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RESEARCH ARTICLE

Inverse Monoids and Rational Schreier Subsets of the Free Group

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Abstract: An inverse monoid M is an idempotent-pure image of the free inverse monoid on a set X if and only if M has a presentation of the form $M = \mathrm{Inv}\langle X : e_i = f_i, i \in I\rangle$, where e_i, f_i are idempotents of the free inverse monoid: every inverse monoid is an idempotent-separating image of one of this type. If R is an $\mathcal{R}$-class of such an inverse monoid, then R may be regarded as a Schreier subset of the free group on X. This paper is concerned with an examination of which Schreier subsets arise in this way. In particular, if I is finite, then R is a rational Schreier subset of the free group. Not every rational Schreier set arises in this way, but every positively labeled rational Schreier set does.

1. Introduction

We shall assume familiarity with the notion of an inverse monoid and we refer the reader to the book of Petrich [9] for basic notation and results concerning inverse monoids. In particular, we shall denote the free inverse monoid on a set X by $FIM(X)$. Thus $FIM(X) \cong (X \cup X^{-1})^*/\rho$ where ρ is the Vagner congruence on the free monoid $(X \cup X^{-1})^*$: here X^{-1} is a set disjoint from X and in one-one correspondence with X by the map $x \mapsto x^{-1}$, $x \in X$. Let $\Gamma(X)$ denote the Cayley graph of the free group $FG(X)$ on X: of course $\Gamma(X)$ is a tree. Recall that if $u = x_1x_2\dots x_n \in (X \cup X^{-1})^*$, then the *Munn tree* of u is the finite subtree of $\Gamma(X)$ traversed when the path labeled by the word u is read in $\Gamma(X)$, starting at the vertex 1 and ending at the vertex $r(u)$ (the reduced form of u). We denote the Munn tree of u by $MT(u)$: clearly $MT(u)$ is a finite birooted labeled subtree of $\Gamma(X)$, the roots being 1 (initial root) and $r(u)$ (terminal root). From Munn [8] we have the following result, which solves the word problem for $FIM(X)$.

Theorem 1.1. (Munn [8], see also Petrich [9], section VIII 3). *For all words $u, v \in (X \cup X^{-1})^*$ we have $u \ \rho \ v$ if and only if $MT(u) = MT(v)$ and $r(u) = r(v)$.*

If $T = \{(u_i, v_i) : i \in I\}$ is a relation on $(X \cup X^{-1})^*$ (i.e. if $u_i, v_i \in (X \cup X^{-1})^*$ for each $i \in I$) then we denote by $Inv\langle X : T\rangle$ or $Inv\langle X : u_i = v_i, i \in I\rangle$ the inverse monoid presented by the set X of generators and the set T of relations: that is, $\mathrm{Inv}\langle X : T\rangle = (X \cup X^{-1})^*/\tau$, where τ is the congruence on $(X \cup X^{-1})^*$ generated by $\rho \cup T$. Presentations of inverse monoids

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have been studied by Margolis, Meakin and Stephen in a series of papers [5], [6], [7], [14], [15]. In particular, in this paper, we shall use the results of Margolis and Meakin [6] concerning inverse monoid presentations of the form $M = Inv\langle X : e_i = f_i, i \in I\rangle$ where e_i, f_i are Dyck words (i.e. idempotents of $FIM(X)$). In [6] it was shown that an inverse monoid has a presentation of this form if and only if it is an idempotent-pure image of $FIM(X)$ and that the word problem for such a presentation is decidable if I is finite.

Recall (Petrich [9], Chapter III) that if τ is any congruence on an inverse monoid S then the *trace* of τ is the equivalence relation $tr(\tau) = \tau \cap (E(S) \times E(S))$ on the set $E(S)$ of idempotents of S and the *kernel* of τ is the subset $\ker \tau = \{a \in S : a\ \tau\ a^2\}$ of elements of S that are τ-related to idempotents of S. Every congruence τ on S is determined by its trace and its kernel. Following Petrich [9] we denote by τ_{min} the smallest congruence on S with the same trace as τ. It is well-known that τ_{min} is the congruence on S generated by $tr(\tau)$, that S/τ is an idempotent-separating image of S/τ_{min} and that τ_{min} is an idempotent-pure congruence on S if S is E-unitary: in fact if S is E-unitary, then $\tau_{min} = \tau \cap \sigma_S$, where σ_S is a minimum group congruence on S. In particular, $FIM(X)$ is E-unitary so if τ is any congruence on $FIM(X)$, τ_{min} is idempotent-pure. From these remarks we have the following.

Lemma 1.2. *Every inverse monoid is an idempotent-separating homomorphic image of some inverse monoid with a presentation of the form* $M = Inv\langle X : e_i = f_i, i \in I\rangle$, *where* e_i, f_i *are idempotents of* $FIM(X)$.

The lemma indicates that inverse monoids are detemined up to idempotent-separating morphisms by inverse monoids of the form $M = Inv\langle X : e_i = f_i, i \in I\rangle$ where e_i, f_i are idempotents of $FIM(X)$. Thus it is of interest to examine this class of monoids in more detail. We shall restrict attention primarily to the case where M is finitely presented (i.e. I is finite). Our concern in the present paper is with describing the $\mathcal{R}$-classes of such a monoid.

Let $M = Inv\langle X : T\rangle \cong (X \cup X^{-1})^*/\tau$ be any inverse monoid with set X of generators and set T of relations and let R be an $\mathcal{R}$-class of M. We define the *Schützenberger graph* $S\Gamma(R) = S\Gamma(X, T, R)$ of R relative to this presentation as follows: the set of *vertices* of $S\Gamma(R)$ is the set R; $S\Gamma(R)$ contains the *edge* $(u\tau, x, v\tau)$ from $u\tau$ to $v\tau$ if $u, v \in (X \cup X^{-1})^*$, $x \in X \cup X^{-1}$, $u\tau$, $v\tau \in R$ and $v\tau = (ux)\tau$. It is easy to see that if $(u\tau, x, v\tau)$ is an edge of $S\Gamma(R)$, then so is $(v\tau, x^{-1}, u\tau)$: the pair of edges $(u\tau, x, v\tau)$ and $(v\tau, x^{-1}, u\tau)$ is usually denoted by $u\tau \overset{x}{\longrightarrow} v\tau$ when sketching a graph of $S\Gamma(R)$. The notion of "graph" that we are using here is that of Serre [13]. We refer the reader to our papers [5] and [6] for more details and notation concerning Schützenberger graphs and Cayley graphs of group presentations. The natural map $\sigma : M = Inv\langle X : T\rangle \twoheadrightarrow G = gp\langle X : T\rangle$ from M onto its maximal group homomorphic image G induces a graph morphism (again denoted by σ) from $S\Gamma(X, T, R)$ into $\Gamma(X, T)$, the Cayley graph of $G = gp\langle X : T\rangle$: the map σ induces an embedding of each graph $S\Gamma(X, T, R)$ into $\Gamma(X, T)$ if and only if M is E-unitary (see Margolis and meakin [6]).

If $M = Inv\langle X : e_i = f_i, i \in I\rangle$ where e_i, f_i are idempotents of $FIM(X)$, then M is E-unitary with maximum group homomorphic image $FG(X)$ (the free group on X), so σ induces an embedding of each corresponding Schützenberger graph $S\Gamma(R)$ into $\Gamma(X)$, the Cayley graph of $FG(X) = gp\langle X : \phi\rangle$. Clearly this embedding σ maps the idempotent in R to the identity 1 of $FG(X)$. Thus we may view the $\mathcal{R}$-classes of such a monoid as subsets of $FG(X)$. In [6], Margolis and Meakin established the following result, the proof of which involves combining the results of Stephen [14] on presentations of inverse monoids and Rabin [11], [12] on the decidability of the second-order monadic logic of the infinite binary tree.

Theorem 1.3. (Margolis and Meakin [6]). *Let* $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$ *where* e_i, f_i *are idempotents of* $FIM(X)$. *Then*

(a) *The map* $\sigma : M \twoheadrightarrow FG(X)$ *embeds each* $\mathcal{R}$*-class of* M *as an effectively constructible rational Schreier subset of* $FG(X)$;

(b) *the word problem for* M *is decidable.*

The primary question of interest in the present paper is the converse: *which rational Schreier subsets of* $FG(X)$ *arise as the set of elements of an* $\mathcal{R}$*-class of some inverse monoid of the form* $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$? We first briefly review the notion of a rational Schreier subset of the free group.

A subset L of the free group $FG(X)$ is called a *Schreier subset* of $FG(X)$ if $1 \in L$ and if L contains every prefix of every reduced word in L. Clearly L is a Schreier subset of $FG(X)$ if and only if there is a subtree Γ of $\Gamma(X)$ such that Γ contains 1 as a vertex and such that L is the set of elements of $FG(X)$ that label the vertices of Γ. Since the correspondence between Schreier subsets of $FG(X)$ and subtrees of $\Gamma(X)$ that contain the vertex 1 is obviously one-one, we often abuse notation slightly and consider Schreier subsets of $FG(X)$ as subtrees of $\Gamma(X)$ containing the vertex 1.

The set $Rat(M)$ of *rational* subsets of a monoid M is the smallest collection of subsets of M such that

(a) $\{m\} \in Rat\ M$ for each $m \in M$;

(b) if $A, B \in Rat\ M$ then $A \cup B \in Rat\ M$, $A \cdot B \in Rat\ M$ and $A^* \in Rat\ M$. (Here $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ and A^* is the submonoid of M generated by A).

The rational subsets of the monoid $FG(X)$ have been classified by Benois [1]. If L is a subset of $FG(X)$ then we let $L\iota = \{u\iota : u \in L\}$ where $u\iota$ is the unique reduced word of the monoid $(X \cup X^{-1})^*$ that is equal to u in $FG(X)$: thus $L\iota \subseteq (X \cup X^{-1})^*$.

If A is a subset of $(X \cup X^{-1})^*$ then we denote by $r(A)$ the set $\{r(a) : a \in A\}$ where $r(a)$ is the reduced form of a. The theorem of Benois can be stated in the following form:

Theorem 1.4. (Benois [1], see also Berstel [2]).

(a) *If* A *is a rational subset of* $(X \cup X^{-1})^*$, *the so is* $r(A)$;

(b) *A subset* $L \subseteq FG(X)$ *is a rational subset of* $FG(X)$ *if and only if* $L\iota$ *is a rational subset of* $(X \cup X^{-1})^*$.

Since rational subsets and recognizable subsets of the free monoid $(X \cup X^{-1})^*$ coincide by Kleene's theorem, it follows that if L is a rational subset of $FG(X)$ then $L\iota$ is recognized by a finite automaton (over the alphabet $X \cup X^{-1}$). We refer the reader to Lallement [4] or Pin [10] or any standard book on language theory or automata theory (e.g. Hopcroft and Ullman [3]) for a discussion of Kleene's theorem and for basic terminology and results concerning automata and languages. We denote by $\mathcal{B}(L)$ (the Benois automaton of L) the minimal automaton of $L\iota$, for each rational subset L of $FG(X)$. Clearly $\mathcal{B}(L)$ accepts only reduced words in $(X \cup X^{-1})^*$: it is also clear that L is a (rational) Schreier subset of $FG(X)$ if and only if all states of $\mathcal{B}(L)$ are terminal. Thus L is a rational Schreier subset of $FG(X)$ if and only if $L\iota$ is the language accepted by a finite automaton $\mathcal{B}(L)$ such that

(a) $\mathcal{B}(L)$ is a minimal automaton over the alphabet $(X \cup X^{-1})^*$;

(b) $\mathcal{B}(L)$ accepts only reduced words in $(X \cup X^{-1})^*$;

(c) $\mathcal{B}(L)$ has one initial state and all states are terminal.

Example 1.5. The language $L = b^* \cup b^* \cdot a \cdot (a \cup b^{-1})^*$ is the set of reduced words in $\{a, b, a^{-1}, b^{-1}\}^*$ corresponding to a rational Schreier subset (again denoted by L) of $FG(\{a, b\})$. The Benois automaton $\mathcal{B}(L)$ is the two state automaton

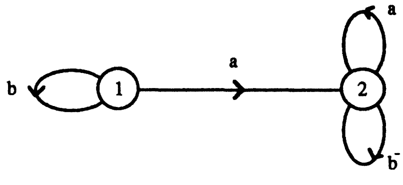

Diagram 1

with initial state (1) and terminal states (1) and (2) . A portion of the subtree of $\Gamma(\{a, b\})$ representing this rational Schreier subset L of $FG(\{a, b\})$ is shown in Diagram 2.

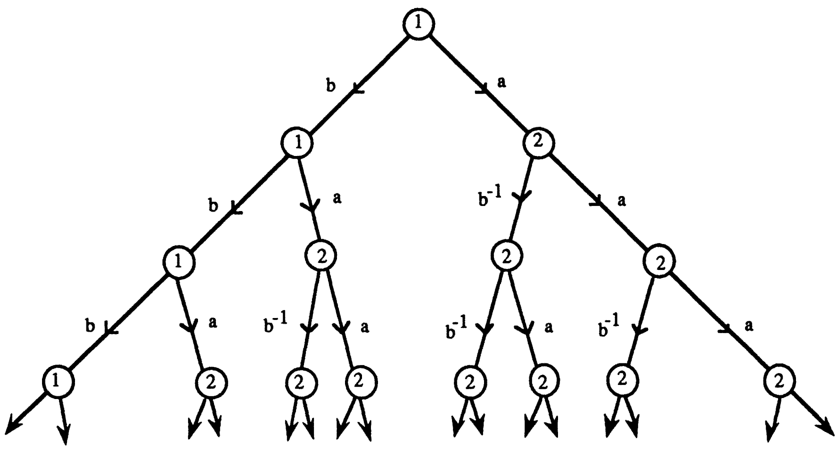

Diagram 2

We may regard the tree depicted in Diagram 2 as a rooted labeled subtree of $\{a, b, a^{-1}, b^{-1}\}^*$ (with initial root at the top). When viewed as a labeled subtree of $\{a, b, a^{-1}, b^{-1}\}^*$, the tree is *directed* with labeled edges directed away from the root as indicated on the diagram. When viewed as a labeled subtree of $\Gamma(\{a, b\})$ it is *undirected*: the edge $\alpha \circ\!\xrightarrow{x}\!\circ\, \beta$ of $\Gamma(\{a, b\})$ is the same as the edge $\alpha \circ\!\xleftarrow{x^{-1}}\!\circ\, \beta$. Given any node (vertex) α of the tree L, the *directed subtree of L rooted at α* has as vertices all those vertices β of L for which α is on the geodesic from the initial root to β (the vertices "below" α in the diagram). On the other hand, the *undirected subtree of L rooted at α* contains *every* vertex of L and has root α. Thus the *directed* subtree of L rooted at α may be viewed as a labeled subtree of L when L is considered as a subtree of the tree $\{a, b, a^{-1}, b^{-1}\}^*$, while the *undirected* subtree of L rooted at α may be viewed as a labeled subtree of L when L is considered a subtree of $\Gamma(\{a, b\})$. Note that there are precisely two isomorphism classes of directed rooted subtrees (those whose roots are labeled by (1) and those whose roots are labeled by (2)) but there are infinitely many isomorphism classes of undirected rooted subtrees. In general, if L is *any* rational Schreier subset of $FG(X)$, then the tree $L\iota$ has finitely many isomorphism classes of directed rooted subtrees, one corresponding to each state of $\mathcal{B}(L)$, but possibly infinitely many isomorphism classes of undirected rooted subtrees. This distinction will be crucial in subsequent parts of this paper. We shall return to Example 1.5 and shall use this notion of directed and undirected rooted subtrees in the sequel. ∎

Let $M = Inv\langle X : T\rangle = (X \cup X^{-1})^*/\tau$ and let $u \in (X \cup X^{-1})^*$. In his paper [14], Stephen provides an iterative procedure for constructing the Schützenberger graph $S\Gamma(X, T, R_{u\tau})$ of the $\mathcal{R}$-class $R_{u\tau}$ of the corresponding element $u\tau \in M$. We briefly summarize his construction below, in the special case in which $T = \{(e_i, f_i) : i \in I\}$ where e_i, f_i are idempotents of $FIM(X)$. As in [6], it is easy to see that we may assume without loss of generality that $e_i \leq f_i$ for all $i \in I$; i.e. $MT(f_i) \subseteq MT(e_i)$ for all $i \in I$. In this special case, the iterative construction of $S\Gamma(X, T, R_{u\tau})$ (for $u \in (X \cup X^{-1})^*$) proceeds as follows.

STEP 1. Starting with the word $u \in (X \cup X^{-1})^*$ we build the Munn tree $MT(u)$: clearly $MT(u)$ is a (birooted) subtree of $\Gamma(X)$, the Cayley graph of the free group $FG(X)$. Let $B_1(u) = MT(u)$.

STEP 2. Suppose that v is a vertex of $MT(u)$ such that $v \cdot MT(f_i) \subseteq MT(u)$ for some $i \in I$: form the tree $MT(u) \cup v \cdot MT(e_i)$. Here the union of the trees $MT(u)$ and $v \cdot MT(e_i)$ is simply the subtree of $\Gamma(X)$ spanned by the union of the vertices of $MT(u)$ and $v \cdot MT(e_i)$. Form such a union of trees for all vertices v in $MT(u)$ and all indices $i \in I$. Let $B_2(u) = B_1(u) \cup \{v \cdot MT(e_i) : v \cdot MT(f_i) \subseteq B_1(u), i \in I\}$. Clearly $B_2(u)$ is a birooted subtree of $\Gamma(X)$ with initial root 1 and terminal root $r(u)$.

INDUCTIVE STEP: Assume that we have constructed the birooted subtree $B_n(u) \subseteq \Gamma(X)$ for some $n \geq 1$. As in Step 2, we construct $B_{n+1}(u)$ from $B_n(u)$ by glueing on to $B_n(u)$ the tree $v \cdot MT(e_i)$ at each vertex v of $B_n(u)$ for which $v \cdot MT(f_i) \subseteq B_n(u)$, some $i \in I$; that is, we let

$$B_{n+1}(u) = B_n(u) \cup \left(\bigcup_{\substack{v \in X_i \\ i \in I}} v \cdot MT(e_i) \right)$$

where $X_i = \{v \in FG(X) : v \cdot MT(f_i) \subseteq B_n(u)\}$. It is clear that

$$MT(u) = B_1(u) \subseteq B_2(u) \subseteq \cdots \subseteq B_n(u) \subseteq B_{n+1}(u) \subseteq \cdots \subseteq \Gamma(X)$$

and that each $B_n(u)$ is a birooted subtree of $\Gamma(X)$ with initial root (1) and terminal root $r(u)$. From Stephen [14] we deduce the following fact.

Theorem 1.6. $$S\Gamma(X, T, R_{u\tau})\sigma = \bigcup_{n=1}^{\infty} B_n(u) \ .$$

2. Schreier Sets Associated with $FIM(X)$

We are now ready to provide a characterization of the $\mathcal{R}$-classes of an idempotent-pure image of $FIM(X)$. We say that a Schreier set $L \subseteq FG(X)$ is *naturally associated with* $FIM(X)$ if there is some inverse monoid of the form $M = Inv\langle X : e_i = f_i, i \in I\rangle$ (for e_i, f_i idempotents of $FIM(X)$) and some

$\mathcal{R}$-class R of M such that $L = R\sigma$ (equivalently, $L = S\Gamma(R)\sigma$). If V is a rooted subtree of $\Gamma(X)$ with root α, then we say that V *embeds in L at w* if there is a labeled graph embedding of V into L that maps α to w (clearly there is precisely one such embedding if there is any). We emphasize that this embedding is in the *undirected* sense. For example, if V is the tree $\alpha \circ\!\xrightarrow{a}\!\circ\, \beta$ with root α and L is the tree $\gamma \xrightarrow{a^{-1}} \delta \xrightarrow{b} \varepsilon$, then V embeds in L at δ. The tree L of Example 1.5 (Diagram 2) embeds in L at each vertex labeled by (1) but not at any vertex labeled by (2). In general, if L is a Schreier subset of $FG(X)$, we call a finite (rooted) subtree V of L a *finite test tree for L* if V has the property that, for all $w \in L$, V embeds in L at w if and only if L embeds in L at w.

Theorem 2.1. *Let L be a Schreier subset of $FG(X)$. Then L is naturally associated with $FIM(X)$ if and only if there is a finite test tree for L.*

Proof. Suppose first that L is naturally associated with $FIM(X)$. Then there is an inverse monoid $M = Inv\langle X : e_i = f_i, i \in I\rangle = (X \cup X^{-1})^*/\tau$ and an $\mathcal{R}$-class R of M such that $L = R\sigma$. Let u be an element of $(X \cup X^{-1})^*$ such that $R = R_{u\tau}$. From Theorem 1.6 it follows that $L = \bigcup_{n=1}^{\infty} B_n(u)$, where the trees $B_n(u)$ are iteratively constructed by the procedure described prior to the statement of Theorem 1.6. Notice that L is thus the smallest subtree of $\Gamma(X)$ such that $MT(u) \subseteq L$ and such that, for all vertices $v \in L$ and all $i \in I$, $v \cdot MT(e_i) \subseteq L$ iff $v \cdot MT(f_i) \subseteq L$. Now let $V = MT(u)$: V is a finite rooted subtree of L. If V embeds in L at w, then $w \cdot V \subseteq L$ and it follows from the iterative construction of L that $w \cdot L \subseteq L$, so L embeds in L at w. Hence $V = MT(u)$ is a finite test tree for L.

Conversely, suppose that L is a Schreier subset of $FG(X)$ with a finite test tree V. Since V is a rooted tree we may regard it as a birooted tree in which the initial and terminal roots coincide, so there is some word $e \in (X \cup X^{-1})^*$ such that $MT(e) = V$: in fact e is an idempotent of $FIM(X)$ since the initial and terminal roots of V coincide. Now let F be any finite rooted subtree of L (with 1 as its root) such that V embeds in F at 1. If w is any vertex of L such that V embeds in L at w, then L embeds in L at w and so certainly F embeds in L at w. Conversely, if F embeds in L at w then certainly V embeds in L at w: thus F is also a finite test tree for L, and V and F embed at the same vertices of L. Corresponding to each such tree F as described above, there exists a word $f \in (X \cup X^{-1})^*$ such that f is an idempotent of $FIM(X)$ and $MT(f) = F$. Let I denote the set of all such finite trees F described above (so that V embeds in F at 1 and F embeds in L at 1) and for each $i \in I$, let f_i denote an idempotent of $FIM(X)$ whose Munn tree is i. Consider the inverse monoid $M = Inv\langle X : e = f_i, i \in I\rangle$ and let $R = R_e$ in M. We claim that $R\sigma = L$. To see this, note first that if v is a vertex in L, then v is contained in some finite rooted subtree F of L such that V embeds in F at 1, so by the iterative construction of $R\sigma$ given in Theorem 1.6 it follows immediately that $v \in R\sigma$. Hence $L \subseteq R\sigma$. For the converse, note that any element of $R\sigma$ must be obtained from $V = MT(e)$ in a finite number of steps by the iterative procedure outlined prior to the statement of Theorem 1.6. Since $MT(e) = V$ and all trees $MT(f_i)$ ($i \in I$) embed in L precisely at those vertices at which V embeds, it follows that if $B_n(u) \subseteq L$ for some n, then $B_{n+1}(u) \subseteq L$. Since $B_1(u) = V \subseteq L$, it follows that $B_n(u) \subseteq L$ for all n and hence $R\sigma = \bigcup_{n=1}^{\infty} B_n(u) \subseteq L$. Hence $L = R\sigma$ and so L is naturally associated with $FIM(X)$. ∎

We next produce an example to show that not every rational Schreier set is naturally associated with $FIM(X)$.

Example 2.2. There are rational Schreier sets that are not naturally associated with $FIM(X)$.

Let L be the rational Schreier set of Example 1.5. The tree L is shown in Diagram 2. Let V be any finite rooted subtree of L. In order for V to be a finite test tree for L we require that V embeds at a vertex $w \in L$ if and only if L embeds at w. Now L embeds precisely at those vertices labeled (1) since if γ is any vertex labeled (2) there is some $n > 0$ such that one cannot read a path labeled by b^n starting at γ. This forces that V embeds only at those vertices labeled by (1) – i.e. at the vertices on the "top left-hand side" of Diagram 2. Let n be the largest integer such that the tree $T_n : \alpha \xrightarrow{b} \xrightarrow{b} \xrightarrow{b} \cdots \xrightarrow{b} \beta$ with n edges and root α embeds in V at the root of V. We can find a vertex w of L with label (2) such that T_n embeds in L at w and it is then easy to see that in fact V embeds in L at w. Thus every finite rooted subtree of L embeds at some vertex with label (2) and it follows that L has no finite test tree. Hence L is not naturally associated with $FIM(X)$, by Theorem 2.1. ■

We remark that Theorem 2.1 does not characterize those rational Schreier sets associated with *finitely presented* monoids of the form $Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$. We provide below an example of a rational Schreier subset of $FG(X)$ that is naturally associated with $FIM(X)$ but which is not associated with any finitely presented inverse monoid of the desired form.

Example 2.3. Consider the language $L = b^* \cup b^*a(b^{-1})^* \subseteq \{a, b, a^{-1}, b^{-1}\}^*$. The minimal automaton of this language L is pictured in Diagram 3 below ((1) is the initial state and both states are terminal).

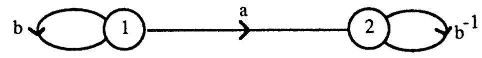

Diagram 3

L is a rational Schreier set represented by the tree sketched in Diagram 4 below.

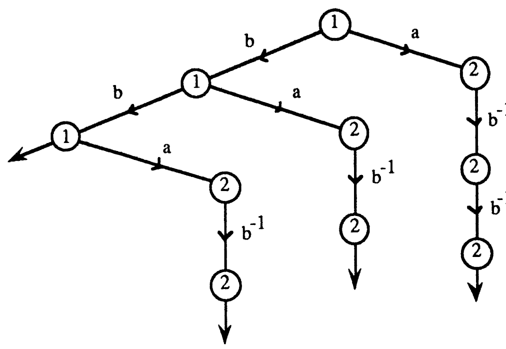

Diagram 4

It is clear that L embeds in L precisely at those vertices labeled by (1) and that the finite tree in Diagram 5 is a finite test tree for L.

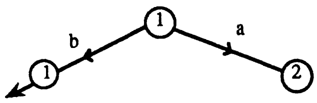

Diagram 5

Hence, by Theorem 2.1, L is naturally associated with $FIM(\{a,b\})$. We claim, however, that there is no finite set $T = \{(e_i, f_i) : i = 1, \ldots, n\}$ of relations on $FIM(X)$ (with $e_i, f_i \in E(FIM(X))$) such that $L = R\sigma$ for some $\mathcal{R}$-class R of $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$. To see this, suppose that such a set T exists and again suppose without loss of generality that $e_i \leq f_i$ for each i (i.e. $MT(f_i) \subseteq MT(e_i)$ for each i). There must exist a word $u \in (X \cup X^{-1})^*$ such that L is obtained from $V = MT(u) \subseteq L$ iteratively in the manner indicated prior to the statement of Theorem 1.6. We claim first of all that M satisfies no relation of the form $b^n b^{-n} = b^m b^{-m}$ for $n > m > 0$: this is because we can always find a vertex γ labeled by (2) (e.g. in the "descending chain" of vertices on the right hand side of Diagram 4) such that the chain labeled by $b^m b^{-m}$ embeds in L at γ but the chain labeled by $b^n b^{-n}$ does not. Similarly (by examining the "descending chain" of vertices on the left hand side of Diagram 6 labeled by (1)) one sees that M satisfies no relation of the form $b^{-n} b^n = b^{-m} b^m$ for $n > m > 0$. A slight extension of this argument shows in fact that if $w_1(b, b^{-1})$ and $w_2(b, b^{-1})$ are two words over the alphabet $\{b, b^{-1}\}$ then $w_1(b, b^{-1}) = w_2(b, b^{-1})$ is a relation in M if and only if $w_1(b, b^{-1})$ and $w_2(b, b^{-1})$ have the same Munn tree. Hence in every relation $e_i = f_i$ in the generating set T for M there must be at least one edge in $MT(f_i)$ labeled by a or a^{-1}. This implies that there is a largest integer $N \geq 0$ with the following property: if $MT(f_i)$ (for $i = 1, \ldots, n$) embeds in L at a vertex γ labeled (2) on the right-hand descending chain in Diagram 4, then the distance in L between γ and the top most vertex (1) must be less than or equal to N. But this forces that there is some number $N_1 \geq N$ such that no vertex labeled by (2) on the right-hand descending chain of Diagram 4, whose distance from the top most vertex (1) is greater than N_1, can ever be reached from V iteratively by applying the relations $e_i = f_i$, $i = 1, \ldots, n$. Hence L does not arise naturally from a finitely presented monoid of the desired form. ∎

It would be interesting to provide an effective procedure for deciding if a rational Schreier subset $L \subseteq FG(X)$ arises naturally as a Schützenberger graph of some finitely presented inverse monoid $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$. In Section 3 we provide an explicit solution to this in the case where the Munn trees $MT(e_i)$ and $MT(f_i)$ are all positively labeled. In the meantime we mention some related results in special cases.

Proposition 2.4. *Let L be a Schreier subset of $FG(X)$. Then there exists an inverse monoid $M = Inv\langle X : e_i = f_i, i \in I\rangle$ (for e_i, $f_i \in E(FIM(X))$) such that $L = R_1\sigma$ (where 1 is the identity of M) if and only if L is a submonoid of $FG(X)$.*

Proof. Suppose first that L is a Schreier submonoid of $FG(X)$. Then for each vertex w of L, $wL \subseteq L$, so L embeds at w. It follows by Theorem 2.1 that the trivial tree 1 is a finite test tree for L, so $L = R\sigma$ for some $\mathcal{R}$-class R of a monoid of the form $M = Inv\langle X : e_i = f_i, i \in I\rangle$. But if 1 is a finite test tree for L, then L may be built from $MT(1)$ iteratively by applying the relations $e_i = f_i$ as in Theorem 1.6. Hence $R = R_1$. The converse is obvious since the $\mathcal{R}$-class R_1 of any inverse monoid M is a submonoid of M. ∎

It is possible to completely characterize the Schützenberger graphs of inverse monoids of the form $M = Inv\langle X : e = 1\rangle$ (where $e = e^2$ in $FIM(X)$). We refer the reader to our paper [6] for details in this case: the Schützenberger graph of R_1 for such a monoid is characterized as a finitely generated Schreier submonoid of $FG(X)$.

3. Positively Labeled Trees

A Schreier set (tree) $L \subseteq FG(X)$ is said to be ***positively labeled*** if $L \subseteq X^*$ (i.e. each vertex $w \neq 1$ of L is labeled by a product of elements of X). In this section we study finitely presented inverse monoids of the form $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$ where e_i, f_i are ***positively labeled idempotents*** of $FIM(X)$ (i.e. $MT(e_i)$, $MT(f_i)$ are positively labeled trees). We are able to classify the Schützenberger graphs of such monoids explicitly. If $\alpha \xrightarrow{x} \beta$ is an edge of L for which α is on the geodesic from 1 to β (i.e. α is "closer to 1 than β ") then we say that this edge is ***positively labeled*** [resp. ***negatively labeled***] if $x \in X$ [resp. $x \in X^{-1}$]. Thus all edges of a positively labeled Schreier set (tree) are positively labeled.

Theorem 3.1. *Let L be a rational Schreier subset of $FG(X)$ (for X a finite set). Then there is some finitely presented inverse monoid of the form $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle$ where e_i, f_i are positively labeled idempotents of $FIM(X)$ and some $\mathcal{R}$-class R of M such that $L = R\sigma$ if and only if L has only finitely many negatively labeled edges.*

Proof. Suppose first that $L = R\sigma$ for R an $\mathcal{R}$-class of the monoid $M = Inv\langle X : e_i = f_i, i = 1, \ldots, n\rangle = (X \cup X^{-1})^*/\tau$ where e_i, f_i are positively labeled idempotents of $FIM(X)$. We already know by Theorem 1.3 that L is a rational Schreier subset of $FG(X)$. Furthermore, if u is an element of $(X \cup X^{-1})^*$ such that $u\tau \in R$, then L is built iteratively from $MT(u)$ by applying the relations $e_i = f_i$ $(i = 1, \ldots, n)$ as described in Theorem 1.6. Now all edges of $MT(e_i)$ and $MT(f_i)$ (for $i = 1, \ldots, n$) are positively labeled. As usual we may assume without loss of generality that $MT(f_i) \subseteq MT(e_i)$ for each i. Let w be a reduced word in L such that $MT(f_i)$ embeds at w (in which case $MT(e_i)$ must also embed at w). If $v_1 \circ\!\xrightarrow{x}\!\circ\, v_2$ is an edge of $MT(e_i)$ then $wv_1 \circ\!\xrightarrow{x}\!\circ\, wv_2$ is an edge of L. Since $x \in X$, this edge is positively labeled if wv_1 is closer to 1 than wv_2. Since $v_1 \in X^*$ and $v_2 = v_1 x$, the only possible way for wv_2 to be closer to 1 than wv_1 is if $w = w_2 x^{-1} v_1^{-1}$, in which case $wv_1 = w_2 x^{-1}$ and $wv_2 = w_2$. This forces the edge $wv_1 \circ\!\xrightarrow{x}\!\circ\, wv_2$ to be on the geodesic from 1 to w. Thus if K is a finite Schreier subset of L and if w is a vertex in K at which $MT(f_i)$ embeds in L, then the only negatively labeled edges of $K \cup w \cdot MT(e_i)$ are those edges of K that are negatively labeled. The inductive construction of L from $MT(u)$ obtained by applying the relations $e_i = f_i$ $(i = 1, \ldots, n)$ thus produces no negatively labeled edges in L that are not in $MT(u)$. Hence L has only finitely many negatively labeled edges.

Suppose conversely that L has only finitely many negatively labeled edges. We may assume without loss of generality that L is infinite, since every finite Schreier subset of $FG(X)$ arises as a Munn tree of some word in $(X \cup X^{-1})^*$, and hence as the Schützenberger graph of an $\mathcal{R}$-class of $FIM(X)$ itself. Let $\mathcal{B}(L) = (Q, q_0, Q)$ be the Benois automaton of L: the notation (Q, q_0, Q) means that Q is the (finite) set of states of $\mathcal{B}(L)$, q_0 is the initial state and every state is terminal. The set Q of states of $\mathcal{B}(L)$ is in one-one correspondence with the set of isomorphism classes of *directed* rooted subtrees of L. Label the vertices of L by the states in Q in such a way that two vertices v_1, v_2 in L have the same label if and only if the directed rooted subtrees of L rooted at v_1 and v_2 are isomorphic. Denote the directed subtree of L rooted at a state labeled by $q \in Q$ by $[q]$. Clearly $[q]$ is a Schreier subset of $FG(X)$. Since L is infinite with only finitely many negatively labeled edges, there exists at least one state $q \in Q$ such that $[q]$ is positively labeled. Let Q' denote the set of all states $q \in Q$ such that $[q]$ is positively labeled. For each integer $m \geq 0$, let $L^{(m)}$ denote the subtree of L spanned by all vertices of L of distance less than or equal to m from 1. Then $L^{(m)}$ is a finite Schreier subset of L. Clearly there exists some integer $n > 0$ such that all negatively labeled edges of L are edges of $L^{(n)}$. Every vertex of $L \backslash L^{(n)}$ is labeled by an element of Q'. For each state $q \in Q$ and each integer $m \geq 0$, let $L_q^{(m)}$ denote the subtree of $[q]$ spanned by the vertices of $[q]$ of distance less than or equal to m from the root of $[q]$. Thus $L_{q_0}^{(m)} = L^{(m)}$. If p and q are distinct elements of Q', then $[p]$ and $[q]$ are non-isomorphic positively labeled trees, so either $[p]$ does not embed in $[q]$ at the root of $[q]$ or else $[q]$ does not embed in $[p]$ at the root of $[p]$. In the former case there is some word $u_{p,q} \in X^*$ such that $u_{p,q}\, u_{p,q}^{-1}$ labels a path in $[p]$ from the root of $[p]$ to the root of $[p]$ but $u_{p,q}\, u_{p,q}^{-1}$ does not label any such path in $[q]$. Dually, there exists such a word $u_{q,p}$ in the latter case. There is at most one state $p_0 \in Q'$ such that $[p_0]$ embeds in $[p]$ at the root of $[p]$ for every $p \in Q'$. If such a state p_0 exists, define $u_{p_0} = 1 \in X^*$. For any state $p \in Q' - \{p_0\}$, there exists at least one state $q \in Q'$ such that $[p]$ does not embed in $[q]$ at the root of $[q]$: in this case, define $u_p = \prod u_{p,q}\, u_{p,q}^{-1}$ where the product is over all $q \in Q'$ such that $[p]$ does not embed in $[q]$ (at the root of $[q]$) and the product is taken in any order. For each $q \in Q'$, there is some integer $m_q > 0$ such that the path $[q]$ labeled by u_q (from the root of $[q]$ to itself) is contained in $L_q^{(m_q)}$. Let $N = \max\{m_q : q \in Q'\}$. The integer N has the following property: if $w \in L \backslash L^{(n)}$ and $L_q^{(N)}$ embeds in L at w, then $[q]$ embeds in L at w. (Notice that for $w \in L \backslash L^{(n)}$, a positively labeled tree embeds in L at w in the undirected sense if and only if it embeds in L at w in the directed sense). We refer to $L_q^{(N)}$ as a *directed finite test tree* for $[q]$. Clearly $L_q^{(t)}$ is also a directed finite test tree for $[q]$ whenever $t \geq N$.

Now suppose that w is a vertex of $L^{(n)}$ and suppose $L_q^{(N)}$ embeds in L at w (in the undirected sense) for some $q \in Q'$. Note that this embedding is not necessarily in the positive direction: that is, it is possible that some vertices of $w \cdot L_q^{(N)}$ may lie on the geodesic from 1 to w in L. Suppose further that $[q]$ does not embed in L at w (in the undirected sense). Then there is some integer $N_q(w) > N$ such that $L_q^{(N_q(w))}$ does not embed in L at w. If $[q]$ does embed at w, define $N_q(w) = N$. Since $N_q(w) \geq N$ for any $w \in L^{(n)}$, the tree $L_q^{(N_q(w))}$ is also a directed finite test tree for $[q]$. Let $\alpha_q = \max\{N_q(w) : w \in L^{(n)}\}$ and let $\alpha = \max\{\alpha_q : q \in Q'\}$. Clearly, $L_q^{(\alpha)}$ is a directed finite test tree for $[q]$: in addition, if $L_q^{(\alpha)}$ embeds in L at any vertex $w \in L^{(n)}$ (in the undirected sense), then $[q]$ also embeds in L at w (in the undirected sense). The same property holds for any vertex $w \in L \backslash L^{(n)}$ since directed and undirected embeddings of positively labeled trees coincide at all such vertices and $L_q^{(\alpha)}$ is positively labeled.

Hence $L_q^{(\alpha)}$ is an *(undirected) finite test tree for* $[q]$: that is, $L_q^{(\alpha)}$ embeds in L at a vertex w of L (in the undirected sense) if and only if $[q]$ embeds in L at w.

It follows that if F_q is any Schreier set with $L_q^{(\alpha)} \subseteq F_q \subseteq [q]$ (for $q \in Q'$) then $L_q^{(\alpha)}$, F_q and $[q]$ all embed in L at the same vertices of L. We construct trees F_q (for $q \in Q'$) as follows. It is clear that if w is any vertex of L labeled by $q \in Q'$ and if $L_q^{(\alpha)}$ is embedded in L at w, then the leaves of $w \cdot L_q^{(\alpha)}$ are all labeled by elements of Q' (and this labeling is independent of the choice of which vertex $w \in L$ we choose, so long as w is labeled by q). Thus there is a well-defined labeling of the leaves of $L_q^{(\alpha)}$ by elements of Q' (labeled a leaf of $L_q^{(\alpha)}$ in the same way as the corresponding leaf of $w \cdot L_q^{(\alpha)}$). For each leaf t of $L_q^{(\alpha)}$, let $q(t)$ denote the label on t. Then define (for $q \in Q'$),

$$F_q = L_q^{(\alpha)} \cup \left(\bigcup_t t \cdot L_{q(t)}^{(\alpha)} \right)$$

where the union is taken over all leaves t of $L_q^{(\alpha)}$: in other words, F_q is formed by glueing the tree $L_{q(t)}^{(\alpha)}$ onto $L_q^{(\alpha)}$ at each leaf t of $L_q^{(\alpha)}$. Clearly $L_q^{(\alpha)} \subseteq F_q \subseteq [q]$, so $L_q^{(\alpha)}$ and F_q embed in L at the same vertices. Let e_q [resp. f_q] be any idempotent of $FIM(X)$ whose Munn tree is $L_q^{(\alpha)}$ [resp. F_q]. Now let M be the inverse monoid $M = inv\langle X : e_q = f_q, q \in Q' \rangle$. Clearly M is presented by a finite number of relations of the form $e_q = f_q$ where e_q and f_q are positively labeled. Now let $\beta = n + \alpha + 1$ and let e be any element of $(X \cup X^{-1})^*$ such that $MT(e) = L^{(\beta)}$: then e is an idempotent of $FIM(X)$. Let $R = R_e$, the $\mathcal{R}$-class of e in the monoid M. We claim that $L = R\sigma$. To see this, note that $MT(e) = L^{(\beta)} \subseteq L$ and also that the Schreier set L is closed under application of the relations $e_q = f_q$ ($q \in Q'$) since if $MT(e_q)$ embeds at a vertex in L, then so does the larger tree $MT(f_q)$. Hence by the iterative construction of $R\sigma$ given in Theorem 1.6, $R\sigma \subseteq L$. On the other hand, we claim that every vertex $w \in L$ is obtained from $MT(e) = L^{(\beta)}$ by iteration involving a finite number of applications of the relations $e_q = f_q$ ($q \in Q'$). In order to see this, note first that all vertices of L of distance less than or equal to β are contained in $R\sigma$ since $MT(u) = L^{(\beta)}$. Thus, in particular, all the negatively labeled edges of L are contained in $R\sigma$. Let w be a reduced word in L with $|w| > \beta$. Then w factors in $(X \cup X^{-1})^*$ as a product $w = w_1 u_1$ where $|w_1| = n+1$ and $u_1 \in X^*$. Suppose that w_1 is labeled by q_1: then $q_1 \in Q'$ since $|w_1| > n$. It follows that $w_1 \cdot L_q^{(\alpha)} \subseteq L^{(\beta)} \subseteq R\sigma$ by definition of β. Now apply the relation $e_{q_1} = f_{q_1}$ at w_1: in other words, $w_1 \cdot F_{q_1} \subseteq R\sigma$ by definition of the iterative construction of $R\sigma$. If $w \in w_1 \cdot F_{q_1}$, we are finished. If not, there are vertices w_2, w_3 of L such that w_2 is on the geodesic from w_1 to w, w_3 is on the geodesic from w_2 to w, $|w_2| = |w_1| + \alpha$ (i.e. w_2 is a leaf of $w_1 \cdot L_{q_1}^{(\alpha)}$) and $|w_3| = |w_2| + \alpha$ (i.e. w_3 is a leaf of $w_2 \cdot L_{q_2}^{(\alpha)}$ where w_2 is labeled by $q_2 \in Q'$). Let w_3 be labeled by $q_3 \in Q'$. Since w_2, $w_3 \in w_1 \cdot F_{q_1} \subseteq R\sigma$ and since $w_2 \cdot L_{q_2}^{(\alpha)} \subseteq R\sigma$, we may apply the relation $e_{q_2} = f_{q_2}$ and obtain $w_2 \cdot F_{q_2} \subseteq R\sigma$. If $w \in w_2 \cdot F_{q_2}$, we are done. Otherwise, there exists w_4 on the geodesic from w_3 to w with $|w_4| = |w_3| + \alpha$ (i.e. w_4 is a leaf of $w_3 \cdot L_{q_3}^{(\alpha)}$). Again, since $w_3 \cdot L_{q_3}^{(\alpha)} \subseteq R\sigma$, we see that $w_3 \cdot F_{q_3} \subseteq R\sigma$. Continue this process: we obtain a sequence of vertices w_1, w_2, $w_3, \ldots, w_k$ on the geodesic from 1 to w such that $|w_{i+1}| = |w_i| + \alpha$ for each i and such that if w_i is labeled by $q_i \in Q'$, then $w_i \cdot L_{q_i}^{(\alpha)} \subseteq R\sigma$. Since $|w|$ is finite, we eventually find an integer k such that $w \in w_k \cdot L_{q_k}^{(\alpha)} \subseteq R\sigma$. Hence $w \in R\sigma$ and so $L \subseteq R\sigma$. Hence $L = R\sigma$ and the result is proved. ∎

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