivial, this implies (3.3).

Let $p_0 \in P$ be the state specified by (3.3). We next prove

$$\text{Every element of } S \text{ is covered by an element of the form } (p_0, \omega, p_0) \text{ in } \Phi. \tag{3.4}$$

Since it is assumed that X is weakly generated by its non-terminal elements, it suffices to prove (3.4) for non-terminal elements only. Let $s \in S$ be non-terminal and suppose s is covered by (p, ω, p') in Φ.

Let $q \in \operatorname{dom} s$. If $(r, p_0)\psi = q$, then

$$(r, p_0)(p, \omega, p')\psi = qs \neq \varnothing$$

and $p_0 = p$. Since s is non-terminal, there is a $q \in \operatorname{rg} s \cap \operatorname{dom} S$, and there exists an $r \in R$ such that

$$\begin{aligned} q &= (r, p_0)(p_0, \omega, p')\psi \\ &= (r\omega, p')\psi. \end{aligned}$$

Since $q \in \operatorname{dom} S$, p' must be p_0. This proves (3.4).

Statement (3.4) implies that

$$X <_\psi (Q_\Phi, \{(p_0, \omega, p_0) \mid p_0\omega \subset p_0\}).$$

To prove the theorem, it suffices to show that ψ when restricted to $p\varphi$ is still surjective. For then

$$X < \mathrm{Tr}_{p_0}.$$

Since X has no isolated states,

$$Q = \operatorname{dom} S \cup (\operatorname{dom} S)S$$

Statement (3.3) shows that $\operatorname{dom} S \subset (p_0\varphi)\psi$. Let $q \in (\operatorname{dom} S)S$. Then

$q = q's$ for some $q \in \operatorname{dom} S$ and $s \in S$. Let $(r, p_0)\psi = q'$ and let (p_0, ω, p_0) cover s. Then

$$\begin{aligned} q = q's &= (r, p_0)(p_0, \omega, p_0)\psi \\ &= (r\omega, p_0)\psi. \end{aligned}$$

Therefore $(\operatorname{dom} S)S \subset (p_0\varphi)\psi$, and ψ restricted to $p_0\varphi$ is surjective. ∎

COROLLARY 3.5. *Let $X = (Q, S)$ be a* ts *with* $\operatorname{dom} S = Q \neq \varnothing$. *Then X is inseparable iff the separation relation of X is trivial.*

Proof. Let X be inseparable. Then either the separation relation is trivial or $X < \mathbf{1}^{\bullet}$. But since $\operatorname{dom} S = Q \neq \varnothing$, X must be $\mathbf{1}^{\bullet}$ and again the separation relation is trivial.

Conversely, assume the separation relation on X is trivial. Since $\operatorname{dom} S = Q \neq \varnothing$, X has no isolated states, every non-empty transformation of X is non-terminal, and X has a non-empty transformation, say s. But the empty transformation is contained in s, so X is weakly generated by its non-terminal elements. Thus by Theorem 3.4, X is inseparable. ∎

COROLLARY 3.6. *Let X be a* ts *with at least one function among its transformations. Then X is inseparable.*

Proof. Let $X = (Q, S)$ and let $s \in S$ with $\operatorname{dom} s = Q$. Then $\operatorname{dom} S = Q \neq \varnothing$ and the separation relation is trivial. ∎

COROLLARY 3.7. *All transformation monoids and all complete* ts's *with transformations are inseparable.*

Since $\bar{\mathbf{n}}$ is a complete ts with transformations, Corollary 3.7 implies Proposition 2.1.

EXAMPLES.

(a) Consider the ts given by the diagram

$$\begin{tikzcd} 1 \arrow[r, "\sigma"] & 2 \arrow[dr, "\tau"] & \\ & & 5 \\ 3 \arrow[r, "\tau"] & 4 \arrow[ur, "\sigma"] & \end{tikzcd}$$

σ and τ are non-terminal, so this ts is weakly generated by its non-terminal elements. However, the separation realtion is not trivial, for $1 \not\equiv 3$ and $2 \not\equiv 4$. Therefore this ts is not inseparable.

(b) Consider the ts

$$\begin{tikzcd} 1 \arrow[d, "\sigma{,}\tau"'] & 2 \arrow[dl, "\tau"'] \arrow[d, bend right, "\sigma"'] \\ 3 & 4 \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

The separation relation is trivial, but τ is terminal and is not contained in any power of σ. Therefore this ts is not weakly generated by its non-terminal elements and hence is not inseparable.

(c) Of the main series of primes

$$\mathbf{0}, \quad \mathbf{1}, \quad \mathbf{2}, \quad E, \quad C, \quad F, \quad \bar{\mathbf{2}}$$

all are inseparable except for $\mathbf{2}$,

$$E: \quad 1 \xrightarrow{\sigma} 0$$

and

$$F: \quad \begin{tikzcd} 1 \arrow[r, bend left, "\sigma"] & 0 \arrow[l, bend left, "\tau"] \end{tikzcd}$$

$\mathbf{2}$ has isolated states, E is not weakly generated by its non-terminal elements (since it has none), and the separation relation of F is not trivial. Nevertheless, $\langle \mathbf{2} \rangle$ and $\langle F \rangle$ are semilocal. This shows that X being inseparable is not necessary in order for $\langle X \rangle$ to be semilocal.

IV. U_n-Free Morphisms

By a morphism of semigroups in this section, we mean a relation

$$\varphi: S \to T$$

satisfying

(i) $\operatorname{dom} \varphi = S$,

(ii) $(s_1\varphi)(s_2\varphi) \subset s_1 s_2 \varphi$ for all $s_1, s_2 \in S$.

If, further, φ is a function, then φ is the usual homomorphism of semigroups. In this case we say φ is a *functional* morphism.

Let **V** be a class of semigroups, i.e., a collection of semigroups closed under division. We say a morphism

$$\varphi: S \to T$$

is a **V**-*morphism* if whenever T' is a subsemigroup of T such that $T' \in \mathbf{V}$, then $T'\varphi^{-1} \in \mathbf{V}$. Note that the composition of **V**-morphisms is again a **V**-morphism.

We will be concerned with **V**-morphisms in the following cases.

(i) **V** is all aperiodic semigroups, denoted **A**,

(ii) **V** is all U_n-free semigroups, denoted $(U_n)_{\mathbf{S}}$.

There is a complexity theory for semigroup morphisms that is intimately related to the complexity of semigroups. Let $\varphi: S \to T$ be a morphism; the complexity of φ is denoted φc and satisfies

$\varphi c \leqslant n$ iff there exists a factorization $\varphi = \alpha_0 \beta_1 \alpha_1 \cdots \alpha_{n-1} \beta_n \alpha_n$ where the α_i are aperiodic morphisms and the α_i are U_2-free morphisms. (4.1)

$$Sc \leqslant \varphi c + Tc. \tag{4.2}$$

(Fundamental Lemma of Complexity) (4.3)

(a) If φ is aperiodic, then $Sc \leqslant Tc$.

(b) If φ is U_2-free, then $Sc \leqslant 1 + Tc$.

The theorem of this section generalizes (4.3b)

THEOREM 4.1. *Let $\varphi: S \to T$ be a U_n-free morphism, $n \geqslant 2$. Then*

$$\varphi c < n.$$

From (4.2) we obtain

COROLLARY 4.2. *Let $\varphi: S \to T$ be a U_n-free morphism, $n \geqslant 2$. Then*

$$Sc < n + Tc.$$

To prove Theorem 4.1 we need the notion of the derived semigroup of a morphism, its traces, and the Rhodes expansion

$$\eta_T: \hat{T} \to T$$

of T. The reader is directed to [4] for a complete treament of this subject. Familiarity with [4] will be assumed from this point forth.

The key relation between complexity of semigroups and complexity of morphisms is

$$\text{Let } \varphi\colon S \to T \text{ be a morphism. Then} \tag{4.4}$$

$$\varphi c = \Psi c,$$

where Ψ is the derived semigroup of

$$\psi\colon S \xrightarrow{\varphi} T \xrightarrow{\eta_T^{-1}} \hat{T}$$

(see [4, Exercise 10.3]).

Thus it suffices, assuming φ is U_n-free, to show that $\Psi c < n$. Since $\eta_T\colon \hat{T} \to T$ is a functional morphism, η_T^{-1} is a U_n-free morphism. Therefore the composition $\psi\colon S \to \hat{T}$ is U_n-free.

Let $\Psi_t = \{(t, s, t) \in \Psi\}$ be a trace of Ψ. Then

$$\Psi_t \approx \hat{T}_t \psi^{-1},$$

where $\hat{T}_t = \{t' \in \hat{T} \mid tt' = t\}$, the stabilizers of $t \in \hat{T}$. Recall that the stabilizers of $\hat{T}$ are U_2-free, hence U_n-free for all $n \geqslant 2$. Since ψ is U_n-free, we can conclude that

$$\Psi_t \in (U_n)_{\mathbf{S}} \qquad \text{for each} \quad t \in \hat{T}. \tag{4.5}$$

To conclude the proof we need to show that Ψ is U_n-free, for then Corollary 2.6 yields

$$\Psi c < n.$$

To this end we present

PROPOSITION 4.3. *Let Φ be the derived semigroup of $\varphi\colon S \to T$. If W is a semigroup without a zero and if $W \prec \Phi$, then $W \prec \Phi_t$ for some $t \in T$.*

Proof. Since $W \prec \Phi$, there exists a subsemigroup W' of Φ and surjective functional morphisms $\theta\colon W' \to W$. Because zeros map onto zeros under homomorphisms, we can conclude that the zero of Φ does not belong to W'. Now let

$$w_i = (t_i, s_i, t_i') \in W', \qquad i = 1, 2.$$

Since $w_i^2 \neq 0$, it must be that $t_i = t_i'$, $i = 1, 2$. Since $w_1 w_2 \neq 0$, it follows that $t_1 = t_2$. In other words, $W' \prec \Phi_t$ for some $t \in T$. Therefore

$$W \prec \Phi_t \qquad \text{for some} \quad t \in T. \quad \blacksquare$$

For $n \geqslant 2$, U_n has no zero. Therefore it follows from (4.5) and Proposition 4.3 that Ψ is U_n-free and

$$\varphi c = \Psi c < n. \quad \blacksquare$$

APPENDIX: Relational Covers, Traces and The Derived ts

Let $X = (Q, S) \lhd_\varphi (P, T) = Y$ be a relational cover. This means

(i) $\varphi: P \to Q$ is a surjective relation, and

(ii) For each $s \in S$, there exists $t \in T$ such that $p\varphi s \subset pt\varphi$ for all $p \in P$. We say *t covers s* in this situation.

If further, φ is a partial function, then $X < Y$.

A parametrization for φ consists of a finite alphabet Ω and a pair of morphisms

$$\alpha: \Omega^+ \to S, \qquad \beta: \Omega^+ \to T$$

such that

(i) α is surjective,

(ii) $\omega\alpha$ is covered by $\omega\beta$ for all $\omega \in \Omega^+$.

Each relational cover has at least one parametrization; set $\Omega = \{(s, t) \in S \times T: t \text{ covers } s\}$ and define $(s, t)\alpha = s$ and $(s, t)\beta = t$.

Given a parametrization (Ω, α, β) of φ, it is convenient to write $q(\omega\alpha) = q\omega$; $p(\omega\beta) = p\omega$ for all $q \in Q$ $p \in P$ and $\omega \in \Omega^+$.

The *derived ts* Φ of φ relative to a given parametrization (Ω, α, β) has its states the graph of φ, that is,

$$Q_\Phi = \{(q, p) \mid p \in P, q \in p\varphi\}.$$

To define S_Φ we consider triples

$$(p, \omega, p')$$

with $p, p' \in P, \omega \in \Omega^+$ and $p\omega \subset p'$. Each triple is regarded as a partial function on Q_Φ by setting

$$\begin{aligned}(q, p'')(p, \omega, p') &= (q\omega, p') &&\text{if } \; p'' = p\\ &= \varnothing &&\text{otherwise.}\end{aligned}$$

S_Φ is then the semigroup generated by all such partial functions, and $\Phi = (Q_\Phi, S_\Phi)$.

We note the following properties of S_Φ. Let θ denote the empty transformation.

$$\begin{aligned}(p, \omega, p')(r, u, r') &= (p, \omega u, r') \qquad \text{if} \quad p' = r \\ &= \theta \qquad\qquad\qquad \text{otherwise.}\end{aligned} \tag{A1}$$

$(p, \omega, p') = \theta$ if $p\omega = \varnothing$. Therefore, every non-empty transformation of S_Φ has the form $(p, \omega, p\omega)$. (A2)

The reason for the derived ts is the following:

THEOREM A1. *Let $X \lhd_\varphi Y$ be a relational cover equipped with a parametrization (Ω, α, β). If $Q_Y \neq \varnothing$, then*

$$X < \Phi \circ Y,$$

where Φ is the derived ts *of φ relative to the given parametrization.*

The important portion of the derived ts Φ is a set of sub-ts's that it shares with X. For each $p \in P$ consider the subsemigroup

$$\Omega_p^+ = \{\omega \in \Omega^+ : p\omega \subset p\}$$

of Ω^+. Since $p\varphi\omega \subset p\omega\varphi \subset p\varphi$, each element of Ω_p^+ defines a partial function on $p\varphi$. The ts represented by

$$(p\varphi, \Omega_p^+)$$

is called the *trace of φ at p* and is denoted Tr_p.

Since $p\varphi \subset Q$ and the action of $\omega \in \Omega^+$ on $p\varphi$ is the action of $\omega\alpha \in S$ restricted to $p\varphi$, we see that

$$\mathrm{Tr}_p \subset X.$$

Also, Tr_p is easily seen to be isomorphic to the sub-ts

$$\Phi_p = (\{(q, p) : q \in p\varphi\}, \{(p, \omega, p)\})$$

of Φ. Thus

$$\mathrm{Tr}_p \subset \Phi.$$

Since $\mathrm{Tr}_p \approx \Phi_p$, we interchange the expressions "trace of φ" and "trace of Φ" at will.

In general, we can only infer that $\mathrm{Tr}_p \subset X$, but for a special relational cover, called the delay cover of X, we can make a stronger statement.

Let e be an idempotent of X. Then by X_e we mean the sub-ts (Qe, eSe) of X. X_e is a transformation monoid (tm).

THEOREM A2 (Trace–Delay Theorem). *Let* $X = (Q, S)$ *be a* ts *with* $S \neq \varnothing$. *Then there exists a relational cover* $X \lhd_{\varphi} Y$ *and a parametrization of* φ *satisfying*

(i) $Y \in [\bar{\mathbf{2}}]$,

(ii) *For each* $p \in P$, *there exists an idempotent* $e \in S$ *such that* $\mathrm{Tr}_p < X_e$.

For proofs and further details of the derived ts, please refer to Chapter III of [1].

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