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Bass–Serre Theory for Groupoids and the Structure of Full Regular Semigroup Amalgams*

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T. E. Hall proved in 1978 that if $[S_1, S_2; U]$ is an amalgam of regular semigroups in which $S_1 \cap S_2 = U$ is a full regular subsemigroup of S_1 and S_2 (i.e., S_1, S_2, and U have the same set of idempotents), then the amalgam is strongly embeddable in a regular semigroup S that contains S_1, S_2, and U as full regular subsemigroups. In this case the inductive structure of the amalgamated free produce $S_1 *_U S_2$ was studied by Nambooripad and Pastijn in 1989, using Ordman's results from 1971 on amalgams of groupoids. In the present paper we show how these results may be combined with techniques from Bass–Serre theory to elucidate the structure of the maximal subgroups of $S_1 *_U S_2$. This is accomplished by first studying the appropriate analogue of the Bass–Serre theory for groupoids and applying this to the study of the maximal subgroups of $S_1 *_U S_2$. The resulting graphs of groups are arbitrary bipartite graphs of groups. This has several interesting consequences. For example if S_1 and S_2 are combinatorial, then the maximal subgroups of $S_1 *_U S_2$ are free groups. Finite inverse semigroups may be decomposed in non-trivial ways as amalgams of inverse semigroups. © 1996 Academic Press, Inc.

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1. INTRODUCTION

If S_1 and S_2 are semigroups such that $S_1 \cap S_2 = U$ is a non-empty subsemigroup of both S_1 and S_2, then $[S_1, S_2; U]$ is called an amalgam of semigroups and U is the core of the amalgam. The amalgam $[S_1, S_2; U]$ is said to be *strongly embeddable* in a semigroup if these exist a semigroup S and injective homomorphisms

$$\phi_i \colon S_i \to S$$

such that

$$\phi_1|_U = \phi_2|_U$$

and

$$S_1\phi_1 \cap S_2\phi_2 = U\phi_1 = U\phi_2.$$

A semigroup S is a *regular* semigroup if for each $a \in S$ there exists $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$: such an element a' is called an *inverse* of a. If each element of S has a *unique* inverse, S is called an *inverse* semigroup: equivalently, an inverse semigroup is a regular semigroup whose idempotents commute. Such semigroups may be faithfully represented as semigroups of partial one–one maps on a set X. We refer the reader to Petrich [13] for this result and many other standard results and ideas about inverse semigroups.

It is well known that a semigroup amalgam $[S_1, S_2; U]$ is not necessarily strongly embeddable. On the other hand, an important theorem of T. E. Hall [3] shows that every amalgam of inverse semigroups is strongly embeddable (in an inverse semigroup) and another theorem of Hall [4] shows that if $[S_1, S_2; U]$ is a semigroup amalgam in which S_1, S_2, and U are regular semigroups and U is a *full* subsemigroup of S_1 and S_2 (i.e., U contains all of the idempotents of S_1 and S_2), the this amalgam is strongly embeddable in a regular semigroup S that contains S_1, S_2, and U as full regular subsemigroups. It follows that in this case the amalgam $[S_1, S_2; U]$ is strongly embeddable in the amalgamated free product $S_1 *_U S_2$ in the category of regular semigroups. (The regular semigroup $S_1 *_U S_2$ is defined by the usual universal diagram of regular semigroups and morphisms.) It is clear that if S_1, S_2, and U are inverse semigroups with U full in S_1 and S_2 then $S_1 *_U S_2$ is also inverse, so it is the amalgamated free product of S_1 and S_2 over U in the category of *inverse* semigroups.

Hall's proofs of his embeddability theorems are via extensions of representations and provide little information about the structure of the amalgamated free product. Considerable additional information has been obtained by Nambooripad and Pastijn [10] in the case where S_1, S_2, and U

are regular semigroups and U is a full subsemigroup of S_1 and S_2. Nambooripad and Pastijn make use of Ordman's work on amalgamated free products in the category of groupoids [11].

A *groupoid* is a small category in which each morphism is an isomorphism. We refer to Higgins [5] for the theory of groupoids. In particular, it is useful to associate with each groupoid G an underlying directed graph whose vertices are the identities (objects) of G and whose directed edges are the morphisms of G. We denote the initial (resp. terminal) vertex of an edge g in G by $i(g)$ [resp. $t(g)$] and we sometimes write $g\colon i(g) \to t(g)$. Then the product gh of two edges g and h in a groupoid G is defined if and only if $t(g) = i(h)$. The inverse of an edge g in G is denoted as usual by g^{-1}: clearly $g^{-1}\colon t(g) \to i(g)$. It is convenient to identify the groupoid G with the set of edges (morphisms) of G—the objects of G are identified with the identities of G. At each vertex v of G the set G_v of morphisms from v to v forms a group with respect to the multiplication in G. We refer to G_v as the *vertex group* of G based at v.

Suppose that G, H, and U are groupoids and that U is a subgroupoid of G and H with $G \cap H = U$. One can define the amalgamated free product $G *_U H$ of G and H amalgamating U in the category of groupoids by the usual universal diagram. Groupoid amalgams have been studied by Ordman [11, 12]. We briefly review Ordman's results here. In order to understand the groupoid $G *_U H$ we consider words of the form $a_1 a_2 \dots a_n$, where $a_i \in G \cup H$ and $t(a_i) = i(a_{i-1})$ is an identity in U. Two such words are *equivalent* if they are connected by a finite string of elementary equivalences of the form:

(E1) If a_i is an identity of U then $a_1 a_2 \dots a_n$ and $a_1 \dots a_{i-1} a_{i+1} \dots a_n$ are elementary-equivalent;

(E2) If $a_i a_{i+1} = a_i^*$, where a_i and a_{i+1} are both in G or both in H, then $a_1 a_2 \dots a_n$ is elementary-equivalent to $a_1 \dots a_{i-1} a_i^* a_{i+2} \dots a_n$.

Equivalence classes of words form a groupoid under the obvious operation of multiplication and the resulting groupoid is isomorphic to the amalgamated free product $G *_U H$ (see [11]).

As in the case of group amalgams, one may refine this somewhat so as to obtain a "normal form" for words in $G *_U H$. We restrict attention for the remainder of the paper to the case where G, H, and U are groupoids with $G \cap H = U$ and U contains all of the identities of G and of H, since this is the only case that we shall need.

Define a relation $\sim_U$ on G by $g \sim_U g'$ (for $g, g' \in G$) if $g = g'u$ for some $u \in U$. (This equation means that $t(g') = i(u)$ and $g = g'u$ in G.) An elementary calculation shows that $\sim_U$ is an equivalence relation on G since every identity of G is in U. Denote the equivalence class containing

$g \in G$ by gU. Thus

$$gU = \{gu : u \in U, t(g) = i(u)\}.$$

By analogy with the situation in group theory, it is natural to refer to gU as the *left coset of* U *containing* g. One may similarly consider the left coset hU of U containing h for each $h \in H$. Representatives for these left cosets of U will be called *coset representatives* for U in G (or H). Choose a system of coset representatives for the left cosets of U in G (resp. H). It is not difficult to see, as in group theory, that every element of $G *_U H$ may be expressed in the form

$$a_1 a_2 \dots a_n u, \tag{1}$$

where the a_i are coset representatives for U in G or H, no a_i is in U, $u \in U$ and $a_i \in G$ if and only if $a_{i+1} \in H$ (i.e., the a_i are alternating coset representatives for U in either G or H). A suitable modification of the usual "van der Waerden" method from combinatorial group theory (see, for example, Cohen [2]) shows that every element of $G *_U H$ may be *uniquely* expressed in such a form. This result is implicit in the paper of Ordman [11].

THEOREM 1. *If U, G, and H are groupoids with $G \cap H = U$ and U contains all the identities of G and of H, then every element of $G *_U H$ may be uniquely expressed in the form* (1) *for suitable choice of coset representatives for U in G or H.*

We remark that a similar result may easily be formulated without the restriction that U contain all the identities of G and H, but the relation $\sim_U$ is not an equivalence relation in this case since $\sim_U$ fails to be reflexive. The corresponding canonical form is somewhat more cumbersome to formulate and will not be needed in this paper.

We also record an essentially equivalent form of Theorem 1 which may be obtained from that theorem by a straightforward argument similar to the usual group-theoretic argument.

COROLLARY 1. *If $a_1, a_2, \dots, a_n \in (G - U) \cup (H - U)$ with $a_i \in G - U$ if and only if $a_{i+1} \in H - U$ then the product $a_1 a_2 \dots a_n$ is not an identity of $G *_U H$.*

We turn now to a very brief description of Nambooripad's theory [9] of inductive groupoids and its connection with the structure of amalgams of regular semigroups. The basic idea is to associate an "inductive" groupoid with each regular semigroup in a canonical way. Here we review only the construction of the groupoid from the semigroup. If S is any regular semigroup then the set $G(S) = \{(x, x') : x'$ is an inverse of $x\}$ forms a groupoid with the set $E(S)$ of idempotents of S as its set of objects (identities) and with (x, x') as a morphism form xx' to $x'x$. Thus a product $(x, x')(y, y')$ is defined in $G(S)$ if and only if $x'x = yy'$. In this case the product is $(xy, y'x')$. Note that the vertex group of $G(S)$ at the idempotent e of S is isomorphic to H_e, the maximal subgroup of S at e. There is also a natural additional structure on $G(S)$ with respect to which $G(S)$ becomes what Nambooripad refers to as an "inductive groupoid" [9]. It suffices for our purposes here to note that the category of inductive groupoids is naturally equivalent to the category of regular semigroups—see [9] for details.

In [10] Nambooripad and Pastijn provided an alternative proof of Hall's theorem [4] on the embeddability of regular semigroups amalgamating a common full regular subsemigroup. In particular, their results show how to construct a natural inductive structure on the groupoid amalgam $G(S_1)*_{G(U)}G(S_2)$ when $S_1 \cap S_2 = U$ is a full regular subsemigroup of S_1 and S_2. We refer to [10] for the details of this. We reformulate that portion of the Nambooripad–Pastijn theorem that we need as follows.

THEOREM 2. *Let $[S_1, S_2; U]$ be an amalgam of regular semigroups with $S_1 \cap S_2 = U$ a full regular subsemigroup of S_1 and S_2. Then $G(U)$ contains all of the identities of $G(S_1)$ and $G(S_2)$ and $G(S_1 *_U S_2) = G(S_1)*_{G(U)}G(S_2)$. In particular, if e is an idempotent of U then the maximal subgroup H_e of e in $S_1 *_U S_2$ is isomorphic to the vertex group of $G(S_1)*_{G(U)}G(S_2)$ at e.*

We remark that in the non-full case, $G(S_1 *_U S_2) \neq G(S_1)*_{G(U)}G(S_2)$ in general. This is because there are more idempotents in the general case and the corresponding amalgam of groupoids must be taken in the category of inductive groupoids. Thus the point of the Nambooripad–Pastijn theorem is that in the case of a full amalgam, the amalgamated free product of the corresponding groupoids *in the category of groupoids* has a natural inductive structure.

While the theorem of Nambooripad and Pastijn implicitly carries complete information about the structure of $S_1 *_U S_2$ (when U is full in S_1 and S_2), we are able to obtain much more explicit structural information about the maximal subgroups of $S_1 *_U S_2$ from the previous theorem by using the Bass–Serre theory of graphs of groups.

Recall [2, 14] that a *graph of groups* $(\mathscr{G}, X)$ consists of

(1) a graph X: let $V(X)$ [resp. $E(X)$] denote the set of vertices [resp. edges] of X: recall from [2] that each edge y of such a graph has an inverse edge denoted by $\bar{y}$;

(2) for each vertex $v \in V(X)$ a group G_v and for each edge $y \in E(X)$ a group G_y such that $G_y = G_{\bar{y}}$; and

(3) for each edge $y \in E(X)$ an embedding τ: $G_y \to G_{t(y)}$.

We define $\sigma\colon G_y \to G_{i(y)}$ to be the embedding of $G_{\bar{y}}$ in $G_{t(\bar{y})} = G_{i(y)}$. Then if X is connected and T is a spanning tree of X, the *fundamental group* of $(\mathscr{G}, X)$ is the group $\pi(\mathscr{G}, X, T)$ generated by the set $E(X)$ and all the vertex groups of $(\mathscr{G}, X)$ subject to the relations that hold in these vertex groups together with the relations $\bar{y} = y^{-1}$, $y\tau(g)y^{-1} = \sigma(g)$, $\forall g \in G_y$, $\forall y \in E(X)$, and $y = 1$, $\forall y \in E(T)$.

This group is independent of the choice of spanning tree T and is usually denoted by $\pi(\mathscr{G}, X)$. See [14, 2] for details.

Recall also that if a group G acts on a tree Γ with quotient graph $X = G/\Gamma$ (the graph of orbits of the action of G on Γ), then there is a natural graph of groups $(\mathscr{G}, X)$ whose vertex (edge) groups are the stabilizers of the corresponding vertex (edge) of Γ. One of the main theorems of Bass–Serre theory [14, 2] then asserts that $G \cong \pi(\mathscr{G}, X)$. Thus a presentation of a group G is known once its action on a tree is understood. We shall exploit this to study the structure of a maximal subgroup of $S_1 *_U S_2$ by finding a natural action of this group on a suitable tree. The resulting graph of groups will be an arbitrary bipartite graph of groups in general, quite in contrast to the situation for amalgamated free products of groups whose associated graph of groups is a segment [14].

2. STRUCTURE OF THE MAXIMAL SUBGROUPS

Throughout this section G, H, and U will denote groupoids with $G \cap H = U$ such that U contains all of the identities of G and of H. Recall that Theorem 1 provides us with a normal form for elements of the amalgamated free product $A = G *_U H$. Fix an identity (object) e of U: we are interested in calculating the structure of the vertex group A_e of $A = G *_U H$ at e. For each element $a \in A$ denote by aU [resp. aG, aH] the left coset in A of U [resp. G, H] containing a. Thus $aU = \{au : u \in U, t(a) = i(u)\}$, etc.

Define a graph X as follows. The set $V(X)$ of vertices of X is

$$V(X) = \{aG : a \in A\} \cup \{aH : a \in A\}$$

and the set $E_+(X)$ of positively oriented edges of X is

$$E_+(X) = \{aU : a \in A\}.$$

The initial [resp. terminal] vertex of the edge aU is $i(aU) = aG$ [resp. $t(aU) = aH$]. The inverse of the edge aU is denoted by $(aU)^{-1}$: clearly $i((aU)^{-1}) = aH$ and $t((aU)^{-1}) = aG$. The set of inverses of positively oriented edges is denoted by $E_-(X)$ and the set of edges of X is $E(X) = E_+(X) \cup E_-(X)$. It is clear from the construction of X that X is a bipartite graph: the vertices of X are naturally partitioned into two disjoint set (cosets of G in A and cosets of H in A) and each edge of X has initial vertex in one of these sets and terminal vertex in the other. We shall show that in fact X is a forest. We first record two easy lemmas for future reference.

LEMMA 1. *If $a, b \in A$ then $aG = bG$ [resp. $aH = bH$, $aU = bU$] if and only if $a^{-1}b \in G$ [resp. $a^{-1}b \in H$, $a^{-1}b \in U$].*

Proof. Suppose $aG = bG$. By definition this means that $a = bg$ for some $g \in G$, where this is an equation in the groupoid A. This implies that $i(a) = i(b)$, $t(a) = t(g)$, and $t(b) = i(g)$ in A. Thus the element $a^{-1}b$ is defined in A, $i(a^{-1}b) = t(g)$ and $t(a^{-1}b) = i(g)$. Furthermore, $a^{-1}bg = a^{-1}a$ is an identity of A so $a^{-1}bgg^{-1} = a^{-1}ag^{-1} = g^{-1}$ and since gg^{-1} is also an identity of A we have $a^{-1}bgg^{-1} = a^{-1}b$, so $a^{-1}b = g^{-1} \in G$. Conversely, if $a^{-1}b \in G$, then $a^{-1}b = g$ for some $g \in G$ so $aa^{-1}b = ag$, so $b = ag$ since aa^{-1} is the identity at $i(b)$. It follows that $aG = bG$, as required.

In order to simplify the statements of several subsequent results it is convenient to introduce the following notation, which is suggested by analogy with standard notation in semigroup theory. If e and f are identities of a groupoid G we write $e\mathscr{D}^G f$ if e and f are in the same connected component of G. It is well known [5] that $e\mathscr{D}^G f$ if and only if there is $g \in G$ such that $gg^{-1} = e$ and $g^{-1}g = f$: in this case $gG_f g^{-1} = G_e$. In the sequel we refer to the connected component of f in G as the $\mathscr{D}$-class of f in G and denote it by D_f^G.

LEMMA 2. *If $a, b \in A$ and $aG = bG$ [resp. $aH = bH$, $aU = bU$] then $i(a) = i(b)$ and $t(a)\mathscr{D}^G t(b)$ [resp. $t(a)\mathscr{D}^H t(b)$, $t(a)\mathscr{D}^U t(b)$].*

Proof. This is clear from the argument used in the proof of the previous lemma.

LEMMA 3. *The graph X is a forest.*

Proof. If X is not a forest it must have a cyclically reduced circuit $(e_1, e_2, \ldots, e_k)$—a sequence of edges without backtracking and with $i(e_s) = t(e_{s-1})$, $i(e_1) = t(e_k)$ and $e_1 \neq e_k^{-1}$. Without loss of generality

$$e_1 = a_1U, e_2 = (a_2U)^{-1}, \ldots, e_k = (a_kU)^{-1}$$

for some $a_i \in A$. We must then have that k is even and

$$a_1H = a_2H, a_2G = a_3G, \ldots, a_kH = a_{k-1}H, a_kG = a_1G,$$

so there exist $h_1, \ldots, h_{k/2} \in H$ and $g_1, \ldots, g_{k/2} \in G$ such that

$$a_1 = a_kg_1, a_k = a_{k-1}h_1, \ldots, a_3 = a_2g_{k/2}, a_2 = a_1h_{k/2}.$$

Furthermore, since $e_1 \neq e_k^{-1}$ we must have $g_1 \notin U$ (or else $a_1U = a_kU$). Similarly since the circuit has no backtracking, $g_i \notin U$ for each i and $h_i \notin U$ for each i, so that

$$h_1, \ldots, h_{k/2} \in H - U, \qquad g_1, \ldots, g_{k/2} \in G - U.$$

Then

$$a_1 = a_k g_1 = a_{k-1} h_1 g_1 = a_{k-2} g_2 h_1 g_1 = \cdots = a_1 h_{k/2} g_{k/2} \ldots h_2 g_2 h_1 g_1.$$

Hence

$$a_1^{-1} a_1 = a_1^{-1} a_1 h_{k/2} g_{k/2} \ldots h_2 g_2 h_1 g_1 = h_{k/2} g_{k/2} \ldots h_2 g_2 h_1 g_1$$

is an identity of A. This contradicts Corollary 1, so X is a forest.

For each identity e of A denote by X_e the connected component of X containing the vertex eG. Clearly X_e is a tree. Let us analyze the nature of this tree in more detail. Choose any element $a \in A$ and write a in normal form (1) $a = a_1 a_2 \ldots a_n u$ as in Theorem 1. Suppose that $i(a) = i(a_1) = e$ (an identity of A). Consider the case in which a_1 is a coset representative for U in G. Then by Lemma 1, $a_1G = eG$, so a_1U is an edge in X_e with $i(a_1U) = a_1G = eG$ and $t(a_1U) = a_1H$. Since $t(a_1) = i(a_2)$ and a_2 is a coset representative of U in H we have $a_1H = a_1a_2H$ and so a_1a_2U is an edge in X_e with $i(a_1a_2U) = a_1a_2G$ and $t(a_1a_2U) = a_1a_2H = a_1H$. Continuing by induction we see

LEMMA 4. *For each element $a \in A$, aG and aH are vertices of X_e and aU is an edge of X_e if and only if $i(a) = e$. The subtrees X_e and X_f (for e and f distinct identities of A) are disjoint.*

There is an obvious (partial) left action of the groupoid A on the forest X. Namely, if $a, b \in A$ then $a.(bU) = abU$ if ab is defined in A (i.e., if $t(a) = i(b)$) and similarly $a.(bG)$ and $a.(bH)$ are defined in this case. From Lemma 4 it follows that if $a \in A_e$ (the vertex group of A at e) then a acts on X_e; i.e., $aX_e \subseteq X_e$ and this action is well defined. Thus the group A_e acts in a natural way on the tree X_e. From the fundamental theorem of Bass–Serre theory [14, 2] we obtain information about the structure of A_e by studying the orbits and stabilizers of this action.

LEMMA 5. *For each $a \in A$ for which $i(a) = e$ the orbit of aG [resp. aH, aU] under the action by A_e is*

$$O(aG) = \{bG \,[\text{resp. } bH, bU] : i(b) = e \text{ and } t(a)\mathscr{D}^G t(b) \\ [\text{resp. } t(a)\mathscr{D}^H t(b), t(a)\mathscr{D}^U t(b)]\}.$$

Proof. Suppose that $bG \in O(aG)$. Then there exists $c \in A_e$ such that $bG = c.aG$. Since $i(c) = t(c) = e = i(a)$ we see immediately by Lemma 2 that $i(b) = e$ and $t(b)\mathscr{D}^G t(ca) = t(a)$. Conversely suppose that $i(a) = i(b) = e$ and $t(a)\mathscr{D}^G t(b)$. Then there exists some edge c in G with $i(c) = t(a)$ and $t(c) = t(b)$. Consider the element $d = acb^{-1} \in A_e$. Clearly $d.bG = acb^{-1}bG = acG = aG$ by Lemma 1 since $c \in G$. Hence $bG \in O(aG)$. A similar argument applies to the calculation of $O(aH)$ and $O(aU)$.

LEMMA 6. *For each $a \in A$ with $i(a) = e$ and $t(a) = f$ the stabilizer of aG [resp. aH, aU] under the action of A_e on X_e is $aG_f a^{-1}$ [resp. $aH_f a^{-1}, aU_f a^{-1}$]; this group is isomorphic to G_f [resp. H_f, U_f].*

Proof. Let $a \in A$ with $i(a) = e$. Suppose $b \in A_e$ is in the stabilizer of aG, i.e., $b.(aG) = aG$. Then by Lemma 1 there exists some $g \in G$ such that $a^{-1}ba = g$. It follows that $i(g) = t(g) = f$, so $g \in G_f$, and that $b = aga^{-1}$ and so $\operatorname{Stab}(aG) = aG_f a^{-1}$. It is easy to check that the map $b \to a^{-1}ba$ is an isomorphism from the stabilizer of aG in A_e onto G_f. A similar argument applies to the stabilizers of aH and aU.

We now construct a graph of groups $(\mathscr{G}, Y)$ associated with the groupoid $A = G *_U H$. Let $V(Y)$ be the disjoint union of the $\mathscr{D}$-classes (i.e., connected components) of G and H. Let $E_+(Y)$ be the set of $\mathscr{D}$-classes of U. For $D \in E_+(Y)$ set

$$D_\sigma = \text{the } \mathscr{D}\text{-class of } G \text{ containing } D, \text{ and}$$
$$D\tau = \text{the } \mathscr{D}\text{-class of } H \text{ containing } D.$$

For each $\mathscr{D}$-class D of G [resp. H, U] let G_D [resp. H_D, U_D] be a specified vertex group at some vertex in D: if $D \in E_-(Y)$ let $U_D = U_{\overline{D}}$. These specified groups will be the vertex and edge groups of $(\mathscr{G}, Y)$. For $D \in E_+(Y)$ let K be the vertex group of $D\sigma$ containing G_D. Fix an element $g \in D\sigma$ such that $g^{-1}Kg = G_{D\sigma}$. The map $\sigma\colon U_D \to G_{D\sigma}$ is then given by $u \to g^{-1}ug$. We define the map $\tau\colon U_D \to H_{D\tau}$ in a similar fashion. This specifies a graph of groups $(\mathscr{G}, Y)$. For each identity e of A note that the $\mathscr{D}$-class of e in G and the $\mathscr{D}$-class of e in H are in the same connected component of Y, which we denote by Y_e. Let $(\mathscr{G}_e, Y_e)$ be the restriction of $(\mathscr{G}, Y)$ to Y_e.

THEOREM 3. *For each identity e of A, $A_e \cong \pi(\mathscr{G}_e, Y_e)$.*

Proof. Consider the action of A_e on the tree X_e defined above. This defines a graph of groups $(\mathscr{P}_e, Z_e)$ in the usual way [14, 2]. We briefly recall the construction of $(\mathscr{P}_e, Z_e)$ here. The graph Z_e is the quotient graph $Z_e = A_e \setminus X_e$; that is, $V(Z_e)$ is the set of orbits of vertices of X_e and $E(Z_e)$ is the set of orbits of edges of X_e under the action by A_e. Let T be a spanning tree of Z_e. There is an embedding $j\colon T \to X_e$ of T in X_e.

For each $x \in V(T) \cup E(T)$ let P_x be the stabilizer of jx. Extend j to a map (not a graph morphism) from Z_e to X_e such that for each $y \in E_+(Z_e)$, $(jy)\sigma \in jT$. The embeddings $P_y \to P_{y\tau}$ for $y \in E(Z_e)$ are defined as follows. If $y \notin E_-(Z_e) - E(T)$ then the embedding is the natural embedding of Stab(jy) into Stab$(j(y\tau))$. If $y \in E_-(Z_e) - E(T)$ then $j(y\tau) \neq (jy)\tau$ but they are in the same orbit: if we choose $\gamma_y \in A_e$ such that $(jy)\tau = \gamma_y j(y\tau)$, then define the embedding $P_y \to P_{y\tau}$ by $p \to \gamma_y p \gamma_y^{-1}$. Then by the fundamental theorem of Bass–Serre theory, $A_e \cong \pi(\mathscr{P}_e, Z_e)$.

Hence it suffices to show that the graphs of groups $(\mathscr{P}_e, Z_e)$ and $(\mathscr{G}_e, Y_e)$ are conjugate isomorphic (see [2, p. 202] for this concept). We need isomorphisms between Z_e and Y_e and between corresponding vertex and edge groups of the two graphs of groups that are compatible up to conjugation with the embeddings of edge groups into vertex groups.

There is a map ϕ: $Z_e \to Y_e$ defined as follows. For each $a \in A$ for which $i(a) = e$ we define

$$\phi\colon O(aG) \to D^G_{t(a)} \qquad \left[\text{resp. } \phi\colon O(aH) \to D^H_{t(a)},\ \phi\colon O(aU) \to D^U_{t(a)}\right].$$

Lemmas 4 and 5 guarantee that ϕ is well-defined, one-to-one, and onto. Since aU: $aG \to aH$ in X_e and $D^U_{t(a)}$: $D^G_{t(a)} \to D^H_{t(a)}$ in Y_e it is clear that ϕ is a graph isomorphism.

We now define isomorphisms between the corresponding vertex groups of the two graphs of groups. We need some notation in order to provide a clear definition of these isomorphisms.

Let T be the spanning tree of Z_e and j: $T \to X_e$ the embedding that is extended to a map j from Z_e to X_e as described above. For each $a \in A$ denote by a_G [resp. a_H, a_U] the identity of $D^G_{t(a)}$ [resp. $D^H_{t(a)}, D^U_{t(a)}$] that is specified in the definition of the graph of groups $(\mathscr{G}, Y)$. Each vertex of Z_e is an orbit of a vertex of X_e under the action by A_e: for the vertex $O(aG)$ [resp. $O(aH)$] the corresponding vertex group of $(\mathscr{P}_e, Z_e)$ is Stab$(j(O(aG)))$ [resp. Stab$(j(O(aH)))$]. To simplify notation, denote $(j(O(aG))$ [resp. $(j(O(aH))$] by a^*_G [resp. a^*_H]. By Lemma 6,

$$\text{Stab}(a^*_G) = a^*_G G_{t(a^*_G)} a^{*^{-1}}_G \qquad \left[\text{resp. Stab}(a^*_H) = a^*_H H_{t(a^*_H)} a^{*^{-1}}_H\right].$$

Since $a^*_G \in O(aG)$ [resp. $a^*_H \in O(aH)$] it follows from Lemma 6 that $t(a^*_G) \in D^G_{t(a)}$ [resp. $t(a^*_H) \in D^H_{t(a)}$] so there exists $g_a \in G$ [resp. $h_a \in H$] such that

$$g_a\colon t(a^*_G) \to a_G \qquad \left[\text{resp. } h_a\colon t(a^*_H) \to a_H\right].$$

Define ϕ: Stab$(a^*_G G) \to G_{a_G}$ by $a^*_G s a^{*^{-1}}_G \to g_a^{-1} s g_a$ for $s \in G_{t(a^*_G)}$. It is easy to check that ϕ is an isomorphism from the vertex group Stab$(a^*_G G)$ of $(\mathscr{P}_e, Z_e)$ to the vertex group G_{a_G} of $(\mathscr{G}_e, Y_e)$. Similarly the map ϕ: Stab$(a^*_H H) \to H_{a_H}$ defined by $a^*_H s a^{*^{-1}}_H \to h_a^{-1} s h_a$ for $s \in H_{t(a^*_H)}$ is an isomorphism.

We now define isomorphisms between the corresponding edge groups of the two graphs of groups. Let y be a positively oriented edge of Z_e and suppose that $jy = aU$, so $y = O(aU)$. The edge group of y in $(\mathscr{P}_e, Z_e)$ is $\mathrm{Stab}(aU) = aU_{t(a)}a^{-1}$ and the map ϕ from this edge group to the corresponding edge group U_{a_U} in $(\mathscr{G}_e, Y_e)$ is given by $asa^{-1} \to u_a^{-1}su_a$, where u_a: $t(a) \to a_U$ is in U. It remains to show that, up to conjugation, the morphisms ϕ between vertex (edge) groups respect the embeddings of edge groups into vertex groups. We provide a proof below: the reader may find it convenient, in the following proof, to sketch a diagram of the relevant arrows and vertices of A indicated in the proof.

Suppose first that the edge $y = O(aU)$ is in T and that $jy = aU$. Then by definition of a_G^* and a_H^* it follows that $aG = a_G^*G$ and $aH = a_H^*H$ so there exist $g_1 \in G$ and $h_1 \in H$ such that $a = a_G^*g_1$ and $a = a_H^*h_1$. Take an element $asa^{-1} \in \mathrm{Stab}(aU)$ for some $s \in U_{t(a)}$. Then

$$(asa^{-1})\sigma\phi = (asa^{-1})\phi = \left(a_G^*g_1sg_1^{-1}(a_G^*)^{-1}\right)\phi = g_a^{-1}g_1sg_1^{-1}g_a,$$

and

$$(asa^{-1})\phi\sigma = \left(u_a^{-1}su_a\right)\sigma = g^{-1}u_a^{-1}su_ag$$

for some $g \in G$. Since $g^{-1}u_a^{-1}$ and $g_a^{-1}g_1$ are both elements of G with initial identity a_G and terminal identity $t(a)$ it follows that $(asa^{-1})\sigma\phi$ and $(asa^{-1})\phi\sigma$ are equal up to conjugation by an element of G_{a_G}. Similarly $(asa^{-1})\tau\phi$ and $(asa^{-1})\phi\tau$ are equal up to conjugation by an element of H_{a_H}.

Suppose finally that $y = O(aU)$ is a positively oriented edge of Z_e that is not necessarily in T and let $aU = j(O(aU))$. As before, if $asa^{-1} \in \mathrm{Stab}(aU)$ for some $s \in U_{t(a)}$, then $(asa^{-1})\sigma\phi = g_a^{-1}g_1sg_1^{-1}g_a$ for some $g_1 \in G$ and $(asa^{-1})\phi\sigma = g^{-1}u_a^{-1}su_ag$ for some $g \in G$, and these are equal up to conjugation by an element of G_{a_G}. Finally, since $\gamma_y.aH = a_H^*H$ there exists some $h_1 \in H$ such that

$$(asa^{-1})\tau\phi = \left(\gamma_y asa^{-1}\gamma_y^{-1}\right)\phi = \left(a_H^*h_1sh_1^{-1}(a_H^*)^{-1}\right)\phi = h_a^{-1}h_1sh_1^{-1}h_a,$$

and

$$(asa^{-1})\phi\tau = \left(u_a^{-1}su_a\right)\tau = h^{-1}u_a^{-1}su_ah$$

for some $h \in H$. Again we see that $h_a^{-1}h_1$ and $h^{-1}u_a^{-1}$ are elements of H with initial identity a_H and terminal identity $t(a)$ so $(asa^{-1})\phi\tau$ and $(asa^{-1})\tau\phi$ are equal to conjugation by an element of H_{a_H}. We have proved that the graphs of groups $(\mathscr{P}_e, Z_e)$ and $(\mathscr{G}_e, Y_e)$ are conjugate isomorphic and hence $A_e \cong \pi(\mathscr{G}_e, Y_e)$, as desired.

As a consequence of the Nambooripad–Pastijn theorem (Theorem 2 above) it is possible to reformulate Theorem 3 in an equivalent form that provides structural information about the maximal subgroups of an amalgamated free product $S_1 *_U S_2$ in the case where S_1, S_2 and U are regular semigroups with U full in S_1 and S_2. For the convenience of the reader, we provide this reformulation below. The proof of the resulting theorem is an immediate consequence of Theorems 2 and 3 and the equivalence between the category of regular semigroups and the category of inductive groupoids.

Let $[S_1, S_2; U]$ be an amalgam of regular semigroups with U full in S_1 and S_2 and let $S = S_1 *_U S_2$. Construct a graph of groups $(\mathscr{S}, W)$ in the following fashion. Let $V(W)$ be the disjoint union of the $\mathscr{D}$-classes of S_1 and S_2. Let $E_+(W)$ be the set of $\mathscr{D}$-classes of U. For $D \in E_+(W)$ set

$$D\sigma = \text{the } \mathscr{D}\text{-class of } S_1 \text{ containing } D, \text{ and}$$
$$D\tau = \text{the } \mathscr{D}\text{-class of } S_2 \text{ containing } D.$$

For each $\mathscr{D}$-class D of S_1 [resp. S_2, U] let $S_D^{(1)}$ [resp. $S_D^{(2)}, U_D$] be a specified maximal subgroup in D: if $D \in E_-(W)$ let $U_D = U_{\overline{D}}$. These specified groups will be the vertex and edge groups of $(\mathscr{S}, W)$. For $D \in E_+(W)$ let K be the maximal subgroup of $D\sigma$ containing $S_D^{(1)}$. Fix an element $g \in D\sigma$ such that $g^{-1}Kg = S_{D\sigma}^{(1)}$. The map $\sigma\colon U_D \to D_{D\sigma}^{(1)}$ is then given by $u \to g^{-1}ug$. We define the map $\tau\colon U_D \to S_{D\tau}^{(2)}$ in a similar fashion. This specifies a graph of groups $(\mathscr{S}, W)$. For each identity e of A note that the $\mathscr{D}$-class of e in S_1 and the $\mathscr{D}$-class of e in S_2 are in the same connected component of W, which we denoted by W_e. Let $(\mathscr{S}_e, W_e)$ be the restriction of $(\mathscr{S}, W)$ to W_e. We have the following theorem as an immediate corollary of Theorems 2 and 3.

THEOREM 4. *Let $[S_1, S_2; U]$ be an amalgam of regular semigroups with U full in S_1 and S_2 and let $S = S_1 *_U S_2$. Then for each idempotent $e \in S$, the maximal subgroup of S containing e is isomorphic to $\pi(\mathscr{S}_e, W_e)$.*

3. SOME APPLICATIONS AND EXAMPLES

In this section we discuss some examples and applications of the theorem proved in the previous section. The examples that we consider are examples of inverse semigroup presentations, where the situation is particularly pleasant. We shall employ standard notation from inverse semigroup theory and refer the reader to Petrich's book [13] for any undefined notation.

EXAMPLE 1. Let $S_1 = S_2$ be the bicyclic monoid B. That is, $B = \mathrm{Mon}\langle a, b : ab = 1\rangle$ (or, when presented as an inverse monoid, $B = \mathrm{Inv}\langle a : aa^{-1} = 1\rangle$). Let U be a full inverse submonoid of B. If U is just the semilattice of idempotents of B then clearly U has infinitely many $\mathscr{D}$-classes. It is well known that all other full inverse subsemigroups of B have only finitely many $\mathscr{D}$-classes. There is exactly one full embedding of U in S_1 and S_2. It is clear that the inverse monoid $S = S_1 *_U S_2$ is bisimple with the same semilattice of idempotents as B (a chain isomorphic as a poset to the negative integers): that is, S is a bisimple ω-semigroup (in the standard notation of semigroup theory). The maximal subgroups of S may be easily calculated from Theorem 4. Since S_1 and S_2 are bisimple the graph W has two vertices (S_1 and S_2). The number of positively oriented edges in W is equal to the number of $\mathscr{D}$-classes in U. Since all vertex and edge groups are trivial (the bicyclic monoid is combinatorial), the graph of groups $(\mathscr{S}, W)$ is just the graph W. Hence its fundamental group is just a free group whose rank is one less than the number of $\mathscr{D}$-classes in U (or this rank is infinite if U is just the semilattice of idempotents of B).

Remark. The semigroup $S = S_2 *_U S_2$ constructed in Example 1 is a bisimple ω-semigroup of the form $B(G, \alpha)$, where G is a (free) group and α is an endomorphism of G (see [13] for an explanation of this notation and for the structure of such semigroups). While the rank of the free group G is determined solely by the number of $\mathscr{D}$-classes in U, the endomorphism α is not. There are n non-isomorphic full inverse subsemigroups of B that contain $n\mathscr{D}$-classes and these all yield a free group of rank $n - 1$, but the corresponding subsemigroups $B(G, d)$ are associated with different endomorphisms α and are non-isomorphic semigroups. It is clear that one may extend Example 1 somewhat to the study of amalgams of the form $S_1 *_U S_2$, where S_1 S_2, and U are all bisimple ω-semigroups (with U full in S_1 and S_2). The nature of the associated endomorphisms determines the structure of the resulting semigroup.

Example 1 illustrates the following observation, which is an immediate consequence of Theorem 4.

COROLLARY 2. *Let $[S_1, S_2; U]$ be an amalgam of regular semigroups with U full in S_1 and S_2 and let $S = S_1 *_U S_2$. If S_1 and S_2 are both combinatorial then every maximal subgroup of S is a free group.*

Proof. If S_1 and S_2 are both combinatorial then all vertex and edge groups of $(\mathscr{S}, W)$ are trivial, so $(\mathscr{S}, W)$ is just a graph, and its fundamental group is a free group.

Similarly one obtains the following results directly from Theorem 4.

COROLLARY 3. *Let $[S_1, S_2; U]$ be an amalgam of regular semigroups with U full in S_1 and S_2 and let $S = S_1 *_U S_2$. If U is combinatorial then every maximal subgroup of S is a free product of maximal subgroups of S_1 and S_2 and a free group.*

COROLLARY 4. *Let $[S_1, S_2; U]$ be an amalgam of regular semigroups with U full in S_1 and S_2 and let $S = S_1 *_U S_2$. Then S is combinatorial if and only if S_1 and S_2 are both combinatorial and the corresponding graph of groups $(\mathscr{S}, W)$ is a forest.*

EXAMPLE 2. Denote by $B_n(G)$ the $n \times n$ Brandt semigroup with maximal subgroup G in its non-zero $\mathscr{D}$-class. If U is an inverse subsemigroup of $B_n(G)$ then clearly every non-zero idempotent of U is primitive, so U is a 0-disjoint union of Brandt semigroups. Suppose now that $S_i = B_n(G_i)$ for $i = 1, 2$ and that U is a full inverse subsemigroup of each S_i with $U = S_1 \cap S_2$. Then all of the non-zero idempotents of S_1 [resp. S_2] are in the same $\mathscr{D}$-class of $S = S_1 *_U S_2$ and each non-zero idempotent of S_1 is identified with some non-zero idempotent of S_2 in S, so in fact S must be an $n \times n$ Brandt semigroup. The structure of the non-trivial maximal subgroup of S is immediately determined from Theorem 4.

For example, suppose that $S_1 = B_2(G_1)$ and $S_2 = B_2(G_2)$, where $G_1 = \mathbf{Z}_4$ (the cyclic group of order 4) and $G_2 = \mathbf{Z}_6$. Let $U = B_2(H)$ where $H = \mathbf{Z}_2$. There is a unique embedding of $\mathbf{Z}_2$ in $\mathbf{Z}_4$ and a unique embedding of $\mathbf{Z}_2$ in $\mathbf{Z}_6$ and these embeddings extend in an obvious way to full embeddings of U in S_1 and S_2. By Theorem 4, the non-zero maximal subgroup of $S = S_1 *_U S_2$ is the fundamental group of a segment whose vertex groups are $\mathbf{Z}_6$ and $\mathbf{Z}_4$ and whose edge group is $\mathbf{Z}_2$, so this group is isomorphic to the amalgamated free product $\mathbf{Z}_6 *_{\mathbf{Z}_2} \mathbf{Z}_4$ in the category of groups, which is isomorphic to $\mathbf{SL}_2(\mathbf{Z})$ by a well-known isomorphism (see [14, p. 11]).

On the other hand, if S_1 and S_2 are as above and U is a full subsemigroup of S_i that is a semilattice of three groups, two of which are isomorphic to $\mathbf{Z}_2$ and the other is trivial then the corresponding graph of groups has three positively oriented edges instead of two and the maximal subgroup of the non-zero $\mathscr{D}$-class of S is the fundamental group of the graph of groups with two vertices and two positively oriented edges between these vertices. The vertex groups are $\mathbf{Z}_4$ and $\mathbf{Z}_6$ and both edge groups are $\mathbf{Z}_2$.

We indicate in the next example how a finite inverse semigroup may be decomposed as an amalgamated free product of simpler inverse semigroups in a non-trivial way. This is quite in contrast to the situation in group theory, where any non-trivial amalgamated free product is necessarily infinite.

EXAMPLE 3. Let S_1 and S_2 be copies of the six element semigroup which is a 0-direct union of the five element combinatorial Brandt semigroup and the two element semilattice. Thus we may write $S_i = \{e_i, f_i, a_i, a_i^{-1}, g_i, 0_i\}$ for $i = 1, 2$ with e_i, f_i and g_i idempotents and

$$a_i a_i^{-1} = e_i,\ a_i^{-1} a_i = f_i,\ a_i e_i = a_i = f_i a_i,\ e_i a_i^{-1} = a_i^{-1} = a_i^{-1} f_i$$

and all other products in S_i are 0_i. Let

$$q\phi_1 = e_1, q\phi_2 = g_2, r\phi_1 = f_1, r\phi_2 = e_2, s\phi_1 = g_1, s\phi_2 = f_2, 0\phi_i = 0_i.$$

The corresponding graph of groups $(\mathscr{S}, W)$ has six vertices and eight edges with incidence given by the ϕ_i maps since the $\mathscr{D}$-relation is trivial in U. Since S_1 and S_2 are combinatorial, $(\mathscr{S}, W)$ is just a graph W. The connected components of W are trees. Hence $S = S_1 *_U S_2$ is combinatorial. Notice that the images of e_1 and g_1 are $\mathscr{D}$-related in S since, in S, $e_1 = a_1 a_2 (a_1 a_2)^{-1}$ and $g_1 = (a_1 a_2)^{-1} a_1 a_2$. Hence S is the ten-element combinatorial Brandt semigroup.

We shall return to this phenomenon of decomposing a finite inverse semigroup as a non-trivial amalgamated free product. We first provide a characterization of the class of graphs of groups belonging to $\mathscr{D}$-classes of amalgamated free products. It is clear from the definition of the graph of groups $(\mathscr{S}, W)$ belonging to $S = S_1 *_U S_2$ that W is bipartite. Hence if $(\mathscr{S}_1, W_1)$ is the component of $(\mathscr{S}, W)$ belonging to the $\mathscr{D}$-class D of S, then W_1 is connected and bipartite. In fact the converse is true.

THEOREM 5. *If $(\mathscr{S}, W)$ belongs to the $\mathscr{D}$-class D of $S = S_1 *_U S_2$ (with S_1, S_2, and U regular and U full in S_1 and S_2), then W is connected and bipartite. Conversely if $(\mathscr{S}, W)$ is a graph of groups and W is connected and bipartite, then there is an inverse semigroup $S = S_1 *_U S_2$ with U full in S_1 and S_2, and a $\mathscr{D}$-class D of S such that $(\mathscr{S}, W)$ belongs to D.*

Proof. Suppose that $(\mathscr{S}, W)$ is a graph of groups with W connected and bipartite, so that $V(W)$ is a disjoint union $V(W) = V_1 \cup V_2$ and edges of W connect vertices in V_1 with vertices in V_2. We construct inverse semigroups S_1, S_2, and U such that $(\mathscr{S}, W)$ belongs to the non-zero $\mathscr{D}$-class of $S = S_1 *_U S_2$.

Define U to be $(\bigcup_{y \in E(W)} G_y) \cup \{0\}$ with multiplication of $g_1 \in G_{y_1}$ and $g_2 \in G_{y_2}$ given by $g_1 . g_2 = g_1 g_2$ if $y_1 = y_2$ and all other products are 0. That is, U is the 0-disjoint union of all the edge groups of W. We build S_1 and S_2 by first constructing a collection of Brandt semigroups and then taking 0-disjoint unions of these Brandt semigroups.

Let v be a vertex of W. Let $\mathrm{Star}(v) = \{y \in E_+(W) : i(y) = v\}$ and let $S_v = B_n(G_v)$, where $n = |\mathrm{Star}(v)|$. Then let S_1 [resp. S_2] be the 0-disjoint union of the S_v with $v \in V_1$ [resp. $v \in V_2$]. For $y \in E(W)$ let $\hat{y}$ denote the edge in $\{y, \bar{y}\}$ whose origin is in V_1. Then define $\phi_1 \colon U \to S_1$ by $0 \to 0$ and if $g \in G_y$ then $g \to (\hat{y}, \sigma(g), \hat{y})$. The map ϕ_2 is constructed in a similar fashion. One checks that $S_1 *_U S_2$ is a Brandt semigroup: denote its non-zero $\mathscr{D}$-class by D. If we let $(\mathscr{H}, X)$ be the graph of groups determined by the amalgam $[S_1, S_2; U]$ then it is routine to see that $(\mathscr{H}_D, X_D) \cong (\mathscr{S}, W)$.

COROLLARY 5. *Every tree of groups* $(\mathscr{G}, T)$ *for which* T *is a tree with at least two vertices arises in this setting.*

Proof. It is clear that a tree with at least two vertices is a bipartite graph.

The reader may contrast this with the situation that occurs with group amalgams, where the underlying graph of the associated graph of groups is a segment. The wide class of graphs of groups that can arise in connection with inverse semigroup amalgams gives rise to multiple decompositions of inverse semigroups (even finite ones) as non-trivial amalgamated free products. Let $B_n = B_n(\{1\})$ be the $n \times n$ combinatorial Brandt semigroup and let $E_n = E(B_n)$ be its semilattice of idempotents. As we saw in Example 3, B_3 can be expressed as a non-trivial amalgamated free product over its semilattice E_3.

Given a tree with n edges we can construct the non-zero $\mathscr{D}$-classes of two inverse semigroups S_1 and S_2 and write $B_n = S_1 *_{E_n} S_2$ as in the proof of Theorem 5. Conversely, suppose that $B_n = S_1 *_{E_n} S_2$ is a decomposition of B_n. Then S_1 and S_2 are combinatorial and by Corollary 4 the graph of groups belonging to the non-zero $\mathscr{D}$-class of B_n is a tree T, which has n edges by definition. The set of idempotents of each $\mathscr{D}$-class D of S_1 or S_2 is in one to one correspondence with the set of edges in the star set of the vertex of T corresponding to D. Hence the structure of the non-zero $\mathscr{D}$-classes of S_1 and S_2 can be read off T as above. We have proved:

THEOREM 6. *The number of non-isomorphic decompositions of* $B_n = S_1 *_{E_n} S_2$ *is equal to the number of non-isomorphic trees having n edges.*

For example, B_9 can be written as a non-trivial amalgamated free product over its semilattice of idempotents in 105 different ways. Of course there are many more ways to write B_9 as an amalgamated free product.

We close by remarking that the structure of amalgamated free products $S_1 *_U S_2$ of inverse semigroups in the case where U is not full in S_1 and S_2 is far from understood. Some recent work along these lines is contained in the thesis of Bennett [1] and several other "ad hoc" cases have been considered, but the general case seems to be very complex. Even the structure of the free product $S_1 * S_2$ of two inverse semigroups is relatively complicated (see [6–8] for details).

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