International Journal of Algebra and Computation, Vol. 12, Nos. 1 & 2 (2002) 179–211

COMBINATORIAL GROUP THEORY, INVERSE MONOIDS, AUTOMATA, AND GLOBAL SEMIGROUP THEORY

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Received 28 November 2000
Revised 1 June 2001
Communicated by J. Meakin

1991 Mathematics Subject Classification: 20F32, 20M18, 20F10, 20M07, 20M35

This paper explores various connections between combinatorial group theory, semigroup theory, and formal language theory.

Let $G = \langle A|R\rangle$ be a group presentation and $\mathcal{B}_{A,R}$ its standard 2-complex. Suppose X is a 2-complex with a morphism to $\mathcal{B}_{A,R}$ which restricts to an immersion on the 1-skeleton. Then we associate an inverse monoid to X which algebraically encodes topological properties of the morphism. Applications are given to separability properties of groups.

We also associate an inverse monoid $M(A, R)$ to the presentation $\langle A|R\rangle$ with the property that pointed subgraphs of covers of $\mathcal{B}_{A,R}$ are classified by closed inverse submonoids of $M(A, R)$. In particular, we obtain an inverse monoid theoretic condition for a subgroup to be quasiconvex allowing semigroup theoretic variants on the usual proofs that the intersection of such subgroups is quasiconvex and that such subgroups are finitely generated. Generalizations are given to non-geodesic combings. We also obtain a formal language theoretic equivalence to quasiconvexity which holds even for groups which are not hyperbolic.

Finally, we illustrate some applications of separability properties of relatively free groups to finite semigroup theory. In particular, we can deduce the decidability of various semidirect and Mal'cev products of pseudovarieties of monoids with equational pseudovarieties of nilpotent groups and with the pseudovariety of metabelian groups.

Keywords: Immersions; coverings; fundamental groups; profinite topology; separability properties; rational sets; automata; graphs; inverse semigroups; quasiconvexity; pseudovarieties; finite semigroups.

If $G = \langle A|R \rangle$ is a group presentation and $\mathcal{B}_{A,R}$ is the standard 2-complex of the presentation, then classical results of combinatorial group theory show that coverings of $\mathcal{B}_{A,R}$ correspond to subgroups of G. But more and more, one finds oneself in the situation of being interested in a subcomplex of such a covering (often a subgraph), or more generally a 2-complex with a morphism to $\mathcal{B}_{A,R}$ which is an immersion on the 1-skeleton. This situation arises in the McCammond–Wise theory of perimeter [37]; it also arises in Scott's topological [48] and Gitik's graph theoretical [19] characterizations of LERF groups; in the case that G is a free group on A, Stallings introduced graph immersions [50] to deal with this situation. Furthermore, there is no language within the realm of combinatorial group theory to deal with such structures; McCammond and Wise [37], and Gitik [19] both consider 2-complexes X with morphisms to $\mathcal{B}_{A,R}$ which are 1-skeleton immersions satisfying the property that any path in X labeled by a word mapping to 1 in G is a loop (this is the condition for embedding the 1-skeleton of X into a cover), but no natural terminology is given there.

In the first section of this paper, we show that underlying a morphism of 2-complexes which is a 1-skeleton immersion there is an inverse monoid. This inverse monoid algebraically encodes such topological properties of the morphism as whether it is a covering or if it can be factored through an embedding into a covering. For the case of a free group, it was shown in [10] that this inverse monoid encodes whether the fundamental group is a root-closed (also called pure) subgroup allowing the deduction that the problem of determining purity for finitely generated subgroups of a free group is PSPACE-complete. The property mentioned at the end of the preceding paragraph, for instance, corresponds to the associated inverse monoid having an E-unitary cover over G. Using this inverse monoid, and a mixture of topological techniques, techniques from formal language theory (already familiar ground to group theorists, thanks to [17]) and global semigroup theory, we give a variant on Scott's [48] and Gitik's [19] topological characterizations of LERF groups as well as a new characterization entirely in terms of inverse monoid theory. We then present a simple proof that LERF passes through free products (along the lines of that of [19]).

Stronger separability properties, such as double coset separability, are then considered. Our inverse monoid theoretic characterizations are motivated by the work which surrounded the Rhodes type II conjecture [28] culminating in the formally equivalent theorems of Ash [8] (proving the tameness [5, 6] of the pseudovariety of finite groups) and Ribes and Zalesskiĭ's theorem proving the n-coset separability of a free group [4, 28, 56]; see [15, 28, 56] for more on this. The arguments here are generalizations from the case of a free group considered in the third author's [56].

In the second section we consider the problem of classifying pointed subgraphs of coverings of $\mathcal{B}_{A,R}$. While the 1-skeleta of coverings are classified by subgroups of G, classification of more general subgraphs lies outside of the realm of group theory. But once again, there is an inverse monoid that is up to the task. To each group presentation $G = \langle A|R \rangle$ we associate an inverse monoid $M(A, R)$ (in inverse monoid theoretic terms: the freest A-generated E-unitary inverse monoid with maximal group image G). We show that pointed subgraphs of coverings of $\mathcal{B}_{A,R}$ are classified by the so-called closed inverse submonoids of $M(A,R)$ which are in turn shown to be encoded by subgroups of G acting invariantly on certain subgraphs of the Cayley graph of G. This work has as its point of departure the second author and Meakin's [35] where the case of a free group is considered.

We also consider quasiconvexity. More precisely we identify the closed inverse submonoid of $M(A,R)$ which corresponds to Gitik's geodesic core of a subgroup [20]. Quasiconvexity then becomes a question of whether this closed inverse submonoid is of finite index. This allows an inverse monoid theoretic proof that the class of quasiconvex subgroups is closed under intersection. It also motivates a generalization of the formal language theoretic characterization of quasiconvexity for word hyperbolic groups to more general groups. We also consider non-geodesic combings.

The final section of the paper turns the tables and examines how separability properties of relatively free groups can be applied to finite semigroup theory. In particular, we are able to use separability properties of polycyclic groups and relatively free metabelian groups to prove decidability results for certain semidirect and Mal'cev products of pseudovarieties of monoids. Such an approach began in the work of the first author [14, 15] following Pin and Reutenauer [41, 42].

This paper attempts to be expository in nature (even though many of the results here presented have never explicitly appeared elsewhere). The first two sections assume little background in semigroup theory but assume some basic familiarity with notions from combinatorial group theory. The last section demands more knowledge of semigroup theory (and less of group theory) but should still be accessible to all.

As a final remark, we mention that just as it is necessary to go from groups to groupoids to fully encode coverings over 2-complexes with multiple vertices, one must go to inverse categories to deal with more general 2-complexes. Such is done in [35] for the case of free groups. We leave the general situation to the reader.

1. Separability Properties via Inverse Monoids

Recall that a group G is said to be *LERF* (locally extended residually finite) or, by some authors, *subgroup separable* if, given any finitely generated subgroup H and $g \in G \setminus H$, there exists a finite index subgroup K with $H \subseteq K$ and $g \notin K$. Equivalently, G is LERF if all its finitely generated subgroups are closed in the profinite topology [26]. It is easy to show that finitely presented LERF groups have decidable generalized word problem.

Stallings showed in [50] that the notion of a graph immersion lends itself to a simple proof of Hall's theorem [25] that free groups are LERF. The second author and Meakin [35] then showed that immersions over a bouquet of circles are nothing more than inverse automata and can thus be understood in terms of their corresponding transition monoids; further connections between inverse automata and profinite topologies were explored by the second author, Sapir, and Weil in [36, 60]. This section is very much influenced by the work surrounding the Rhodes type II conjecture which eventually led to a proof that free groups satisfy a certain generalization [44] of LERF which we shall discuss below; see [28]. The immediate antecedents of this section, however, are the third author's [55, 56].

This section builds on these ideas to obtain two equivalent conditions to a group being LERF. One condition, which appears to be a combinatorial analog to a topological condition of Scott [48] (used to prove that surface groups are LERF), involves graph immersions and basically says that Stallings' proof of Hall's theorem is completely general (see also [19]); the other condition is inverse monoid theoretic involving the notions of E-unitary covers and idempotent pure relational morphisms. Using these equivalences, we give a short proof, essentially due to Gitik [19], of the theorem of Burns [11] and Romanovskii [47] that the free product of two LERF groups is LERF. We shall then consider more general separability properties which will be useful in Sec. 3 for applications to finite semigroup theory.

The main result of this section is the following.

Theorem 1.1. *For a group $G = \langle A|R\rangle$, the following are equivalent*:

(1) *G is LERF.*
(2) *Given a covering $\psi : \tilde{X} \to \mathcal{B}_{A,R}$ and a finite connected subgraph $C \subseteq \tilde{X}$, there exists a finite sheeted covering $\psi' : \tilde{X}' \to \mathcal{B}_{A,R}$ such that the immersion $\psi|_C : C \to \mathcal{B}_{A,R}$ factors through an inclusion followed by ψ'.*
(3) *Given a finitely generated, idempotent pure relational morphism $\varphi : I \multimap G$ with I a finite inverse monoid, there exists a group homomorphism $\psi : G \to G'$ with G' finite and $\psi\varphi$ idempotent pure.*

To prove this theorem we develop an inverse monoid theoretic calculus of 2-complexes with morphisms to $\mathcal{B}_{A,R}$ which are immersions on the 1-skeleton. This calculus will allow us to determine when such a complex embeds in a cover.

1.1. *The topological characterization*

This section takes the point of view that the reader is more familiar with the techniques of combinatorial group theory than with the techniques of automata and inverse semigroup theory. With this in mind we will take for granted the (combinatorial) definitions of 2-complexes, the fundamental group of a 2-complex, and coverings of 2-complexes. Also we will assume basic properties of such; the reader is referred to [32, 58] for more. While we will give the definition of an immersion, we will use their basic properties, found in [50], without comment. If X is a 2-complex, $X^{(1)}$ will denote the 1-skeleton of X. We shall assume all morphisms of 2-complexes to be combinatorial. For a 2-complex X, $\pi_1(X, v_0)$ will denote the fundamental group of X at v_0. If X has a single vertex, we will omit the base point. If $f : X \to Y$ is a morphism of 2-complexes, then we shall also use f for the corresponding homomorphism $f : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

To set up notation, we define a *graph* C to consist of a set $V(C)$ of *vertices*, $E(C)$ of *edges*, an involution of $E(C)$, $e \mapsto e^{-1}$ such that $e^{-1} \neq e$, and functions $i, t : E(C) \to V(C)$ (selecting the *initial*, respectively, *terminal* vertex of an edge) such that $i(e) = t(e^{-1})$. An *orientation* of C consists of a subset $E^+(C)$ of *positively oriented* edges of C with the property that $E^+(C) \cap E^+(C)^{-1} = \emptyset$ and $E(C) = E^+(C) \cup E^+(C)^{-1}$. We shall assume all graphs to be oriented. The notion of a morphism of oriented graphs is then obvious. If v is a vertex, the *star* of v is the set $i^{-1}(v)$. A morphism of graphs is called an *immersion* if it is injective when restricted to all star sets and a *covering* if it is bijective when so restricted.

We now give a topological condition for the fundamental group of a 2-complex to be LERF. Stallings' proof of Hall's theorem [50] proves just this for a bouquet of circles.

Theorem 1.2. *Let (X, v_0) be a pointed 2-complex with the property: given a covering $\psi : \tilde{X} \to X$ and a finite connected subgraph $C \subseteq \tilde{X}$, there exists a finite sheeted covering $\psi' : \tilde{X}' \to X$ such that the immersion $\psi|_C : C \to X$ factors through an inclusion followed by ψ'. Then $\pi_1(X, v_0)$ is LERF.*

Proof. Let H be a finitely generated subgroup of $G = \pi_1(X, v_0)$ and suppose $g \in G \setminus H$. Let $\psi : (\tilde{X}, \widetilde{v_0}) \to (X, x_0)$ be the covering map associated to H, $\{h_1, \ldots, h_n\}$ be a finite generating set for H, and let C be the subgraph of $\tilde{X}$ consisting of the edges and vertices of the lifts of $h_1, \ldots, h_n$ and g starting at $\widetilde{v_0}$. Then C is finite and connected so, by assumption, there is a finite sheeted covering $\psi' : (\tilde{X}', \widetilde{v_0}') \to (X, v_0)$ such that $\psi|_C$ factors through an embedding followed by ψ'. But then $\pi_1(\tilde{X}', \widetilde{v_0}')$ is a finite index subgroup of G containing H but not g. □

1.2. *Rational sets*

In this subsection, we introduce some tools from formal language theory which we shall need to prove Theorem 1.1. We shall again make use of these tools when we study quasiconvexity. If A is a set, A^* will denote the free monoid on A. Setting $\tilde{A} = A \cup A^{-1}$, we view $\tilde{A}^*$ as a monoid with involution in the obvious way (and it is, in fact, the freest such monoid). A subset of a monoid M is called *rational* if it is in the smallest collection $\mathcal{R}$ of subsets of M satisfying the following properties:

(1) $\emptyset \in \mathcal{R}$;
(2) $\{m\} \in \mathcal{R}$ for all $m \in M$;
(3) $L_1, L_2 \in \mathcal{R} \Longrightarrow L_1 \cup L_2 \in \mathcal{R}$ and $L_1 L_2 \in \mathcal{R}$;
(4) $L \in \mathcal{R} \Longrightarrow L^* = \bigcup_{n \in \mathbb{N}} L^n \in \mathcal{R}$.

We shall denote the set of rational subsets of M by $Rat(M)$.

A subset L of a monoid M is called *recognizable* if there is a homomorphism $\varphi : M \to N$ with N a finite monoid, and a subset $F \subseteq N$ such that $L = \varphi^{-1}(F)$. The collection of recognizable subsets of M will be denoted $Rec(M)$.

Kleene's theorem [9, 16, 17] states that, for a finite set A, $Rat(A^*) = Rec(A^*)$. Indeed, one can show that both collections are precisely those languages accepted by finite state automata over A. The following proposition is straightforward [9, 16].

Proposition 1.3. *Let $\varphi : M \to N$ be a monoid homomorphism. Then $L \in Rat(M)$ implies $\varphi(L) \in Rat(N)$ and $L \in Rec(N)$ implies $\varphi^{-1}(L) \in Rec(M)$.*

Using the above proposition and Kleene's theorem one obtains the following well-known corollary [9, 16].

Corollary 1.4. *For M a finitely generated monoid, $Rec(M) \subseteq Rat(M)$.*

The next theorem, due to Anissimov and Seifert [7, 9, 17], will be of use in relating these concepts to LERF.

Theorem 1.5. *Let G be a group. Then, for a subgroup $H \subseteq G$, $H \in Rec(G)$ if and only if it is of finite index and $H \in Rat(G)$ if and only if it is finitely generated.*

1.3. *Immersions and inverse monoids*

Our presentation is from the point of view of [55], where the case of graphs is covered in greater detail, although the connection between graph immersions and inverse monoids was first drawn in [35]; see also [36, 53] for related results.

If A is a set, $\mathcal{B}_A$ will denote the oriented graph consisting of a single vertex, with $E(\mathcal{B}_A) = \tilde{A}$, and with $E^+(\mathcal{B}_A) = A$. We think of each edge as being labeled by the corresponding element of $\tilde{A}$. A *labeling* of a graph C over A is then a morphism $\ell : C \to \mathcal{B}_A$. We call such a graph an A-*graph* and we think of each edge e of C as having the label $\ell(e)$ written on it. If, in addition, ℓ is an immersion, we call C an *inverse* A-graph. Suppose C is an inverse A-graph; then given $v \in V(C)$ and $a \in \tilde{A}$, there is at most one lift of a starting at v. We define va to be the endpoint of this lift if it exists, and otherwise va is undefined. In this manner, we obtain a partial injective function $\cdot a : V(C) \to V(C)$. Hence there is a natural homomorphism τ from $\tilde{A}^*$ to the monoid of all partial injective functions from $V(C)$ to $V(C)$ (acting on the right). We denote $\tau(\tilde{A}^*)$ by $I(C)$ and call it the *transition monoid* of C. Note that if $V(C)$ is a finite set, then $I(C)$ is finite.

Observe that, for $w \in \tilde{A}^*$, $\tau(ww^{-1}w) = \tau(w)$ and that, for $u, v \in \tilde{A}^*$, $\tau(uu^{-1}vv^{-1}) = \tau(vv^{-1}uu^{-1})$. A monoid with involution satisfying the identities $xx^{-1}x = x$ and $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$ is called an *inverse monoid* (see [30, 40]) and one says that x^{-1} is the *inverse* of x. It can be shown that in an inverse monoid, x^{-1} is the unique element y such that both $yxy = y$ and $xyx = x$ hold. Any group, for instance, is an inverse monoid. The theory of finite inverse monoids is very closely connected to the theory of finitely generated subgroups of a free group as is evidenced by [10, 35, 36, 56, 60]. This sections shows that, in fact, finite inverse monoids are closely connected to the study of finitely generated subgroups of any group.

An inverse monoid M is said to be *generated* by A if there is a surjective homomorphism $\varphi : \tilde{A}^* \to M$ (preserving the involution). What we have so far done is to have associated to each inverse A-graph C (that is, to each graph immersion $\ell : C \to \mathcal{B}_A$) an A-generated inverse monoid $I(C)$. It is straightforward to check that the labeling $\ell : C \to \mathcal{B}_A$ is a covering if and only if $I(C)$ is a group. This is but one of the many ways that the transition monoid encodes algebraically information about the immersion. The second author and Meakin [35] showed how graph immersions can be classified in terms of closed inverse submonoids of a free inverse monoid in much the same way as graph coverings are classified in terms of subgroups of a free group. This will be generalized in the next section.

We remark that there is an obvious notion of a morphism of A-graphs (namely, a morphism of graphs which preserves labels). However the correspondence $C \mapsto I(C)$ is by no means a functor (in general). An exception is the case of a covering. A (surjective) covering morphism of graphs $C \to C'$ does induce a (surjective) homomorphism $I(C) \to I(C')$.

If A is a set and $R \subseteq \tilde{A}^*$, then $\mathcal{B}_{A,R}$ will denote the 2-complex obtained from $\mathcal{B}_A$ by attaching $|R|$ 2-cells via the elements of R. A complex X with a (*labeling*) morphism $\ell : X \to \mathcal{B}_{A,R}$ will be called an (A,R)-*complex.* There is an obvious notion of a morphism of (A,R)-complexes. Notice that if the labeling is an immersion on the 1-skeleton, then $X^{(1)}$ is an inverse A-graph in a natural way. We call such an (A,R)-complex X an *inverse* (A,R)-complex and denote $I(X^{(1)})$ by $I(X)$ (which we designate the *transition monoid* of X). Thus coverings of $\mathcal{B}_{A,R}$ give rise to inverse (A,R)-complexes with transition monoids which are groups. In fact, the transition monoid, in this case, is nothing more than the action group for the usual right action of the fundamental group on the fiber, over the base point, of a covering; hence it is a quotient of the group $G = \langle A|R\rangle$. In general, for an inverse (A,R)-complex X, $I(X)$ is a group if and only if the labeling is a covering when restricted to $X^{(1)}$. We shall call a connected inverse (A,R)-complex X an (A,R)-*cover* if the labeling morphism is a covering and we shall call it *finite sheeted* if the covering morphism is a finite sheeted covering.

We now give the appropriate notion of a congruence on an inverse (A,R)-complex; see [36, 55] for the case of graphs. If X is an inverse (A,R)-complex, then a *congruence* is an equivalence relation $\equiv$ on $V(X)$ such that if v, $w \in V(X)$ and $a \in \tilde{A}$ are such that $v \equiv w$ and va, wa are defined, then $va \equiv wa$. The quotient complex $X/\equiv$ is defined as follows. The vertex set is $V(X)/\equiv$. Using square brackets for equivalence classes, there is an edge labeled by $a \in \tilde{A}$ from $[v]$ to $[w]$ if and only if there is an edge labeled by a from v' to w' for some $v' \in [v]$, $w' \in [w]$. One can verify this gives an inverse A-graph which is a quotient of $X^{(1)}$. One then attaches the 2-cells of X to this graph by composing the original attaching maps with the projection. The resulting 2-complex is $X/\equiv$. It is straightforward to verify that every morphism of inverse (A,R)-complexes factors as a quotient by a congruence followed by an inclusion.

Note that if X is an inverse (A,R)-complex, then the intersection of any two congruences is another congruence. Indeed, it is the congruence associated to the natural map from X to the pullback of the quotients over $\mathcal{B}_{A,R}$. It follows that if C is an inverse A-graph, then C has a least congruence so that the quotient is embeddable in an (A, R)-cover. Indeed, the universal congruence is one such. Moreover, by considering pullbacks, the set of such congruences is easily seen to be closed under intersection. Thus there is a least such congruence, which we call the *G-embeddable congruence*. We remark that this congruence does, in fact, depend on the presentation; that is, the term G-embeddable congruence should only be applied with a fixed presentation in mind. Suppose now that C is finite vertex inverse A-graph. Then we claim that there is a least congruence so that the quotient is embeddable in a finite-sheeted (A, R)-cover. Indeed, as before there is at least one such and the set Y of such is closed under finite intersections. However, since there are only finitely many congruences on C, Y is a finite set so it has a minimal element (namely the intersection of all its members). We call this the *G-finite-embeddable congruence*. Theorem 1.1(2) then has the following simple restatement which we set down as a corollary (to be proved when Theorem 1.1 is proved).

Corollary 1.6. *A group $G = \langle A|R\rangle$ is LERF if and only if, for each finite inverse A-graph C, the G-embeddable congruence and G-finite-embeddable congruence coincide.*

We shall give explicit descriptions of these congruences later in this section. Similar congruences were considered for the case of a free group equipped with a variant on the profinite topology [36].

1.4. *Relational morphisms*

We introduce several algebraic notions from semigroup theory to prove that if a group G is LERF, Theorem 1.1(3) holds. In the next subsection, we interpret these notions geometrically. Our primary tool will be relational morphisms, introduced by B. Tilson [16].

A *relational morphism* $\varphi : I \multimap I'$ of inverse monoids I, I' is a relation $\varphi \subseteq I \times I'$ which is an inverse submonoid of $I \times I'$ projecting onto I. For example, homomorphisms are relational morphisms, as are inverses of surjective homomorphisms. It is easy to check that the composition of relational morphisms (using the ordinary composition of relations) is again a relational morphism. Relational morphisms φ arise by considering diagrams

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, two heads, "\tau"'] & I' \\ I & \end{tikzcd}$$

where $\tau : R \to I$ is a surjective homomorphism and $\psi : R \to I'$ is a homomorphism, and letting $\varphi = \psi\tau^{-1}$. By setting τ and ψ equal to the projections from φ to I and I', respectively, we see that all relational morphisms arise in this manner.

The condition that $\varphi : I \mathrel{-\!\!\circ\!\!\!\rightarrow} I'$ is a relational morphism can be described more concretely as follows.

Proposition 1.7. *Let φ be a relation from I to I'. Then φ is a relational morphism if and only if:*

(1) $\varphi(m) \neq \emptyset$ *for all* $m \in I$;
(2) $1 \in \varphi(1)$;
(3) $\varphi(m)^{-1} = \varphi(m^{-1})$;
(4) $\varphi(m_1)\varphi(m_2) \subseteq \varphi(m_1 m_2)$.

Suppose that I and I' are both generated as inverse monoids by a set A. Then the *canonical* relational morphism $\varphi : I \mathrel{-\!\!\circ\!\!\!\rightarrow} I'$ is the submonoid of $I \times I'$ generated by pairs $([a]_I, [a]_{I'})$ with $a \in A$ and is diagrammed as

$$\begin{tikzcd} \tilde{A}^* \arrow[r, two heads, "\psi"] \arrow[d, two heads, "\tau"'] & I' \\ I \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where τ and ψ are the canonical projections. In particular, given two inverse (A, R)-complexes, there exists a canonical relational morphism between their transition monoids.

A relational morphism $\varphi : I \mathrel{-\!\!\circ\!\!\!\rightarrow} I'$ will be called *finitely generated* if φ is a finitely generated inverse monoid. If A is a finite set, then the canonical relational morphism between any two A-generated inverse monoids is finitely generated.

If I is an inverse monoid, $E(I)$ will denote the set of idempotents of I; that is, $E(I) = \{e \in I | e^2 = e\}$. It is easy to verify that the idempotents of I commute and hence $E(I)$ is a submonoid of I. Note that I is a group precisely when $|E(I)| = 1$; in particular, inverse submonoids of groups are subgroups. It is easy to deduce from Proposition 1.7 the following lemma.

Lemma 1.8. *Let $\varphi : I \mathrel{-\!\!\circ\!\!\!\rightarrow} I'$ be a relational morphism of inverse monoids. Then for $e \in E(I')$, $\varphi^{-1}(e)$ is an inverse submonoid of I and, for $e \in E(I)$, $\varphi(e)$ is an inverse submonoid of I'.*

If $\varphi : I \mathrel{-\!\!\circ\!\!\!\rightarrow} G$ is a relational morphism of inverse monoids with G a group, we define $\ker \varphi = \varphi^{-1}(1)$. The above remarks show that $\ker \varphi$ is an inverse submonoid of I. It is easy to see that $E(I) \subseteq \ker \varphi$; indeed, if $e \in E(I)$, then $e^{-1} = e$ and so, for $g \in \varphi(e)$, $g^{-1} \in \varphi(e^{-1}) = \varphi(e)$ whence

$$1 = gg^{-1} \in \varphi(e)\varphi(e) \subseteq \varphi(e^2) = \varphi(e)\,.$$

We say that $\varphi : I \mathrel{-\!\!\circ\!\!\!\rightarrow} G$ is *idempotent pure* if $\ker \varphi = E(I)$. Following the literature, one says in this case that I has an *E-unitary cover* over G.

We now study in more detail relational morphisms between inverse monoids and groups. For the rest of this section, we adopt the convention that I (and its variants) will always denote an inverse monoid and G (and its variants) a group.

Proposition 1.9. *Let $\varphi : I \multimap G$ be a relational morphism and let $e \in E(I)$. Then $\varphi(e)$ is a subgroup of G. If I is finite and φ is finitely generated, then $\varphi(e)$ is finitely generated.*

Proof. By Lemma 1.8, we know that $\varphi(e)$ is an inverse submonoid and hence a subgroup of G. For the second statement, let $\pi_1 : \varphi \to I$ and $\pi_2 : \varphi \to G$ be the projections. Then since I is finite, $\pi_1^{-1}(e) \in Rec(\varphi)$. But φ is finitely generated as an inverse monoid and hence as a monoid, so $\pi_1^{-1}(e) \in Rat(\varphi)$ by Corollary 1.4. Hence, by Proposition 1.3, we see that $\varphi(e) = \pi_2(\pi_1^{-1}(e)) \in Rat(G)$. Thus, by Theorem 1.5, $\varphi(e)$ is finitely generated. □

Lemma 1.10. *Let $\varphi : I \multimap G$ be a relational morphism, $m \in I$, and $g \in \varphi(m)$. Then $\varphi(m)$ is the coset $\varphi(mm^{-1})g$.*

Proof. Since

$$(mm^{-1})(mm^{-1}) = (mm^{-1}m)m^{-1} = mm^{-1}\,,$$

we see, by Proposition 1.9, that $H = \varphi(mm^{-1})$ is a subgroup of G. Now

$$Hg \subseteq \varphi(mm^{-1})\varphi(m) \subseteq \varphi(mm^{-1}m) = \varphi(m)$$

so we just need to show that $\varphi(m) \subseteq Hg$. Let $g' \in \varphi(m)$. Then

$$g'g^{-1} \in \varphi(m)\varphi(m^{-1}) \subseteq \varphi(mm^{-1})$$

so $g' \in Hg$ as desired. □

The following is a trivial consequence of the fact that multiplication is continuous in the profinite topology [26].

Lemma 1.11. *Let G be a group. Then G is LERF if and only if, given $g \in G$, H a finitely generated subgroup, and $g' \notin Hg$, there exists a homomorphism $\psi : G \to G'$ with G' finite so that $\psi(g') \notin \psi(Hg)$.*

We now are in a position to prove that LERF implies the inverse monoid theoretic characterization.

Theorem 1.12. *Suppose G is a LERF group, I is a finite inverse monoid, and $\varphi : I \multimap G$ is a finitely generated, idempotent pure relational morphism. Then there is a homomorphism $\psi : G \to G'$ with G' finite such that $\psi\varphi$ is idempotent pure.*

Proof. Let $m \in I \setminus \ker \varphi$. Then $1 \notin \varphi(m)$. Now, by Lemma 1.10, $\varphi(m) = Hg$, where $g \in \varphi(m)$ and $H = \varphi(mm^{-1})$ is, by Proposition 1.9, a finitely generated subgroup of G. Hence, by Lemma 1.11, there is a homomorphism $\psi_m : G \to G_m$ with G_m finite and

$$1 = \psi_m(1) \notin \psi_m(Hg) = \psi_m(\varphi(m))\,.$$

Let $G' = \prod_{m \in I \setminus \ker\varphi} G_m$ and $\psi = \prod_{m \in I \setminus \ker\varphi} \psi_m$. Since I is finite, so is G'. But, by construction of G', $\ker(\psi\varphi) \subseteq \ker\varphi = E(I)$. It follows that $\psi\varphi$ is idempotent pure. $\square$

1.5. *Subgraphs of coverings*

We now interpret idempotent pure relational morphisms geometrically. Let $G = \langle A|R\rangle$ be a group. We adopt the notation that if I is an A-generated inverse monoid and $w \in \tilde{A}^*$, then $[w]_I$ denotes the evaluation of w in I.

Proposition 1.13. *Suppose C is an inverse A-graph such that, for each connected component C_i of C, C_i embeds in an (A,R)-cover X_i. Then the canonical relational morphisms $\varphi : I(C) \multimap G$ and $\varphi_i : I(C_i) \multimap I(X_i)$ are idempotent pure.*

Proof. Suppose $w \in \tilde{A}^*$ is such that $[w]_{I(X_i)} = 1$. Then w reads a loop in $X_i^{(1)}$ at every vertex. Hence at any vertex v of $C_i \subseteq X_i$, either w cannot be read, or w reads a loop: that is, $vw = v$. Thus w acts as a partial identity on the vertices of C_i whence $[w]_{I(C_i)} \in E(I(C_i))$. It follows then that φ_i is idempotent pure. Suppose now that $w \in \tilde{A}^*$ is such that $[w]_G = 1$. Then since, for all i, $I(X_i)$ is a quotient of G, it follows that $[w]_{I(X_i)} = 1$ for all i. The above argument then shows that w acts as a partial identity on each C_i and hence on C; thus $[w]_{I(C)} \in E(I(C))$ establishing that φ is idempotent pure. $\square$

We now aim towards proving the converse of the above proposition.

Lemma 1.14. *Let C be a graph, $\ell : C \to \mathcal{B}_{A,R}$ an immersion, $v \in V(C)$, and $\varphi : I(C) \multimap G$ the canonical relational morphism. Suppose $\psi : G \to G'$ is a homomorphism such that $\psi\varphi$ is idempotent pure. Then if $w \in \tilde{A}^*$ is such that $[w]_G \in \ell(\pi_1(C,v))(\ker\psi)$ and vw is defined, $vw = v$.*

Proof. Let $H = \ell(\pi_1(C,v))$ and $K = \ker\psi$. Then since $w \in HK$ and ℓ is an immersion, there exists u, $u' \in \tilde{A}^*$ such $vu = v$, $[u']_G \in K$, and $[uu']_G = [w]_G$ (that is, u reads a loop at v, $\psi([u']_G) = 1$, and uu' is homotopic to w in $\mathcal{B}_{A,R}$). But then $[u^{-1}w]_G \in K$, whence $[u^{-1}w]_{I(C)} \in E(I(C))$ by the assumption that $\psi\varphi$ is idempotent pure. It follows that $u^{-1}w$ acts a partial identity on $V(C)$. But $u^{-1}w$ can be read from v, so

$$vw = (vu^{-1})w = v(u^{-1}w) = v\,. \qquad \square$$

The following lemma is standard and we omit the proof.

Lemma 1.15. *Let C be a graph, $\ell : C \to \mathcal{B}_{A,R}$ an immersion, and v_1, $v_2 \in V(C)$. Suppose $w \in \tilde{A}^*$ is such that $v_1 w = v_2$. Then $\ell(\pi_1(C, v_1)) = [w]_G \ell(\pi_1(C, v_2))[w]_G^{-1}$.*

We prove one last result on the way to the converse of Proposition 1.13. This result, in the case that ψ is the identity map, has appeared in the inverse semigroup literature in various guises. A topological proof for that case, similar to the one below, was given in [55].

Theorem 1.16. *Let C be a connected graph, $\ell : C \to \mathcal{B}_{A,R}$ an immersion, and $v_0 \in V(C)$. Let $\varphi : I(C) \multimap G$ be the canonical relational morphism and $\psi : G \to G'$ be a homomorphism such that $\psi\varphi$ is idempotent pure. Then (C, v_0) embeds in the (A, R)-cover $(\tilde{X}, \widetilde{v_0})$ corresponding to the subgroup $\ell(\pi_1(C, v_0))(\ker \psi)$.*

Proof. Let $K = \ker \psi$ and let ℓ be the labeling morphism for C. By choice of $\tilde{X}$, there is a lift $\tilde{\ell} : (C, v_0) \to (\tilde{X}, \widetilde{v_0})$. Since $\tilde{\ell}$ must be an immersion, it is an embedding if and only if it is injective on vertices. First observe that since C is connected, given $v \in V(C)$, there exists $w \in \tilde{A}^*$ labeling a path from v_0 to v. It follows that w labels a path from $\widetilde{v_0}$ to $\tilde{\ell}(v)$. Hence two applications of Lemma 1.15 and the fact that K is a normal subgroup show that $\ell(\pi_1(C, v))K = \pi_1(\tilde{X}, \tilde{\ell}(v))$.

So now suppose that v_1, $v_2 \in V(C)$ are such that $\tilde{\ell}(v_1) = \tilde{\ell}(v_2)$. Since C is connected, there exists $u \in \tilde{A}^*$ labeling a path from v_1 to v_2. Then u labels a loop at $\tilde{\ell}(v_1)$ whence

$$[u]_G \in \pi_1(\tilde{X}, \tilde{\ell}(v_1)) = \ell(\pi_1(C, v_1))K\,.$$

But then u reads a loop at v_1 by Lemma 1.14 so $v_1 = v_2$. It follows that $\tilde{\ell}$ is an embedding. $\square$

It is worth remarking that, for an inverse (A, R)-complex C which does not have "too many 2-cells" (in a sense to be made precise below), any 1-skeleton embedding into a covering extends to an embedding of the entire complex. An inverse (A, R)-complex has "too many 2-cells" if there is a relator $r \in R$ such that, in the free group, $r = s^n$ where s is not a proper power, and there are at least $n+1$ 2-cells of C with a common boundary cycle labeled by r. This is an obstruction to embedding because any cover will have exactly n 2-cells at any occurrence of a cyclic path with label r.

We now have the following theorem on embedding graphs.

Theorem 1.17. *Suppose C is an inverse A-graph. Then each connected component C_i of C embeds in an (A, R)-cover X_i if and only if the canonical relational morphism $\varphi : I(C) \multimap G$ is idempotent pure.*

Proof. Necessity is Proposition 1.13. We show that, for each connected component C_i of C, the canonical relational morphism $\varphi_i \multimap G$ is idempotent pure. The result will then follow from Theorem 1.16 with ψ the identity map. Indeed, if $w \in \tilde{A}^*$ and

$[w]_G = 1$, then $[w]_{I(C)}$ is an idempotent by hypothesis. But then w acts as a partial identity on C and hence on C_i. Thus $[w]_{I(C_i)}$ is an idempotent. □

We are now able to prove that Theorem 1.1(3) implies Theorem 1.1(2).

Theorem 1.18. *Let G be a group generated by a set A satisfying Theorem* 1.1(3)*. Then given an (A, R)-cover $\tilde{X}$ and a finite connected subgraph C of $\tilde{X}$, there is a finite sheeted (A, R)-cover $\tilde{X}'$ such that C embeds (as an (A, R)-complex) in $\tilde{X}'$.*

Proof. Let C be such an inverse A-graph. Then, by Theorem 1.17, the canonical relational morphism $\varphi : I(C) \multimap G$ is idempotent pure. Since C is finite, $I(C)$ is finite and only a finite subset $A_0 \subseteq A$ appears as labels of elements of C. Consider the inverse submonoid of φ generated by the pairs $([a]_{I(C)}, [a]_G)$ with $a \in A_0$. This defines a finitely generated, idempotent pure relational morphism $\varphi' : I' \multimap G$ where I' is the inverse submonoid of $I(C)$ generated by A_0. Hence, by hypothesis on G, there is a homomorphism $\psi : G \to G'$ with G' finite and $\psi\varphi'$ idempotent pure. We claim that $\psi\varphi$ is idempotent pure. Indeed, it $w \in \tilde{A}^*$ and w contains a letter outside of A_0, then $[w]_{I(C)} = 0$ and hence is idempotent. On the other hand, if $w \in \tilde{A}_0^*$, then $\psi([w]_G) = 1$ implies that $[w]_{I(C)} = [w]_{I'}$ is an idempotent. It now follows, by Theorem 1.16, that C embeds in the (A, R)-cover $\tilde{X}'$ corresponding to $H = \ell(\pi_1(C, v_0))(\ker \psi)$. But since G' is finite, $\ker \psi$ is of finite index whence H is of finite index. Thus $\tilde{X}'$ is finite sheeted. □

All the ingredients of Theorem 1.1 have now been proved: Theorem 1.12 shows that LERF implies the third condition, Theorem 1.18 shows that the third condition implies the second condition, and Theorem 1.2 shows that the second condition implies LERF. Corollary 1.6 now follows as well.

We remark that if C is an inverse A-graph and $G = \langle A|R \rangle$, then Theorem 1.17 makes it clear that the G-embeddable congruence on C is given by defining vertices p, q to be equivalent if they are connected by a path whose label w is 1 in $G = \langle A|R \rangle$. Using similar techniques to the above, one can show that the G-finite-embeddable congruence is defined as follows. Let $\varphi : \tilde{A}^* \to G$ be the projection associated with the presentation $G = \langle A|R \rangle$. For p, $q \in V(C)$, let $L_{p,q} = \{w \in \tilde{A}^* | pw = q\}$. Then p and q are identified if and only if $1 \in \overline{\varphi(L_{p,q})}$ (where the closure is with respect to the profinite topology).

1.6. *An application to free products*

We are now in a position to give a version of Gitik's [19] topological proof of the Burns–Romanovskii theorem [11, 47] that free products of LERF groups are LERF.

Lemma 1.19. *Let $G = \langle A|R \rangle$ and C be an A-graph such that: $\ell : C \to \mathcal{B}_A$ is a covering and the group $I(C)$ satisfies R. Then each connected component of C embeds in an (A, R)-cover. Furthermore, if C has finitely many vertices, these covers can be taken to be finite sheeted.*

Proof. Let Y be a connected component of C, $\varphi : I(Y) \multimap G$ the canonical relational morphism, and $\psi : G \to I(C)$ the canonical surjection. Then, for $w \in \tilde{A}^*$, $\psi([w]_G) = 1$ implies w reads a loop at each vertex of C and hence at each vertex of Y. Thus $\psi\varphi$ is idempotent pure. The result then follows from Theorem 1.16 and the fact that if C has finitely many vertices, then $I(C)$ is finite and so $\ker\psi$ is of finite of index. □

In fact, each connected component of C must be the 1-skeleton of an (A, R)-cover. We observe that this lemma also applies to inverse (A, R)-complexes which do not have too many 2-cells.

Theorem 1.20. *Let G_1 and G_2 be LERF groups. Then their free product $G_1 * G_2$ is LERF.*

Proof. Let $G_1 = \langle A_1 | R_1\rangle$ and $G_2 = \langle A_2 | R_2\rangle$ (with $A_1 \cap A_2 = \emptyset$). Set $A = A_1 \cup A_2$ and $R = R_1 \cup R_2$; then $G_1 * G_2 = \langle A | R\rangle$. Let C be a finite connected subgraph of an (A, R)-cover $\tilde{X}$. We show C embeds in a finite sheeted (A, R)-cover.

First observe that if X is an inverse (A, R)-complex, then one can construct inverse (A_i, R_i)-complexes X_i $(i = 1, 2)$, each having the same vertex set as X, but keeping only edges with labels in A_i and 2-cells with boundary cycles in R_i. If X is an (A, R)-cover, then the connected components of X_i, $i = 1, 2$, are (A_i, R_i)-covers. Since, for $i = 1, 2$, C_i embeds in $\tilde{X}_i$, we see that each connected component of C_i embeds in an (A_i, R_i)-cover which, by Theorem 1.1, can be taken to be finite sheeted (since C_i is finite and G_i is LERF). For each i, the union of these covers will be denoted Y_i. Without loss of generality, we may assume that $V(Y_1) \cap V(Y_2) = V(C)$. Let Y be the 1-skeleton of $Y_1 \cup Y_2$; that is, Y is formed by gluing the graphs Y_1 and Y_2 along the vertices of C via the respective embeddings of $V(C)$ into $V(Y_1)$ and $V(Y_2)$. Then Y is an inverse A-graph with finitely many vertices, containing C. We form a new inverse A-graph Y' by adding loops at each vertex of $Y_2 \setminus C$ labeled by the elements of A_1 and at each vertex of $Y_1 \setminus C$ labeled by the elements of A_2. The labeling $\ell : Y' \to \mathcal{B}_A$ is then a covering and Y' has only finitely many vertices.

We claim that $I(Y')$ satisfies R. Indeed, if $w \in R_1$, then w reads a loop at every vertex of Y_1 (since Y_1 is a union of (A_1, R_1)-covers). But w also reads a loop at every vertex of $Y_2 \setminus C$ by construction. So it follows that $[w]_{I(Y')} = 1$. A symmetric argument shows that $I(Y')$ satisfies R_2. Thus, by Lemma 1.19 (and since Y' has finitely many vertices), the connected graph Y' embeds in a finite sheeted (A, R)-cover. The result now follows by Theorem 1.1. □

It is instructive for the reader to interpret this proof in terms of idempotent pure relational morphisms of inverse monoids. This proof technique is generalized in [21] to obtain more precise separability results for free products of LERF groups.

1.7. *Stronger separability properties*

We call a group G *n-coset separable* if, given finitely generated subgroups $H_1, \ldots, H_n$ of G and $w \notin H_1 \cdots H_n$, there exists a finite index normal subgroup $N \lhd G$ such that $w \notin NH_1 \cdots H_n$ or, equivalently, $H_1 \cdots H_n$ is closed in the profinite topology. Note that a group is LERF if and only if it is 1-coset separable. It is straightforward to verify that if G is n-coset separable, one can replace any of the H_i by one of its left or right cosets and the conclusion still holds.

Double coset separability was introduced by Gitik and Rips [22] where the result was proved for free groups. This was extended to surface groups by Niblo [39] and to free products of word hyperbolic surface groups (or, more generally, free products of pinched one-relator groups where the amalgamation occurs over malnormal cyclic subgroups) by the second and third authors together with Gitik [21]. Ribes and Zalesskiĭ [44] proved that free groups are n-coset separable for all n; see [29] for a model theoretic proof with an inverse semigroup theory flavor. Ash [8] proved an inverse monoid theoretic result formally equivalent to the n-coset separability of a free group [4, 8, 56]. Coulbois [13] proved that surface groups are n-coset separable for all n. He also showed [12, 13] that n-coset separability passes through free products using a mixture of model theoretic and inverse semigroup theoretic tools; the underlying argument of his proof is the same as the proof of Theorem 1.20. Our current goal is to give an inverse monoid theoretic characterization of n-coset separability.

For the rest of this section, fix a group G. Let I be a finite inverse monoid and $(m_1, \ldots, m_n) \in I^n$. We say that $(m_1, \ldots, m_n)$ is a *G-liftable n-tuple* if, given any relational morphism $\psi : I \multimap G'$ with G' a finite image of G, $1 \in \psi(m_1) \cdots \psi(m_n)$. The following argument is a generalization of the approach of [56] for free groups.

Theorem 1.21. *Let G be a group. Then the following are equivalent*:

(1) *G is n-coset separable.*
(2) *Given a finite inverse monoid I, a G-liftable n-tuple $(m_1, \ldots, m_n) \in I^n$, and a relational morphism $\psi : I \multimap G$, $1 \in \psi(m_1) \cdots \psi(m_n)$.*

Proof. Suppose first that G is n-coset separable and that I is a finite inverse monoid. Let $(m_1, \ldots, m_n) \in I^n$ be a G-liftable n-tuple and $\psi : I \multimap G$ be a relational morphism. For each $m \in I$, choose $g_m \in \psi(m)$. Let $\psi' : I \multimap G$ be the subrelational morphism generated by the (m, g_m), $m \in I$. Then it clearly suffices to prove the second statement for ψ'. Thus we may assume without loss of generality that ψ is finitely generated. The fact that $(m_1, \ldots, m_n)$ is a G-liftable n-tuple implies that, for any finite index normal subgroup $N \lhd G$, $1 \in N\psi(m_1) \cdots \psi(m_n)$. Now, by Lemma 1.10, each $\psi(m_i)$ is a right coset of a finitely generated subgroup. Hence, by n-coset separability, if $1 \notin \psi(m_1) \cdots \psi(m_n)$, there would exist $N \lhd G$ of finite index such that $1 \notin N\psi(m_1) \cdots \psi(m_n)$, a contradiction. We conclude 2 holds.

Now suppose 2 holds. Let $H_1, \ldots, H_n$ be finitely generated subgroups of G and $w \in G \setminus H_1 \cdots H_n$. Then

$$1 \notin H_1 \cdots H_n w^{-1} .$$

Choose a presentation $\langle A|R \rangle$ of G. For each i, choose a finite subgraph C_i of the pointed (A, R)-cover associated to H_i containing a lift from the base point of each of its generators. For H_n, choose C_n so that there is also a lift of w^{-1}. Let $C = \bigcup_i C_i$ be the disjoint union and set $I = I(C)$.

Now let $\psi : I \multimap G$ be the canonical relational morphism. Also let, for $i = 1, \ldots, n-1$, F_i be the set of $m \in I$ fixing the base point of C_i, and let F_n be the set of $m \in I$ taking the base point of C_n to the endpoint of the lift of w^{-1}. Then it is straightforward to check

$$H_1 \cdots H_n w^{-1} = \bigcup_{m_1 \in F_1} \psi(m_1) \cdots \bigcup_{m_n \in F_n} \psi(m_n) .$$

We claim that no $(m_1, \ldots, m_n) \in F_1 \times \cdots \times F_n$ is a G-liftable tuple. For if one of them were, then, by 2,

$$1 \in \psi(m_1) \cdots \psi(m_n) \subseteq H_1 \ldots H_n w^{-1} ,$$

a contradiction. Thus, for each such n-tuple, there is a relational morphism $\psi_{(m_1,\ldots,m_n)} : I \to G_{(m_1,\ldots,m_n)}$ with $G_{(m_1,\ldots,m_n)}$ a finite image of G and $1 \notin \psi_{(m_1,\ldots,m_n)}(m_1) \cdots \psi_{(m_1,\ldots,m_n)}(m_n)$. By choosing, for each $a \in A$, $g_a \in \psi_{(m_1,\ldots,m_n)}(a)$ and taking the subrelational morphism generated by the (a, g_a), $a \in A$, it follows we may assume that the $\psi_{(m_1,\ldots,m_n)}$ are canonical relational morphisms. Hence for each such n-tuple we can find a normal subgroup $N_{(m_1,\ldots,m_n)}$ such that $\psi_{(m_1,\ldots,m_n)}$ is the composition of ψ followed by the projection to $G/N_{(m_1,\ldots,m_n)}$. Let $N = \bigcap_{(m_1,\ldots,m_n) \in F_1 \times \cdots \times F_n} N_{(m_1,\ldots,m_n)}$. Since I^n is finite, N is a finite intersection of finite index normal subgroups and is hence such a subgroup. It is straightforward to verify

$$1 \notin N H_1 \cdots H_n w^{-1}$$

whence $w \notin N H_1 \cdots H_n$ as desired. □

Ash proved [8] that the second statement holds for free groups.

2. Classifying Subgraphs of Covers via Inverse Monoids

Our next application of inverse monoids to combinatorial group theory is to classify connected subgraphs of coverings. In this section, all 2-complexes are assumed to be connected unless otherwise stated. Classical results of combinatorial group theory tell us that if $G = \langle A|R \rangle$, then pointed coverings of $\mathcal{B}_{A,R}$ (or even their 1-skeleta) correspond to subgroups of G. We now canonically associate to each group presentation $G = \langle A|R \rangle$ an A-generated inverse monoid $M(A, R)$ (which we shall denote $M(G)$ when the presentation is clear from the context). Pointed subgraphs of (A, R)-covers will then be in one-to-one correspondence with closed inverse submonoids of $M(G)$. The work in this section is a generalization of the results of [35] for the case of a free group. We also identify the closed inverse submonoid corresponding to the geodesic core [20] of a subgroup allowing us to apply our techniques to quasiconvexity.

2.1. *Classification*

Observe that Theorem 1.17 shows that an inverse A-graph C embeds in an (A, R)-cover if and only if the canonical relational morphism of $I(C)$ with G is idempotent pure. Thus, it seems natural that such inverse A-graphs should be classified by the freest A-generated inverse monoid with such a relational morphism; that is, by the inverse monoid $M(A, R)$ given by the inverse monoid presentation

$$\langle A | w = w^2 \text{ whenever } [w]_G = 1 \rangle \,. \tag{2.1}$$

Notice that these relations are satisfied by G, so there is a natural surjective homomorphism $\psi : M(A, R) \to G$. Moreover, $\ker \psi = E(M(A, R))$ by (2.1) so ψ is idempotent pure. It follows that the word problem for G reduces to the membership problem for $E(M(A, R))$ which is decidable if the word problem for $M(A, R)$ is decidable.

On the other hand, it is shown [34] that two words of $\tilde{A}^*$ represent the same element of $M(A, R)$ if and only if they map to the same element of G and traverse the same set of edges in the Cayley graph $\Gamma_A(G)$ of G with respect to A when read from 1 (or, equivalently, any vertex). Here we consider an edge and its inverse as a single geometric edge. We remind the reader that $\Gamma_A(G)$ is the 1-skeleton of the universal cover of $\mathcal{B}_{A,R}$ [17]. Since the decidability of the word problem for G is equivalent to the effective constructibility of arbitrary finite parts of $\Gamma_A(G)$, it follows that the word problems for G and $M(G)$ are equivalent.

For example, if $G = FG(A)$ is the free group on $A = \{x, y\}$, then $M(FG(A))$ is the free inverse monoid on A [30, 34, 40]. Consider the words 1 and xx^{-1}. These words are not equivalent in $M(FG(A))$ even though they are in $FG(A)$ because, when read from 1, xx^{-1} traverses the edge $(1, x, x)$ while 1 does not. On the other hand, $xx^{-1}yy^{-1}x$ and $yy^{-1}x$ do represent the same element of $M(FG(A))$ since they both map to x in $FG(A)$ and they both transverse the set of edges $\{(1, x, x), (1, y, y)\}$ when read from 1.

An explicit model of $M(G)$, due to the second author and Meakin [34], is as follows. Consider all pairs (X, g) where X is a finite, connected subgraph of $\Gamma_A(G)$ containing 1 and g. Define a product by

$$(X, g)(Y, h) = (X \cup gY, gh)$$

and an involution by $(X, g)^{-1} = (g^{-1}X, g^{-1})$. One can verify [34] that this inverse monoid is $M(G)$: a generator a is sent to $(\{(1, a, [a]_G)\}, [a]_G)$. The structure of this inverse monoid is studied in detail in [34]. In particular, it is residually finite.

Let (C, b) be a pointed, connected subgraph of an (A, R)-cover X. The lifting property of covering maps shows that the inclusion of (C, b) factors through the covering associated to $\ell(\pi_1(C, b))$ where ℓ is the labeling function. Hence we may assume without loss of generality that (C, b) is embedded in the pointed (A, R)-cover corresponding to $\ell(\pi_1(C, b))$.

Observe that $I(C)$ satisfies the relations (2.1) by Theorem 1.17. Hence there is a natural homomorphism $\varphi : M(G) \to I(C)$. In analogy to combinatorial group theory, define

$$Stab(b) = \{[w]_{M(G)} | bw = b\}\,.$$

This is clearly an inverse submonoid of $M(G)$. Furthermore, $\psi(Stab(b))$ is easily verified to be $\ell(\pi_1(C, b))$. Our goal is to show that (C, b) can be reconstructed from $Stab(b)$ and to characterize the submonoids of $M(G)$ which can arise in this manner. This is essentially a special case of the theory of transitive representations of an inverse monoid, so we shall only sketch the details (more can be found in [35, 40], though we recommend computing some examples to gain some intuition about inverse monoids).

If M is any inverse monoid, the natural partial order on M is defined by $m \leq n$ if $m = mm^{-1}n$ or, equivalently, $m \in E(M)n$. One can show that $\leq$ is compatible with multiplication and inversion [30, 40]. The intuition comes from thinking about $I(C)$. For $m, n \in I(C)$, $m \leq n$ says exactly that if vm is defined, then vn is defined and $vm = vn$ or, more succinctly, $\cdot m$ is a restriction of $\cdot n$. If $X \subseteq M$, we set $X\!\uparrow = \{m \in M | m \geq x,\ \text{some } x \in X\}$. A subset X is said to be *closed* if $X\!\uparrow = X$. It is an elementary exercise to verify that $Stab(b)$ is closed (think of $\leq$ in terms of restriction).

Suppose N is a closed inverse submonoid of $M(G)$. We construct the unique pointed, connected inverse A-graph (C, b) embeddable in an (A, R)-cover such that $N = Stab(b)$. It will follow immediately from the construction that morphisms of such inverse A-graphs correspond to inclusions of closed inverse submonoids. Also we shall see that (C, b) is the 1-skeleton of an (A, R)-covering precisely when $E(N) = E(M(G))$ or, in inverse monoid terminology, N is a *full*, closed inverse submonoid. Details will be suppressed; the reader is referred to [35, 40].

If N is a closed inverse submonoid of $M(G)$, then the *right ω-cosets* of N are the sets of the form $N[w]_{M(G)}\!\uparrow$ with $[ww^{-1}]_{M(G)} \in N$. The *index* of N is the number of right ω-cosets. We use $N/M(G)$ to denote the set of right cosets. Intuitively, if $N = Stab(b)$, then the set $N[w]_{M(G)}\!\uparrow$ is a valid right ω-coset of N precisely when bw is defined. Furthermore, $N[w]_{M(G)}\!\uparrow = N[u]_{M(G)}\!\uparrow$ precisely when $bu = bw$. Thus there is a one-to-one correspondence between vertices of C (we are using connectivity here) and right ω-cosets of $Stab(b)$.

In general, if N is a closed inverse submonoid, we can define an inverse A-graph with vertices $N/M(G)$. If $N[w]_{M(G)}\!\uparrow$ is a right coset and $a \in \tilde{A}$, we define an edge labeled by a from $N[w]_{M(G)}\!\uparrow$ to $N[wa]_{M(G)}\!\uparrow$ if $waa^{-1}w^{-1} \in N$. One can check that this gives a pointed, connected inverse A-graph $(C(N), N)$, that the transition monoid is a quotient of $M(G)$ (and thus satisfies (2.1) and hence is embeddable in an (A,R)-cover), that $N = \mathit{Stab}(N)$, and that if $N = \mathit{Stab}(b)$ as above, then $(C(N), N)$ is isomorphic to (C, b). It follows (C, b) has finitely many vertices if and only if $\mathit{Stab}(b)$ is of finite index.

As an example, let G be a free Abelian group on $A = \{x, y\}$ and let C be the subgraph of $\Gamma_A(G)$ consisting of the square given by the path $xyx^{-1}y^{-1}$ read from 1; choose 1 as the base point. The elements of $\mathit{Stab}(1)$ correspond to pairs $(X, 1)$ where X is a connected subgraph of C containing 1. There are eleven such elements: one choice of X with no edges, two choices with one edge, three choices with two edges, four choices with three edges, and one choice with four edges.

One can verify that N has four cosets: N, $N[x]_G\!\uparrow$, $N[xy]_G\!\uparrow$, and $N[y]_G\!\uparrow$. The corresponding inverse A-graph is exactly the square C.

It remains to characterize when (C, b) is the 1-skeleton of an (A, R)-cover. In this case, for any $w \in \tilde{A}^*$, $bww^{-1} = b$, so $[ww^{-1}]_{M(G)} \in \mathit{Stab}(b)$. It follows that $E(\mathit{Stab}(b)) = E(M(G))$. Conversely, if N is a full, closed inverse submonoid (that is, $E(N) = E(M(G))$) then $N[w]_{M(G)}\!\uparrow$ is a right ω-coset for all $w \in \tilde{A}^*$ whence the inverse A-graph $C(N)$ constructed above is an A-cover satisfying (2.1) and hence R. Thus $C(N)$ is the 1-skeleton of an (A, R)-cover by the remark after Lemma 1.19.

For any inverse submonoid $N \subseteq M(G)$, $\psi(N)$ is a subgroup of G (where $\psi : M(G) \to G$ is the projection). Note that if N is closed, then right ω-cosets of N project to right cosets of $\psi(N)$. This gives an immediate way to embed (C, N) in the pointed (A, R)-cover associated to $\psi(N)$. We now characterize the full, closed inverse submonoids of $M(G)$.

Proposition 2.1. *Let $N \subseteq M(G)$ be an inverse submonoid. Then N is closed and full if and only if $N = \psi^{-1}(H)$ for a subgroup H of G.*

Proof. Since $\psi(E(M(G))) = 1$, it follows easily that $\psi^{-1}(H)$ is closed and full for any subgroup. For the converse, let $H = \psi(N)$. Suppose $m \in \psi^{-1}(H)$. Then there exists $n \in N$ with $\psi(m) = \psi(n)$ whence $\psi(n^{-1}m) = 1$. Since ψ is idempotent pure, $n^{-1}m \in E(M(G)) \subseteq N$. Thus $nn^{-1}m \in N$. But $m \geq nn^{-1}m$, so $m \in N$, N being closed. $\square$

We now show that closed inverse submonoids of $M(G)$ are in one-to-one correspondence with subgroups $H \subseteq G$ acting invariantly on a connected subgraph of $\Gamma_A(G)$ containing 1. Such a situation is considered, for instance, in [37].

Theorem 2.2. *Let N be a closed inverse submonoid and $H = \psi(N)$. Then there is an H-invariant, connected subgraph Y of $\Gamma_A(G)$ containing 1 such that*

$$N = \{(X, h) \in M(G) | h \in H, X \subseteq Y\}\,. \tag{2.2}$$

Conversely, given a subgroup H and an H-invariant, connected subgraph of $\Gamma_A(G)$ containing 1, (2.2) defines a closed inverse submonoid of $M(G)$. Moreover, these are inverse constructions.

Furthermore, if we view $(C(N), N)$ as sitting in the pointed (A, R)-cover Z associated to H and we let $\varphi : \Gamma_A(G) \to Z$ be the natural map, then $Y = \varphi^{-1}(C(N))$.

Proof. Suppose first N is a closed inverse submonoid of $M(G)$ and define Y as in the final statement of the theorem. Since the projection φ is just taking the quotient under the left action of H, Y is H-invariant; moreover, since N is the base point of Z, $1 \in Y$. To see that Y is connected, by standard lifting properties of covers, it suffices to show that each lift of the right ω-coset N can be connected to 1 by a path in Y. But such lifts correspond to elements of H. Since $\psi(N) = H$, for each $h \in H$, we can find $w \in \tilde{A}^*$ with $[w]_G = h$ and $[w]_{M(G)} \in N$. But then w labels a loop at N in $C(N)$, so the path labeled by w from 1 to h in $\Gamma_A(G)$ is contained in Y.

Let $w \in \tilde{A}^*$ represent an element of N. Then w reads a loop at vertex N in $C(N) \subseteq Z$ whence $[w]_G \in H$ and the lift in $\Gamma_A(G)$ of w at 1 uses only edges in Y. Thus $[w]_{M(G)} = (X, h)$ with $h \in H$ and $X \subseteq Y$. Conversely, suppose $(X, h) \in M(G)$ is such that $X \subseteq Y$ and $h \in H$. Then if $w \in \tilde{A}^*$ represents this element, $[w]_G = h \in H$ and uses precisely the edges of X when read from 1 in $\Gamma_A(G)$, all of which are in Y. It follows that in Z, w reads a loop at N using only edges of $C(N)$; that is $w \in Stab(N) = N$.

Conversely, suppose Y is an H-invariant, connected subgraph of $\Gamma_A(G)$ containing 1. Let $(C, b) = (H/Y, H) \subseteq Z$. We claim that $Stab(b) = N$ (defined as per (2.2)). Indeed, suppose $w \in \tilde{A}^*$ represents an element of $Stab(b)$. Then the lift of w to 1 in $\Gamma_A(G)$ ends at a point $h \in H$ and uses only edges of Y. Thus w corresponds to $(X, h) \in N$ where X is the set of edges used by w. Conversely, if $(X, h) \in N$ and w is a word with $[w]_G = h$ using precisely the edges of $X \subseteq Y$ in its run from 1 in $\Gamma_A(G)$, then w reads a loop in C at b so $(X, h) \in Stab(b)$. The result follows. $\square$

We remark that N is full precisely when $Y = \Gamma_A(G)$.

2.2. *Applications to quasiconvexity*

For the remainder of this section, we take the alphabet A to be finite. We also assume familiarity with the usual metric on $\Gamma_A(G)$ which defines the *distance* between two vertices as the minimal length of a path connecting them [17]. Recall that if $G = \langle A|R\rangle$, then a word $w \in \tilde{A}^*$ is called a *geodesic* if it is a minimal length representative of $[w]_G$, or, equivalently, all paths in $\Gamma_A(G)$ labeled by w are geodesics (in the usual sense). We use $\mathcal{G}$ to denote the set of geodesic words in $\tilde{A}^*$. If G is word hyperbolic, $\mathcal{G}$ is a rational subset [17].

A subgroup $H \leq G$ is called *K-quasiconvex* [17] for an integer K if any geodesic between members of H is contained in the K-neighborhood of H; equivalently, any geodesic from 1 to H is contained in the K-neighborhood of H. A subgroup is *quasiconvex* if it is K-quasiconvex for some K.

Gitik defined the *geodesic core* [20] of H, $\mathcal{G}$-*core*(H), to be the smallest subgraph of the pointed (A, R)-cover associated to H which contains all loops at the base point labeled by geodesic representatives of H. She proved that H is K-quasiconvex if and only if $\mathcal{G}$-*core*(H) is contained in the K-neighborhood of the base point. In particular, H is quasiconvex if and only if $\mathcal{G}$-*core*(H) is finite.

More generally, if $\mathcal{C} \subseteq \tilde{A}^*$ is a subset mapping onto G, one calls $\mathcal{C}$ a *combing*. One then gives $\mathcal{G}$ the appellation, *geodesic combing*. The notions of K-$\mathcal{C}$-*quasiconvexity* and $\mathcal{C}$-*quasiconvexity* are defined as above, replacing geodesics with elements of $\mathcal{C}$. One can also define, for a subgroup $H \leq G$, $\mathcal{C}$-*core*(H) analogously and we can deduce the same relations between $\mathcal{C}$-quasiconvexity and boundedness of $\mathcal{C}$-*core*(H); the third author and Silva used this more general construction in [49] to study rational subgroups of automatic monoids; see [23] for quasiconvexity of more general subsets with respect to a combing, and for relations with formal language theory.

Fix a combing $\mathcal{C}$ and a subgroup $H \leq G$. We now proceed to identify the closed inverse submonoid associated to $\mathcal{C}$-*core*(H). Then we shall prove that $\mathcal{C}$-quasiconvexity is closed under finite intersection. We shall also obtain a formal language theoretic equivalence to $\mathcal{C}$-quasiconvexity which will imply the result that quasiconvex subgroups of word hyperbolic groups are precisely the $\mathcal{G}$-rational subgroups (and its generalization to automatic structures [17]) as well as closure under finite intersection. In addition, we shall be able to deduce the result that $\mathcal{C}$-quasiconvex subgroups are finitely generated.

First we give another characterization of $(C(N), N)$ for a closed inverse submonoid of $M(G)$. Let $H = \psi(N)$. The following proposition is clear.

Proposition 2.3. *$(C(N), N)$ is the subgraph of the pointed (A, R)-cover associated to H consisting of all loops at the base point with label w such that $[w]_{M(G)} \in N$.*

It now follows that, for a subgroup H, $\mathcal{C}$-*core*(H) corresponds to the smallest closed inverse submonoid of $M(G)$ containing $\{[w]_{M(G)} | w \in \mathcal{C}, [w]_G \in H\}$ which we denote (abusing notation) by $\mathcal{C} \cap H$. We then have

Theorem 2.4. *Let $\mathcal{C}$ be a combing for $G = \langle A|R \rangle$. Then $H \leq G$ is $\mathcal{C}$-quasiconvex if and only if $\mathcal{C} \cap H$ is a finite index, closed inverse submonoid of $M(G)$.*

The following result is well known [40].

Proposition 2.5. *Suppose that N_1, N_2 are closed inverse submonoids of an inverse monoid M of finite indices n_1, n_2, respectively. Then $N_1 \cap N_2$ is a closed inverse submonoid of index at most $n_1 n_2$.*

The proof in our case is that if one considers the pullback of $(C(N_1), N_1)$ and $(C(N_2), N_2)$ over $\mathcal{B}_{A,R}$, then $Stab((N_1, N_2)) = N_1 \cap N_2$.

Proposition 2.6. *Suppose that $N_1 \subseteq N_2$ are closed inverse submonoids of $M(G)$ with $\psi(N_1) = \psi(N_2) = H$ and N_2 of finite index. Then N_1 is also of finite index.*

Proof. Both $(C(N_1), N_1)$ and $(C(N_2), N_2)$ have pointed embeddings into the (A, R)-cover associated to H. Since $N_1 \subseteq N_2$, it follows that $(C(N_1), N_1) \subseteq (C(N_2), N_2)$ and hence has finitely many vertices. We conclude N_1 is of finite index. $\square$

The following generalizes results from the cases of quasiconvexity [17, 20] and of $\mathcal{C}$-quasiconvexity for an automatic structure given by a rational combing $\mathcal{C}$ [17].

Corollary 2.7. *Suppose H_1, $H_2 \leq G$ are $\mathcal{C}$-quasiconvex subgroups of G. Then so is $H_1 \cap H_2$.*

Proof. We have $N_1 = \mathcal{C} \cap H_1$, $N_2 = \mathcal{C} \cap H_2$ are of finite index in $M(G)$, whence $N_1 \cap N_2$ is of finite index by Proposition 2.5. But it is clear that $\mathcal{C} \cap (H_1 \cap H_2) \subseteq N_1 \cap N_2$, so $\mathcal{C} \cap (H_1 \cap H_2)$ is of finite index by Proposition 2.6. The result follows from Theorem 2.4. ◻

We now generalize the theorem that if $\mathcal{C}$ is a combing giving an automatic structure, then the $\mathcal{C}$-quasiconvex subgroups are the $\mathcal{C}$-rational subgroups [17] (in particular, for hyperbolic groups, this is the case for $\mathcal{G}$).

Theorem 2.8. *Suppose $G = \langle A|R\rangle$ and that $\mathcal{C}$ is a combing. Let $\varphi : \tilde{A}^* \to G$ be the canonical projection. Then, for $H \leq G$ a subgroup, the following are equivalent:*

(1) *H is $\mathcal{C}$-quasiconvex.*
(2) *$\mathcal{C}$-core(H) is finite.*
(3) *$\mathcal{C} \cap H$ is of finite index in $M(G)$.*
(4) *There is a recognizable subset $L \subseteq \varphi^{-1}(H)$ such that $\mathcal{C} \cap \varphi^{-1}(H) \subseteq L$.*

Proof. We already have the equivalence of the first three conditions. Suppose $\mathcal{C}$-*core*(H) is finite. Then we can view it as a finite state automaton over $\tilde{A}$ with the base point as the start state and the unique final state. The language of this automaton is a recognizable subset $L \subseteq \varphi^{-1}(H)$ which contains $\mathcal{C} \cap \varphi^{-1}(H)$ by definition of $\mathcal{C}$-*core*(H).

Suppose now that the last condition holds. Let K be the number of states of a finite state automaton $\mathcal{A}$ accepting L and suppose $w \in \mathcal{C}$ is read from 1 to H in $\Gamma_A(G)$. Then if u is a prefix of w, u read from the start state of $\mathcal{A}$ does not go to a fail state (since $w \in \mathcal{C} \cap \varphi^{-1}(H)$ is accepted by $\mathcal{A}$). Thus there exists $v \in \tilde{A}^*$ such that $|v| \leq K$ and uv is accepted by $\mathcal{A}$. But then $[uv]_G \in H$. It follows that w is in the K-neighborhood of H whence H is $\mathcal{C}$-quasiconvex. ◻

We now give a formal language theoretic proof that the intersection of $\mathcal{C}$-quasiconvex subgroups is $\mathcal{C}$-quasiconvex. Suppose H_1, H_2 are $\mathcal{C}$-quasiconvex and choose recognizable subsets L_1, L_2 as in Theorem 2.8. Since the class of recognizable subsets is closed under intersection [9, 16, 17], $L_1 \cap L_2$ is a recognizable subset of $\tilde{A}^*$ which verifies Theorem 2.8(4) for $H_1 \cap H_2$.

In particular, if $\mathcal{C}$ is recognizable, so is $L \cap \mathcal{C} = \varphi^{-1}(H) \cap \mathcal{C}$. This explains the results on rational subgroups in [17]. We observe that condition 4 of the theorem implies condition 1 even in the infinitely generated case.

Theorem 2.9. *Let $\mathcal{C}$ be a combing of $G = \langle A|R\rangle$. Then $\mathcal{C}$-quasiconvex subgroups are finitely generated.*

Proof. Let H be $\mathcal{C}$-quasiconvex and let L be a recognizable subset as per Theorem 2.8(4). Then $\varphi(L) = H$. Since L is rational by Kleene's theorem, $H = \varphi(L)$ is rational by Proposition 1.3 and hence finitely generated by Theorem 1.5. Alternatively, one could use the fact that $\mathcal{C}$-*core*(H) is finite, its fundamental group maps onto H via the labeling, and that the fundamental group of a finite graph is finitely generated. □

3. Applications to Finite Semigroup Theory

This section is concerned with the consequences of separability properties of groups, mainly relatively free groups, to the decidability of membership for certain classes of finite monoids.

A *variety* of groups is a class of groups closed under the formation of subgroups, homomorphic images and direct products. For a set A and a variety of groups $\mathcal{H}$, $F_{\mathcal{H}}(A)$ will denote the free group on A in the variety $\mathcal{H}$ and is referred to as a *relatively free group*. See [38] for more on varieties of groups.

In this section, we shall assume that $\mathcal{H}$ is a variety of groups such that each finitely generated relatively free group $F_{\mathcal{H}}(A)$ is residually finite, i.e. $\mathcal{H}$ is generated by its finite members. Many of the most common varieties of groups satisfy these assumptions. For instance, the variety of all groups and the variety of all Abelian groups (in fact any variety of Abelian groups) are well-known examples. The variety of metabelian groups (i.e. solvable groups of derived length 2) is also an example, since P. Hall [27] proved that any finitely generated metabelian group is residually finite.

A *pseudovariety of monoids* is a class of finite monoids closed under the formation of submonoids, homomorphic images and finite direct products. The class $\mathbf{M}$ of all finite monoids, the classes $\mathbf{R}$, $\mathbf{L}$, $\mathbf{J}$ of all finite, respectively, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$-trivial monoids, and the class $\mathbf{Sl}$ of all finite semilattices (i.e. idempotent, commutative monoids) are examples of pseudovarieties of monoids. A *pseudovariety of groups* is a pseudovariety of monoids consisting entirely of groups. The classes $\mathbf{G}$ of all finite groups, $\mathbf{G_p}$ of all finite p-groups, p a prime, $\mathbf{Ab}$ of all finite Abelian groups and $\mathbf{G_{nil}}$ of all finite nilpotent groups are all examples of pseudovarieties of groups. In this section we shall assume all pseudovarieties to be recursively enumerable.

It is well known that varieties are defined by equations (or identities). This is not the case for pseudovarieties (see [2]). For example, there is no set of equations defining $\mathbf{G_p}$. Those pseudovarieties of groups that are defined by equations are said to be *equational*. For example, $\mathbf{Ab}$ and the class of all finite metabelian groups are equational pseudovarieties.

A *relational morphism* $\varphi : M \multimap N$ *of monoids* M, N is a relation $\varphi \subseteq M \times N$ which is a submonoid of $M \times N$ projecting onto M. If $\varphi \subseteq M \times G$ is a relational morphism, with G a group, we set $\ker \varphi = \varphi^{-1}(1)$. Let $\mathbf{H}$ be a pseudovariety of groups and M a monoid. The $\mathbf{H}$-*kernel of* M is the submonoid $K_{\mathbf{H}}(M) = \bigcap \ker \varphi$, with the intersection being taken over all relational morphisms of monoids

$\varphi : M \multimap G$, $G \in \mathbf{H}$. A subset $P \subseteq M$ is said to be *pointlike* for a relational morphism $\varphi : M \multimap N$ if there exists $n \in N$ such that $P \subseteq \varphi^{-1}(n)$. One says that P is $\mathbf{V}$-*pointlike* if it is pointlike for every relational morphism from M into a monoid in the pseudovariety $\mathbf{V}$; $\mathbf{V}$-*pointlike pairs* are $\mathbf{V}$-pointlike sets with two elements.

Among the most studied operators between pseudovarieties of monoids are the semidirect and Mal'cev products. The *semidirect product* $\mathbf{V} * \mathbf{W}$ of the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the pseudovariety generated by all monoid semidirect products $M * N$, where $M \in \mathbf{V}$ and $N \in \mathbf{W}$. For example, the pseudovariety $\mathbf{Ab} * \mathbf{Ab}$ is the pseudovariety of all finite metabelian groups. Now we give the definition of Mal'cev product. As in all cases considered in this paper the second factor is a pseudovariety $\mathbf{H}$ of groups, rather than giving the standard definition, we opt for one that is equivalent in these cases (see [28]). Let $\mathbf{V}$ be a pseudovariety of monoids and $\mathbf{H}$ a pseudovariety of groups. The *Mal'cev product* of $\mathbf{V}$ and $\mathbf{H}$, $\mathbf{V} \textcircled{m} \mathbf{H}$, is the set $\{M \in \mathbf{M} : K_{\mathbf{H}}(M) \in \mathbf{V}\}$. The two operators just defined are related by the following well-known inclusion

$$\mathbf{V} * \mathbf{H} \subseteq \mathbf{V} \textcircled{m} \mathbf{H}\,.$$

The reverse inclusion, although valid in some important cases (such as when $\mathbf{V}$ is a pseudovariety of groups, or, more generally, $\mathbf{V}$ is local [59]), is not true in general [28, 55].

When $\mathcal{H}$ is a variety of groups generated by a pseudovariety $\mathbf{H}$, we denote the relatively free group $F_{\mathcal{H}}(A)$ by $F_{\mathbf{H}}(A)$. Note that $F_{\mathbf{G}}(A) = FG(A)$.

We define an equational pseudovariety $\mathbf{H}$ of groups to be n-*coset separable* if all the relatively free groups $F_{\mathbf{H}}(A)$ are n-coset separable. The terminology LERF for pseudovarieties of groups is also used with the obvious meaning.

Several results concerning n-coset separability have already been mentioned in this paper. For instance, M. Hall [25] proved that free groups (and thus the pseudovariety $\mathbf{G}$) are LERF and, with Hall's theorem as a starting point, Ribes and Zalesskiĭ [44] proved that they are, in fact, n-coset separable for all integers $n > 0$. The reason for the terminology RZ, used by some authors to express the property of being n-coset separable for all positive integers n, now becomes clear. We remark that the motivation for Ribes and Zalesskiĭ's result arose out of semigroup theory. Indeed, it was conjectured by Pin and Reutenauer in [42] where they reduced the Rhodes type II conjecture (which proposed an algorithm to compute the $\mathbf{G}$-kernel of a finite monoid) to this problem.

Another important class of groups, all of whose members are known to be LERF, is the class of polycyclic groups [33], [46, Theorem 5.4.16].

Theorem 3.1. *Polycyclic groups are LERF.*

This result was generalized by Lennox and Wilson [31] as follows.

Theorem 3.2. *Polycyclic groups are* 2*-coset separable.*

An obvious question that one could now ask (are polycyclic groups 3-coset separable?) was answered negatively by Lennox and Wilson in the same paper.

Since finitely generated nilpotent groups are polycyclic, they are 2-coset separable whence we can deduce the following corollary.

Corollary 3.3. *Any equational pseudovariety of nilpotent groups is* 2*-coset separable.*

In particular, the pseudovariety **Ab** is LERF. In fact, since the product of subgroups of an Abelian group is again a subgroup, it is indeed n-coset separable for all positive integers n; that is, the pseudovariety **Ab** is RZ.

Coulbois [13] proved the following theorem.

Theorem 3.4. *Free metabelian groups are LERF.*

As an immediate corollary, since the pseudovariety of metabelian groups generates a (in fact "the", after P. Hall's result) variety of metabelian groups, we have

Corollary 3.5. *The pseudovariety of metabelian groups is LERF.*

As Coulbois also observed, these results cannot be extended in the natural directions: Agalakov [1] proved that free solvable groups of derived length ≥ 3 are not LERF; on the other hand, free metabelian groups of rank ≥ 2 are not 2-coset separable [13].

A standard argument (see, for example [13, Proposition 1.14] or [44]) shows the following

Proposition 3.6. *If $G \leq H$ is a finite index subgroup and G is n-coset separable, then so is H.*

In order to apply this to pseudovarieties of groups, we prove the following proposition. Recall that a pseudovariety **K** of groups is said to be *locally finite* if, for any finite set A, the relatively free group $F_{\mathbf{K}}(A)$ on A is finite.

Proposition 3.7. *Suppose* **H** *is an equational pseudovariety of groups and* **K** *a locally finite pseudovariety of groups. Then $\mathbf{H} * \mathbf{K}$ is equational. Moreover, if* **H** *has a recursively enumerable basis of identities, so does $\mathbf{H} * \mathbf{K}$.*

Proof. We first observe that the relatively free groups for **K** are computable by Zel'manov's positive solution to the restricted Burnside's problem [61]. Hence the set of identities satisfied by **K** is recursive. Let Σ be a basis of identities for **H**. We give an effective procedure to construct a basis of identities Ω for $\mathbf{H} * \mathbf{K}$. It follows that if **H** has a recursively enumerable basis of identities, then so does $\mathbf{H} * \mathbf{K}$.

To construct Ω, for each identity $(w = 1)$ in Σ over an alphabet B and each homomorphism $\sigma : FG(B) \to FG(A)$ such that $\mathbf{K} \models (\sigma(b) = 1)$ for all $b \in B$, we place in Ω the identity $(\sigma(w) = 1)$ over A.

That Ω is satisfied by $\mathbf{H} * \mathbf{K}$ follows from standard results on product varieties [2, 38].

For the converse, let G be an A-generated group satisfying Ω and consider the canonical relational morphism φ of G with $F_{\mathbf{K}}(A)$ diagrammed

$$\begin{tikzcd} FG(A) \arrow[r, two heads, "\psi"] \arrow[d, "\tau"'] & F_{\mathbf{K}}(A) \\ G \arrow[ur, "\varphi"'] & \end{tikzcd} .$$

Then $\ker\varphi = \{g \in G | g = \tau(u), \mathbf{K} \models (u = 1)\}$. Suppose $(w = 1) \in \Sigma$ over an alphabet B and $\alpha : FG(B) \to \ker\varphi$ is an homomorphism. Then we can choose, for each $b \in B$, $u_b \in FG(A)$ such that $\tau(u_b) = \alpha(b)$ and $\mathbf{K} \models (u_b = 1)$. Define $\sigma : FG(B) \to FG(A)$ by $\sigma(b) = u_b$. Then $(\sigma(w) = 1) \in \Omega$, so $\tau(\sigma(w)) = 1$. It follows that $\alpha(w) = \tau\sigma(w) = 1$ so $\ker\varphi \in \mathbf{H}$, whence $K_{\mathbf{K}}(G) \in \mathbf{H}$. We conclude $G \in \mathbf{H} * \mathbf{K}$. $\square$

Corollary 3.8. *If* $\mathbf{H}$ *and* $\mathbf{K}$ *are equational pseudovarieties of groups,* $\mathbf{H}$ *is n-coset separable and* $\mathbf{K}$ *is locally finite, then* $\mathbf{H} * \mathbf{K}$ *is n-coset separable.*

Proof. Since $\mathbf{K}$ is locally finite, $\mathbf{H} * \mathbf{K}$ is equational by Proposition 3.7 and $F_{\mathbf{K}}(A)$ is finite for all finite sets A. But $F_{\mathbf{H}*\mathbf{K}}(A)$ embeds in $F_{\mathbf{H}}(F_{\mathbf{K}}(A) \times A) * F_{\mathbf{K}}(A)$ where $*$ denotes the free product (see [2]). The result follows. $\square$

These kinds of separability properties for pseudovarieties of groups lead to many interesting consequences as we shall see in the sequel. The attention to the role played by LERF-like properties of pseudovarieties of groups in finite semigroup theory started with work of Pin and Reutenauer [42] (following work of Pin [41]) and continued with work of the first author [14, 15]. The key result, proved by Pin for the pseudovariety $\mathbf{G}$ and by the first author for $\mathbf{Ab}$, is stated in Proposition 3.9 below for any equational pseudovariety $\mathbf{H}$ of groups.

The pro-$\mathbf{H}$ topology on a group F is the weakest topology rendering continuous all homomorphisms from F into a group in $\mathbf{H}$. Of course, the profinite topology is the pro-$\mathbf{G}$ topology. In order to avoid confusion, we denote by $\mathrm{cl}_{\mathrm{H}}(X)$ the closure of X in the pro-$\mathbf{H}$ topology and by $\bar{X}$ the closure in the profinite topology. We remark that, for an equational pseudovariety of groups, the pro-$\mathbf{H}$ topology on $F_{\mathbf{H}}(A)$ is precisely the profinite topology. For those unfamiliar with pro-$\mathbf{H}$ topologies, we note that if a group F is endowed with the pro-$\mathbf{H}$ topology, then, for a subset $X \subseteq F$, $x \in \mathrm{cl}_{\mathrm{H}}(X)$ if and only if $\tau(x) \in \tau(X)$ for any homomorphism $\tau : F \to G$, $G \in \mathbf{H}$.

For the remainder of this section, A shall be a finite alphabet and M a finite A-generated monoid i.e. a finite monoid together with a surjective homomorphism $\tau : A^* \to M$. For an equational pseudovariety $\mathbf{H}$ of groups, let $\gamma_{\mathbf{H}} : A^* \to F_{\mathbf{H}}(A)$ be the canonical homomorphism from the free monoid into the relatively free group. With the notation just introduced, we have the following proposition.

Proposition 3.9. *Let $x \in M$. Then $x \in K_{\mathbf{H}}(M)$ if and only if $1 \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$.*

Proof. Let $x \in K_{\mathbf{H}}(M)$. We need to show that, for any group $G \in \mathbf{H}$ and any homomorphism $\pi : F_{\mathbf{H}}(A) \to G$, one has $1 = \pi(1) \in \pi(\gamma_{\mathbf{H}}(\tau^{-1}(x)))$. We then have the situation indicated in the following diagram

$$\begin{tikzcd} A^* \arrow[r, "\gamma_{\mathbf{H}}"] \arrow[d, "\tau"'] & F_{\mathbf{H}}(A) \arrow[d, "\pi"] \\ M \arrow[r, "\circ" description, "\varphi"'] & G \end{tikzcd}$$

where $\varphi = \pi\gamma_{\mathbf{H}}\tau^{-1} : M \multimap G$ is a relational morphism. Thus $1 \in \varphi(x) = \pi(\gamma_{\mathbf{H}}(\tau^{-1}(x)))$ by hypothesis.

Conversely, let $\varphi : M \multimap G$ be a relational morphism with $G \in \mathbf{H}$ and suppose that $1 \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$. Let α and β be the projections from φ to M and G, respectively.

Observe that there is a homomorphism f such that the following triangle commutes.

$$\begin{tikzcd} A^* \arrow[r, "f"] \arrow[d, "\tau"'] & \varphi \arrow[dl, "\alpha"] \\ M & \end{tikzcd}$$

It suffices to define f on A; this can be done by defining, for $a \in A$, $f(a)$ to be any element of $\alpha^{-1}(\tau(a))$.

There is then a homomorphism ψ such that the following diagram commutes.

$$\begin{tikzcd} A^* \arrow[rr, "\gamma_{\mathbf{H}}"] \arrow[dr, "f"] \arrow[dd, "\tau"'] & & F_{\mathbf{H}}(A) \arrow[dd, "\psi"] \\ & \varphi \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M \arrow[rr, "\circ" description, "\varphi"'] & & G \end{tikzcd}$$

It suffices to define ψ on $\gamma_{\mathbf{H}}(A)$, since $\gamma_{\mathbf{H}}(A)$ generates the relatively free group $F_{\mathbf{H}}(A)$. So define, for $a \in A$, $\psi(\gamma_{\mathbf{H}}(a)) = \beta(f(a))$.

As $1 \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$, we have $1 \in \psi(\gamma_{\mathbf{H}}(\tau^{-1}(x)))$ i.e. there exists $w \in \gamma_{\mathbf{H}}(\tau^{-1}(x))$ such that $\psi(w) = 1$. Let $u \in \tau^{-1}(x)$ be such that $\gamma_{\mathbf{H}}(u) = w$. Then $\beta(f(u)) = \psi(\gamma_{\mathbf{H}}(u)) = 1$. But $\alpha(f(u)) = \tau(u) = x$, so $1 \in \varphi(x)$. As this happens for every relational morphism φ, we have that $x \in K_{\mathbf{H}}(M)$. $\square$

Let us now remark that if $\Phi : FG(A) \to F_{\mathbf{H}}(A)$ is the canonical homomorphism, then, for any homomorphism $\psi : FG(A) \to G$ into a group $G \in \mathbf{H}$, there exists a homomorphism $\tau : F_{\mathbf{H}}(A) \to G$ such that $\psi = \tau\Phi$. On the other hand, given a homomorphism $\tau : F_{\mathbf{H}}(A) \to G$, $G \in \mathbf{H}$, $\psi = \tau\Phi$ is a homomorphism $\psi : FG(A) \to G$. This observation, together with the remark before Proposition 3.9, make the following lemma clear.

Lemma 3.10. *Let $X \subseteq FG(A)$. Then g belongs to $\mathrm{cl}_{\mathrm{H}}(X)$ if and only if $\Phi(g)$ belongs to $\overline{\Phi(X)}$.*

A result of the third author [55, Corollary 7.16] shows that if $x \in M$ is regular, then $\mathrm{cl}_{\mathrm{H}}(\gamma_{\mathbf{G}}(\tau^{-1}(x))) = \mathrm{cl}_{\mathrm{H}}(H)w$, where $w \in \tau^{-1}(x)$ (which we view as an element of $FG(A)$) and H is a finitely generated subgroup of $FG(A)$ with an effectively constructible, finite generating set. As a consequence we have the following

Lemma 3.11. *If $\mathbf{H}$ is LERF and $x \in M$ is regular, then $\overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))} = \Phi(H)\Phi(w)$.*

Proof. First observe that, since $\gamma_{\mathbf{H}}(\tau^{-1}(x)) = \Phi(\gamma_{\mathbf{G}}(\tau^{-1}(x)))$, we have that $\overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))} = \overline{\Phi(\gamma_{\mathbf{G}}(\tau^{-1}(x)))}$. Thus, using Lemma 3.10, $h \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$ occurs precisely when $g \in \mathrm{cl}_{\mathrm{H}}(\gamma_{\mathbf{G}}(\tau^{-1}(x)))$, for some $g \in \Phi^{-1}(h)$. But $\mathrm{cl}_{\mathrm{H}}(\gamma_{\mathbf{G}}(\tau^{-1}(x))) = \mathrm{cl}_{\mathrm{H}}(H)w$ for H and $w \in A^*$ as above. Again using Lemma 3.10, we have that $h \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$ if and only if $h = \Phi(g) \in \overline{\Phi(H)}\Phi(w)$. As $\mathbf{H}$ is LERF, the result follows. ◻

A straightforward generalization of an argument of Evans [18] gives

Lemma 3.12. *Let $L \subseteq \tilde{A}^*$ be rational and G be a residually finite group having a recursively enumerable set of finite images. Suppose $\psi : \tilde{A}^* \to G$ is a surjective homomorphism with $\psi^{-1}(1)$ recursively enumerable and such that $\psi(L)$ is closed in the profinite topology. Then L is recursive.*

For a monoid M, we denote by $\mathrm{Reg}(M)$ the set of regular elements of M.

Theorem 3.13. *If $\mathbf{H}$ is LERF and has a recursively enumerable basis of identities, then $\mathrm{Reg}(K_{\mathbf{H}}(M))$ is computable.*

Proof. Since it is well known that $\mathrm{Reg}(K_{\mathbf{H}}(M)) = \mathrm{Reg}(M) \cap K_{\mathbf{H}}(M)$ (see, for instance, [55]), it suffices, by Proposition 3.9, to show that we can test, for $x \in \mathrm{Reg}(M)$, whether $1 \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$. By Lemma 3.11, we have to show that we can test, for $x \in \mathrm{Reg}(M)$, whether $1 \in \Phi(H)\Phi(w)$, for H finitely generated and $w \in A^*$ as above. But, using Lemma 3.12, we conclude that $\Phi(H)\Phi(w)$ is recursive, and this completes the proof. ◻

As a consequence, one has the decidability of a number of Mal'cev and semidirect products. We denote by **Reg V** the pseudovariety of all monoids M such that $\mathrm{Reg}(M)$ generates a monoid in the pseudovariety of monoids $\mathbf{V}$. Theorem 3.13 implies

Corollary 3.14. *If* $\mathbf{V}$ *is decidable and* $\mathbf{H}$ *is LERF with a recursively enumerable basis of identities, then* $\mathbf{Reg}\ \mathbf{V} \textcircled{m} \mathbf{H}$ *is decidable.*

The pseudovarieties $\mathbf{A}$ of finite aperiodic monoids, $\mathbf{DA}$ of finite monoids whose regular $\mathcal{D}$-classes are aperiodic semigroups and $\mathbf{DS}$ of finite monoids whose regular $\mathcal{D}$-classes are subsemigroups are easily seen to be examples of pseudovarieties $\mathbf{V}$ such that $\mathbf{Reg}\ \mathbf{V} = \mathbf{V}$. As each of these is local (see, [59] for $\mathbf{A}$, [3] for $\mathbf{DA}$ and [24] for $\mathbf{DS}$), we have

Corollary 3.15. *If* $\mathbf{H}$ *is LERF and has a recursively enumerable basis of identities, then* $\mathbf{A} * \mathbf{H}$, $\mathbf{DA} * \mathbf{H}$ *and* $\mathbf{DS} * \mathbf{H}$ *are decidable.*

Examples of pseudovarieties of groups satisfying the hypotheses of the above corollaries include the pseudovariety of metabelian groups and equational pseudovarieties of nilpotent groups definable by finitely many identities.

As another application, we have the following proposition [55] which, in conjunction with Theorem 3.13, gives the decidability of some more Mal'cev products.

Proposition 3.16. *The membership problems for the pseudovarieties of monoids* $\mathbf{R} \textcircled{m} \mathbf{H}$, $\mathbf{L} \textcircled{m} \mathbf{H}$ *and* $\mathbf{J} \textcircled{m} \mathbf{H}$ *are each equivalent to the computability of* $\mathrm{Reg}(K_{\mathbf{H}}(M))$ *for all finite monoids* M.

Theorem 3.17. *If* $\mathbf{H}$ *is RZ, then the* $\mathbf{H}$*-kernel and* $\mathbf{H}$*-pointlike pairs of an* A*-generated finite monoid* M *are decidable.*

Proof. With an entirely analogous proof to the one given for the case of a free group in [42] one can show that, for an element $x \in M$, the set $\overline{\gamma_{\mathbf{H}}(\tau^{-1}(x))}$ is a finite union of sets of the form $gH_1 \cdots H_r$, $r \geq 0$, where the element $g \in F_{\mathbf{H}}(A)$ and finite generating sets for the subgroups $H_1, \ldots, H_r$ are effectively constructible. So testing whether $x \in K_{\mathbf{H}}(M)$ is reduced to testing whether $1 \in gH_1 \cdots H_r$. We may then use Lemma 3.12 to conclude the decidability of the $\mathbf{H}$-kernel. The decidability of the $\mathbf{H}$-pointlike pairs is given in [14] for $\mathbf{H} = \mathbf{Ab}$; the general case is similar. $\square$

We say that an n-tuple $(x_1, \ldots, x_n) \in M^n$ is an $\mathbf{H}$-*liftable* n-tuple if, given any relational morphism $\psi : M \multimap G$, with $G \in \mathbf{H}$, $1 \in \psi(x_1) \cdots \psi(x_n)$. For liftable n-tuples one has a statement analogous to Proposition 3.9 and its proof is similar; see also Theorem 1.21.

Proposition 3.18. *Let* $(x_1, \ldots, x_n) \in M^n$. *Then* $(x_1, \ldots, x_n)$ *is an* $\mathbf{H}$*-liftable* n*-tuple if and only if* $1 \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x_1)) \cdots \gamma_{\mathbf{H}}(\tau^{-1}(x_1))}$.

Observe that a subset $\{a, b\} \subseteq \mathrm{Reg}(M)$ is $\mathbf{H}$-pointlike if and only if (a, c) is a liftable 2-tuple where c is an inverse of b [28].

Theorem 3.19. *If* $\mathbf{H}$ *is* n*-coset separable and has a recursively enumerable basis of identities, then* $\mathbf{H}$*-liftable* k*-tuples* $(k \leq n)$ *of regular elements are computable.*

Proof. It suffices, by Proposition 3.18, to show that we can test whether $1 \in \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x_1)) \cdots \gamma_{\mathbf{H}}(\tau^{-1}(x_k))}$ for a k-tuple $(x_1, \ldots, x_k)$ of regular elements. We claim that

$$\overline{\gamma_{\mathbf{H}}(\tau^{-1}(x_1)) \cdots \gamma_{\mathbf{H}}(\tau^{-1}(x_k))} = \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x_1))} \cdots \overline{\gamma_{\mathbf{H}}(\tau^{-1}(x_k))}\,. \tag{3.1}$$

Since the righthand side of (3.1) is contained in the lefthand side by continuity of multiplication, it suffices to show the righthand side is closed. But, by Lemma 3.11, the righthand side is $\Phi(H_1)\Phi(w_1) \cdots \Phi(H_k)\Phi(w_k)$ for finitely generated subgroups H_i and $w_i \in A^*$ $(1 \leq i \leq k)$ as above and hence closed (since $\mathbf{H}$ is n-coset separable).

Hence we just have to show we can test, for a k-tuple $(x_1, \ldots, x_k)$ of regular elements, whether $1 \in \Phi(H_1)\Phi(w_1) \cdots \Phi(H_k)\Phi(w_k)$. But this set is recursive by Lemma 3.12 allowing us to complete the proof. □

As a consequence we have that if $\mathbf{H}$ is 2-coset separable and has a recursively enumerable basis of identities, the $\mathbf{H}$-pointlike pairs of regular elements are decidable. This implies that $\mathbf{J} * \mathbf{H}$ is decidable [57]. Since $\mathbf{J} * \mathbf{H} = \Diamond\mathbf{H}$ [43, 54], where $\Diamond\mathbf{H}$ is the Schützenberger product, $\Diamond\mathbf{H}$ is decidable.

A monoid is said to be *completely regular* if each of its elements belongs to a subgroup. A pseudovariety consisting of completely regular monoids is termed *completely regular.*

Proposition 3.20. *If* $\mathbf{W}$ *is a completely regular, locally finite pseudovariety with computable relatively free monoids and* $\mathbf{H}$ *is* 2*-coset separable with a recursive basis of identities, then the pseudovariety join* $\mathbf{W} \vee \mathbf{H}$ *is decidable.*

Proof. Since $\mathbf{W} \vee \mathbf{H}$ consists entirely of completely regular monoids, we just need to test membership for such. The result then follows from [51, Theorem 4.2] and the proof of [51, Theorem 5.1] since we are only dealing with regular monoids. □

Acknowledgments

The first and third author were supported in part by FCT through *Centro de Matemática da Universidade do Porto* and by the Project PDCTI/32817/MAT/2000 in participation with the European Community Fund FEDER; the second and third authors were partially supported by the Emmy Noether Research Institute for Mathematics, Bar-Ilan University and the Minerva Foundation of Germany, and by the Excellency Center "Group Theoretic Methods in the study of Algebraic Varieties" of the Israel Science Foundation; the second author was partially supported by NSF grant DMS-9801357; the third author was supported in part by NSF-NATO postdoctoral fellowship DGE-9972697 and by Praxis XXI scholarship BPD 16306 98. All the authors received support from INTAS project 99-1224.

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