

ON SEMIGROUPS WHOSE IDEMPOTENT-GENERATED SUBSEMIGROUP IS APERIODIC

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We show that if S is a finite semigroup with aperiodic idempotent-generated subsemigroup and \mathbf{H} is a pseudovariety of groups, then the sequence of iterated \mathbf{H} -kernels of S stops at the idempotent-generated subsemigroup if and only if each subgroup of S belongs to the wreath product closure of \mathbf{H} . Applications are given to Mal'cev products.

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1. Introduction

In this paper, all semigroups are assumed to be finite unless otherwise stated. If \mathbf{H} is a pseudovariety of groups (i.e. a class of groups closed under formation of finite

direct products, subgroups and quotients groups), then it is natural to define a group to be **H**-solvable if it has a subnormal series, each of whose quotients belongs to **H**. For instance, a group is solvable in the classical sense if and only if it is **Ab**-solvable where **Ab** is the pseudovariety of Abelian groups. A group is a *p*-group if and only if it is **Z_p**-solvable where **Z_p** is the pseudovariety generated by a cyclic group of order *p*. One should not confuse **Z_p**-solvability with the classical notion of *p*-solvability [16]. Recall [9] that a group *G* is *p*-solvable if it has a normal series each of whose quotients is a *p*-group or a *p*'-group (the latter meaning the order is relatively prime to *p*). If **H** is the pseudovariety generated by **Z_p** and the simple *p*'-groups, then the *p*-solvable groups are precisely the **H**-solvable groups in our sense.

It is easy to see that a group is **H**-solvable if and only if its simple group divisors belong to **H** and moreover the collection of **H**-solvable groups is the smallest pseudovariety of groups containing **H** and closed under extension (or equivalently wreath product); see [6]. This pseudovariety is denoted **WH**.

Alternatively a group is **H**-solvable if iteration of the operation of taking the verbal subgroup corresponding to **H** eventually yields the trivial subgroup. For instance, a group is solvable if and only if iteration of the derived subgroup eventually arrives at the trivial subgroup.

One possible way to generalize these notions to semigroups is to consider all semigroups whose simple group divisors belong to **H**. Equivalently, one is considering all semigroups whose subgroups belong to **WH**. This set forms a pseudovariety of semigroups (defined analogously to the group case), which is normally denoted $\overline{\mathbf{WH}}$. The Krohn–Rhodes theorem [3, 6] implies that this pseudovariety consists precisely of all semigroups that divide a wreath product of groups in **H** and copies of the monoid obtained by adjoining a unit to the two element right zero semigroup. However, we do not choose this as the definition of **H**-solvability for semigroups. Instead we choose a definition introduced by the first and second authors in [5].

In [5], motivated by the fact that the verbal subgroup associated to **H** of a group can be generalized to semigroups via the notion of the **H**-kernel, they defined a semigroup to be **H**-solvable if iterating the **H**-kernel operator eventually arrives at the subsemigroup generated by the idempotents. In [5], they showed that a semigroup with commuting idempotents is **Ab**-solvable if and only if its subgroups are solvable groups. This paper supersedes [5] by proving a much more general result that is in some sense optimal. This paper can be read independently of [5], although that paper is a source for several ideas in this paper.

For a pseudovariety **V** of semigroups, we denote by **EV** the pseudovariety consisting of all semigroups whose idempotent-generated subsemigroup belongs to **V**.

Let **A** be the pseudovariety of all aperiodic semigroups and let **V** be a subpseudovariety of **A**. Recall that \circledast denotes the Mal'cev product. Our main result states that, for a non-trivial pseudovariety **H** of groups,

$$\mathbf{EV} \cap \overline{\mathbf{WH}} = \bigcup (\dots ((\mathbf{V} \circledast \mathbf{H}) \circledast \mathbf{H}) \circledast \mathbf{H} \dots) \circledast \mathbf{H} = \mathbf{V} \omega \circledast \mathbf{H}.$$

In particular,

$$\mathbf{EA} = \mathbf{A}^{\omega(\mathfrak{m})}\mathbf{G}.$$

There are many examples of pseudovarieties $\mathbf{V} \subseteq \mathbf{A}$ such that the stronger equality $\mathbf{EV} = \mathbf{V}(\mathfrak{m})\mathbf{G}$ holds. Most of them are direct consequences of the main result of [15]. Denote by $\text{Reg}(S)$ the set of regular elements of S and let \mathbf{DA} be the pseudovariety consisting of all semigroups whose regular elements are idempotents. Let

$$\mathbf{RV} = \{S \mid \langle \text{Reg}(S) \rangle \in \mathbf{V}\}.$$

Results of [15] imply that $\text{Reg}(\langle E(S) \rangle) = \text{Reg}(\mathbf{K}_{\mathbf{G}}(S))$, for $S \in \mathbf{EDA}$ (where $\mathbf{K}_{\mathbf{G}}(S)$ is the \mathbf{G} -kernel of S). Hence, if $\mathbf{V} \subseteq \mathbf{DA}$ is such that $\mathbf{V} = \mathbf{RV}$, then

$$\mathbf{EV} = \mathbf{V}(\mathfrak{m})\mathbf{G}.$$

Clearly $\mathbf{RV} = \mathbf{V}$ for \mathbf{V} any of \mathbf{J} , \mathbf{R} , \mathbf{L} or \mathbf{DA} , where \mathbf{J} , \mathbf{R} and \mathbf{L} are respectively the classes of all \mathcal{J} -trivial, \mathcal{R} -trivial, and \mathcal{L} -trivial semigroups. See [10] for more information on pseudovarieties of the form $\mathbf{V}(\mathfrak{m})\mathbf{G}$. See also [2] for some new examples and syntactic proofs.

It is natural to ask: does $\mathbf{EA} = \mathbf{A}(\mathfrak{m})\mathbf{G}$? The answer is negative. In fact, although all semigroups in $\mathbf{A}(\mathfrak{m})\mathbf{G}$ have complexity 1, Rhodes and Tilson [15] have shown the existence of regular semigroups of arbitrary complexity in \mathbf{EA} . The subpseudovariety $(\mathbf{A}(\mathfrak{m})\mathbf{G})(\mathfrak{m})\mathbf{G}$ of \mathbf{EA} also has semigroups of larger complexity. Rhodes has a published example [13] of a semigroup of complexity 2 in $(\mathbf{A}(\mathfrak{m})\mathbf{G})(\mathfrak{m})\mathbf{G}$ and has unpublished examples of semigroups with arbitrary complexity in this pseudovariety. A construction of Rhodes and Tilson [15] implies that, for all positive integers n and any non-trivial pseudovariety \mathbf{H} of groups, $\mathbf{A}^n(\mathfrak{m})\mathbf{H} \neq \mathbf{A}^{n-1}(\mathfrak{m})\mathbf{H}$. It follows that the hierarchy $\mathbf{EA} \cap \overline{\mathcal{WH}} = \mathbf{A}^{\omega(\mathfrak{m})}\mathbf{H}$ is strict, that is $\mathbf{EA} \cap \overline{\mathcal{WH}} \neq \mathbf{A}^n(\mathfrak{m})\mathbf{H}$, for any positive integer n .

2. Definitions, Notation and Preliminaries

For basic notions related to pseudovarieties of semigroups or monoids we refer the reader to Almeida’s book [1]. For background on Green’s relations and Rees matrix semigroups see the book of Clifford and Preston [4] and the chapters of Krohn, Rhodes and Tilson in [3]; see also Lallement’s book [12] and Tilson’s chapters in Eilenberg [6].

Let S be a semigroup. As usual, denote by S^I the monoid obtained from S by adjoining an identity element and, for a subset X of S , denote by $E(X)$ the subset of all idempotents of S contained in X and by $\langle X \rangle$ the subsemigroup of S generated by X .

The quasi-order $\leq_{\mathcal{J}}$ associated to Green’s relation \mathcal{J} is defined by: for all $u, v \in S$,

$$u \leq_{\mathcal{J}} v \quad \text{if and only if} \quad S^I u S^I \subseteq S^I v S^I.$$

Of course, for every $u, v \in S$, $u\mathcal{J}v$ if and only if $u \leq_{\mathcal{J}} v$ and $v \leq_{\mathcal{J}} u$. Denote by J_u the \mathcal{J} -class of the element $u \in S$. As usual, a partial order relation $\leq_{\mathcal{J}}$ is defined on the set S/\mathcal{J} by setting, for all $u, v \in S$, $J_u \leq_{\mathcal{J}} J_v$ if and only if $u \leq_{\mathcal{J}} v$. Given $u, v \in S$, we write $u <_{\mathcal{J}} v$ and $J_u <_{\mathcal{J}} J_v$ if and only if $u \leq_{\mathcal{J}} v$ and $(u, v) \notin \mathcal{J}$.

Similar notations are used for Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H} = \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$.

Let S be a semigroup and J be a \mathcal{J} -class of S . We denote by J^0 the semigroup with support $J \cup \{0\}$, with $0 \notin J$, and multiplication defined by:

$$a \cdot b = \begin{cases} ab & \text{if } a, b \in J \text{ and } ab \in J \\ 0 & \text{otherwise,} \end{cases}$$

for all $a, b \in J \cup \{0\}$.

Now, let A and B be non-empty sets, G a group and $C : B \times A \rightarrow G^0$ a $B \times A$ matrix over G^0 . The *Rees matrix semigroup* $\mathcal{M}^0(G, A, B, C)$ is the semigroup with support $(A \times G \times B) \cup \{0\}$ and multiplication defined by

$$(a, g, b)0 = 0(a, g, b) = 00 = 0$$

and

$$(a, g, b)(a', g', b') = \begin{cases} (a, gC(b, a')g', b') & \text{if } C(b, a') \in G \\ 0 & \text{otherwise (i.e. } C(b, a') = 0), \end{cases}$$

for all $a, a' \in A, b, b' \in B$ and $g, g' \in G$.

It is well-known that $\mathcal{M}^0(G, A, B, C)$ is a regular semigroup if and only if C is a *regular* matrix, i.e. each row and column of C has at least one non-zero entry, and in this case $\mathcal{M}^0(G, A, B, C)$ is a (completely) 0-simple semigroup.

Given semigroups S and T , a *relational morphism of semigroups* $\tau : S \dashrightarrow T$ is a function from S into $\mathcal{P}(T)$, such that $\tau(s_1) \neq \emptyset$ and $\tau(s_1)\tau(s_2) \subseteq \tau(s_1s_2)$, for all $s_1, s_2 \in S$.

A relational morphism $\tau : S \dashrightarrow T$ is, in particular, a relation from S to T and we may compose relational morphisms in the obvious way. Homomorphisms, viewed as relations, and inverses of onto homomorphisms are examples of relational morphisms.

From now on, \mathbf{H} will always denote a pseudovariety of groups; we use $\overline{\mathbf{H}}$ for the pseudovariety of semigroups whose subgroups are in \mathbf{H} .

The \mathbf{H} -kernel of a semigroup S is the subsemigroup $K_{\mathbf{H}}(S) = \bigcap \tau^{-1}(1)$, with the intersection being taken over all relational morphisms $\tau : S \dashrightarrow G$, with $G \in \mathbf{H}$. It is well-known [15] that there always exists a relational morphism $\tau : S \dashrightarrow G$, with $G \in \mathbf{H}$ such that $\tau^{-1}(1) = K_{\mathbf{H}}(S)$.

One of the most widely studied operators on the lattice of pseudovarieties is the Mal'cev product; see [14] for more on operators and iteration of operators on the lattice of pseudovarieties. Its definition, when the rightmost factor is a pseudovariety of groups, may be formulated as follows: for a pseudovariety \mathbf{V} , the *Mal'cev product* of \mathbf{V} and \mathbf{H} is the pseudovariety

$$\mathbf{V} \circledast \mathbf{H} = \{S \mid K_{\mathbf{H}}(S) \in \mathbf{V}\}.$$

We refer the reader to [10] for some history concerning the Mal'cev product and for connections with the all-important semidirect product.

Next, we define recursively $K_{\mathbf{H}}^{(n)}(S)$ as follows:

- $K_{\mathbf{H}}^{(0)}(S) = S$;
- $K_{\mathbf{H}}^{(n)}(S) = K_{\mathbf{H}}(K_{\mathbf{H}}^{(n-1)}(S))$, for $n \geq 1$.

Since S is finite and the operator $K_{\mathbf{H}}$ is non-increasing, it follows that the sequence $K_{\mathbf{H}}^{(n)}(S)$ is eventually constant; we denote this constant value by $K_{\mathbf{H}}^{(\omega)}(S)$. Observe that $K_{\mathbf{H}}^{(\omega)}(S)$ is the largest subsemigroup of S fixed by $K_{\mathbf{H}}$.

We say that S is \mathbf{H} -solvable if $K_{\mathbf{H}}^{(\omega)}(S) = \langle E(S) \rangle$.

Herein, we consider iterated Mal'cev products. More specifically, we consider iterations of the operator $(-) \circledast \mathbf{H}$. For a pseudovariety \mathbf{V} and $n \geq 1$, we define the operator $(-) \circledast^n \mathbf{H}$ recursively as follows:

- $\mathbf{V} \circledast^1 \mathbf{H} = \mathbf{V} \circledast \mathbf{H}$;
- $\mathbf{V} \circledast^{n+1} \mathbf{H} = (\mathbf{V} \circledast^n \mathbf{H}) \circledast \mathbf{H}$;
- $\mathbf{V} \circledast^\omega \mathbf{H} = \cup_{n \geq 1} \mathbf{V} \circledast^n \mathbf{H}$.

We use left exponentiation to distinguish iteration of the operator $(-) \circledast \mathbf{H}$ from iteration of the operator $\mathbf{V} \circledast (-)$.

It is easy to see that we have $\mathbf{V} \circledast^n \mathbf{H} = \{S \mid K_{\mathbf{H}}^{(n)}(S) \in \mathbf{V}\}$.

3. The Main Result

In this section, \mathbf{H} denotes a non-trivial pseudovariety of groups.

First we state some preliminary results. The following lemma concerning the idempotent-generated subsemigroup of a semigroup is due to Fitzgerald [7].

Lemma 3.1. *Let S be a (perhaps infinite) semigroup and let $s \in \langle E(S) \rangle$ be an element of a regular \mathcal{D} -class D . Then, there exist $e_1, e_2, \dots, e_m \in E(D)$ such that*

$$e_1 \mathcal{R} e_2 \mathcal{L} e_3 \mathcal{R} \cdots \mathcal{L} e_{m-1} \mathcal{R} e_m$$

and $s = e_1 e_2 \cdots e_m$. Hence $\langle E(S) \rangle \cap D = \langle E(D) \rangle \cap D$.

Recall that a \mathcal{J} -class is called *null* [3] if it is not regular.

Lemma 3.2. *Let S be a semigroup. If J is a null \mathcal{J} -class of S such that $J \not\subseteq \langle E(S) \rangle$ and $J' >_{\mathcal{J}} J$ implies $J' \subseteq \langle E(S) \rangle$, then $J \cap \langle E(S) \rangle = \emptyset$.*

Proof. Suppose that there exists $t \in J \cap \langle E(S) \rangle$. If $s \in J$, then $s = xty$, for some $x, y \in S^I$. As J is null, we must have x, y strictly \mathcal{J} -above J . Thus $x, y \in \langle E(S) \rangle^I$, which implies that $s \in \langle E(S) \rangle$, and therefore $J \subseteq \langle E(S) \rangle$, a contradiction. □

Lemma 3.3. *If $S \in \mathbf{EA}$ is \mathbf{H} -solvable, then the subgroups of S are \mathbf{H} -solvable.*

Proof. Let G be a subgroup of S . Then $K_{\mathbf{H}}^{(\omega)}(G) \subseteq K_{\mathbf{H}}^{(\omega)}(S) = \langle E(S) \rangle$. As $S \in \mathbf{EA}$, the semigroup $\langle E(S) \rangle$ is aperiodic and so its subgroup $K_{\mathbf{H}}^{(\omega)}(G)$ must be trivial, as required. \square

In the sequel we shall make extensive use of the following theorem of Graham [8].

Theorem 3.4. *Let S be a semigroup and let J be a regular \mathcal{J} -class of S . Then there exists an isomorphism (i.e. a Rees coordinatization)*

$$\iota : J^0 \rightarrow \mathcal{M}^0(G, A, B, C)$$

from J^0 to a Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$ such that:

- A and B can be partitioned into subsets A_1, \dots, A_n and B_1, \dots, B_n , respectively.
- The matrix C is the direct sum of the n blocks C_1, \dots, C_n , as suggested by the following picture (all the entries outside these blocks are zeros):

$$\begin{array}{c}
 A_1 \quad A_2 \quad \dots \quad A_n \\
 \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_n \end{array} \left(\begin{array}{cccc}
 \boxed{C_1} & & & \mathbf{0} \\
 & \boxed{C_2} & & \\
 & & \ddots & \\
 \mathbf{0} & & & \boxed{C_n}
 \end{array} \right)
 \end{array}$$

- Each matrix $C_i : B_i \times A_i \rightarrow G^0$ is regular and

$$\langle E(J^0) \rangle = \cup_{i=1}^n \mathcal{M}^0(G_i, A_i, B_i, C_i),$$

where G_i is the subgroup of G generated by all non-zero entries of C_i , for $i = 1, \dots, n$.

A Rees coordinatization satisfying the properties described in the above theorem is called a *Graham normalization* of J and can be computed in polynomial time from the multiplication table for J^0 [8].

Lemma 3.1 and Theorem 3.4 then combine to give the following result characterizing membership in \mathbf{EA} .

Corollary 3.5. *A semigroup S belongs to \mathbf{EA} if and only if, for each regular \mathcal{J} -class J , there exists a Rees coordinatization such that all the matrix entries are 0 or 1 if and only if, for each Graham normalization of such a \mathcal{J} -class J , all the entries of the matrix are 0 or 1.*

Now, we are in a position to establish our main result.

Theorem 3.6. *Let \mathbf{H} be a non-trivial pseudovariety of groups. Then every semigroup in $\mathbf{EA} \cap \overline{\mathbf{WH}}$ is \mathbf{H} -solvable.*

Proof. Let $S \in \mathbf{EA} \cap \overline{\mathbf{WH}}$. The proof is by induction on $|S \setminus \langle E(S) \rangle|$. Clearly, the statement is true when $S = \langle E(S) \rangle$. Suppose now that $S \neq \langle E(S) \rangle$. Then, it suffices to construct a relational morphism $\tau : S \dashrightarrow G \in \mathbf{H}$ such that $|\tau^{-1}(1)| < |S|$. Indeed, this would imply

$$|\mathbf{K}_{\mathbf{H}}(S)| \leq |\tau^{-1}(1)| < |S|$$

and so

$$|\mathbf{K}_{\mathbf{H}}(S) \setminus \langle E(\mathbf{K}_{\mathbf{H}}(S)) \rangle| < |S \setminus \langle E(\mathbf{K}_{\mathbf{H}}(S)) \rangle| = |S \setminus \langle E(S) \rangle|.$$

Hence, by induction,

$$\mathbf{K}_{\mathbf{H}}^{(\omega)}(\mathbf{K}_{\mathbf{H}}(S)) = \langle E(\mathbf{K}_{\mathbf{H}}(S)) \rangle,$$

that is,

$$\mathbf{K}_{\mathbf{H}}^{(\omega)}(S) = \langle E(S) \rangle.$$

Let J be a \mathcal{J} -class of S which is $\leq_{\mathcal{J}}$ -maximal amongst the \mathcal{J} -classes of S not contained in $\langle E(S) \rangle$. To prove our result it suffices to construct a relational morphism $\tau : S \dashrightarrow H$ such that $|\tau^{-1}(1) \cap J| < |J|$. Consider the subsets $T(J) = \{s \in S \mid J <_{\mathcal{J}} J_s\}$ and $B(J) = S \setminus T(J)$ of S . Figure 1 may help the reader to visualize the situation. Note that both $B(J)$ and $B(J) \setminus J$ are ideals of S . Two cases may arise: J may be regular or null.

If J is null, then by Lemma 3.2, $J \cap \langle E(S) \rangle = \emptyset$. It is then straightforward to verify that, for any non-trivial group $H \in \mathbf{H}$ and any fixed element $g \in H \setminus \{1\}$, the relation $\tau : S \dashrightarrow H$ defined by

$$\tau(x) = \begin{cases} \{1\} & \text{if } x \in T(J) \\ \{g\} & \text{if } x \in J \\ H & \text{otherwise (i.e. if } x \in B(J) \setminus J), \end{cases}$$

for all $x \in S$, is a relational morphism such that $0 = |\tau^{-1}(1) \cap J| < |J|$.

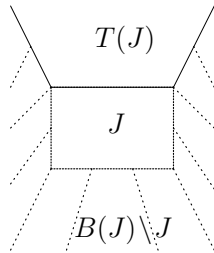


Fig. 1.

Thus, it remains to consider the case where J is regular. Let us consider a Graham normalization $\iota : J^0 \rightarrow \mathcal{M}^0(G, A, B, C)$ of J and suppose that C is the direct sum of blocks C_1, \dots, C_n , as in Graham's theorem. Without loss of generality we take A and B to be disjoint. Let $N = \mathbf{K}_{\mathbf{H}}(G)$. Since \mathbf{H} is non-trivial, there is a prime number p such that the cyclic group \mathbb{Z}_p belongs to \mathbf{H} . We may then consider the vector space \mathbb{Z}_p^n over \mathbb{Z}_p with base $\{v_1, \dots, v_n\}$. The construction of the required relational morphism is divided into several steps.

Step 1. First, we construct a partial homomorphism $\varphi : J \rightarrow G/N \times \mathbb{Z}_p^n$. Recall that φ is a partial homomorphism if

$$s, t, st \in J \Rightarrow \varphi(s)\varphi(t) = \varphi(st).$$

To this effect we consider the auxiliary function $\mathbf{bl} : A \cup B \rightarrow \{v_1, \dots, v_n\}$ defined by $\mathbf{bl}(x) = v_i$, if $x \in A_i \cup B_i$. Then, we define φ by $\varphi(a, g, b) = (Ng, \mathbf{bl}(a) - \mathbf{bl}(b))$, for all $(a, g, b) \in J$. Next, we prove that φ is, indeed, a partial homomorphism. First, we observe that $(a, g, b)(a', g', b') \in J$ if and only if $C(b, a') \neq 0$ (i.e. $C(b, a') = 1$, by Corollary 3.5). Since C is the direct sum of the C_i , from $C(b, a') = 1$ it follows that $\mathbf{bl}(b) = \mathbf{bl}(a')$ and $(a, gg', b') = (a, g, b)(a', g', b')$. Thus

$$\begin{aligned} \varphi(a, g, b)\varphi(a', g', b') &= (Ng, \mathbf{bl}(a) - \mathbf{bl}(b))(Ng', \mathbf{bl}(a') - \mathbf{bl}(b')) \\ &= (Ngg', \mathbf{bl}(a) - \mathbf{bl}(b) + \mathbf{bl}(a') - \mathbf{bl}(b')) \\ &= (Ngg', \mathbf{bl}(a) - \mathbf{bl}(b')) \\ &= \varphi(a, gg', b') \\ &= \varphi((a, g, b)(a', g', b')). \end{aligned}$$

Note that, for an idempotent e , we have

$$\varphi(e) = \varphi(ee) = \varphi(e)\varphi(e),$$

whence $\varphi(e) = 1 = (N, 0)$. It follows that if $s \in \langle E(J) \rangle \cap J$ then $\varphi(s) = (N, 0)$.

Step 2. Next, we extend φ to a relational morphism $\tau : B(J) \dashrightarrow G/N \times \mathbb{Z}_p^n$ by defining $\tau(s) = G/N \times \mathbb{Z}_p^n$, for all $s \in B(J) \setminus J$. Since $B(J) \setminus J$ is an ideal, τ is clearly a relational morphism.

Step 3. Finally, we extend τ to a relational morphism $\tilde{\tau} : S \dashrightarrow G/N \times \mathbb{Z}_p^n$ by defining $\tilde{\tau}(s) = \{1\}$, for all $s \in T(J)$.

In order to verify that $\tilde{\tau}$ is in fact a relational morphism, choose $s, t \in S$. Let us prove that $\tilde{\tau}(s)\tilde{\tau}(t) \subseteq \tilde{\tau}(st)$. Since τ is a relational morphism and $B(J) \setminus J$ is an ideal, the only non-immediate cases are: $s, t \in T(J)$; $s \in T(J)$ and $t \in J$; and $t \in T(J)$ and $s \in J$. First, let $s, t \in T(J)$. Then $\tilde{\tau}(s)\tilde{\tau}(t) = \{1\}$. If $st \in T(J)$ or $st \in B(J) \setminus J$, then clearly $1 \in \tilde{\tau}(st)$. On the other hand, noting that $s, t \in \langle E(S) \rangle$, by the maximality

of J , we conclude that if $st \in J$, then $st \in \langle E(S) \rangle \cap J = \langle E(J) \rangle \cap J$, by Lemma 3.1. Thus $\tilde{\tau}(st) = \varphi(st) = 1$ and, in both cases, we have $\tilde{\tau}(s)\tilde{\tau}(t) \subseteq \tilde{\tau}(st)$.

Next, consider $s \in T(J)$ and $t \in J$. Then $st \in B(J) \setminus J$ or $st \in J$. In the first case, it is obvious that $\tilde{\tau}(s)\tilde{\tau}(t) \subseteq \tilde{\tau}(st)$. So, suppose that $st \in J$. Then, we will prove that $\varphi(st) = \varphi(t)$. Let t' be an inverse of t . Then

$$st = stt't = (stt')t.$$

Since $s \in \langle E(S) \rangle$ and $tt' \in E(S)$, we have

$$stt' \in \langle E(S) \rangle \cap J = \langle E(J) \rangle \cap J,$$

by Lemma 3.1. Thus $\varphi(stt') = 1$. Therefore, $\varphi(st) = \varphi(stt')\varphi(t) = \varphi(t)$.

Finally, the case $t \in T(J)$ and $s \in J$ is dual to the previous one and so $\tilde{\tau}$ is, indeed, a relational morphism.

Now, one immediately sees from the definition that

$$\tilde{\tau}^{-1}(N, 0) \cap J = \{(a, g, b) \mid \text{bl}(a) = \text{bl}(b) \text{ and } g \in N\}$$

and so we must consider two cases. First, if J has more than one block, then

$$|\tilde{\tau}^{-1}(N, 0) \cap J| < |J|,$$

as required. On the other hand, if J has only one block, then as $J \not\subseteq \langle E(S) \rangle$, by Graham's theorem, we must have that G is non-trivial. Therefore, since $G \in \mathcal{WH}$ (whence $\mathcal{K}_{\mathbf{H}}^{(\omega)}(G) = 1$), we may conclude that $N = \mathcal{K}_{\mathbf{H}}(G)$ is a proper subgroup and so $\tilde{\tau}^{-1}(N, 0) \cap J$ is again strictly contained in J , completing the proof. \square

We remark that this proof is effective (assuming the decidability of \mathbf{H}) in the sense that we have explicitly constructed relational morphisms to groups in \mathbf{H} witnessing that $\mathcal{K}_{\mathbf{H}}^{(\omega)}(S) = \langle E(S) \rangle$. Moreover, no computability hypotheses on $\mathcal{K}_{\mathbf{H}}$ were necessary.

Lemma 3.3, Theorem 3.6 and the fact that $\mathbf{V} \omega_{\mathcal{M}} \mathbf{H} \subseteq \mathbf{EV}$ (see [5, 10]) imply the following result:

Corollary 3.7. *Let $\mathbf{V} \subseteq \mathbf{A}$ and $1 \subsetneq \mathbf{H} \subseteq \mathbf{G}$. Then*

$$\mathbf{EV} \cap \overline{\mathcal{WH}} = \mathbf{V} \omega_{\mathcal{M}} \mathbf{H}.$$

In particular, for $\mathbf{V} = \mathbf{SI}$ and $\mathbf{H} = \mathbf{Ab}$, we recover the main result from [5], where \mathbf{G}_{sol} denotes the pseudovariety of all solvable finite groups.

Corollary 3.8. $\mathbf{ESl} \cap \overline{\mathbf{G}_{\text{sol}}} = \mathbf{Sl} \omega_{\mathcal{M}} \mathbf{Ab} = (\dots((\mathbf{Sl} \omega_{\mathcal{M}} \mathbf{Ab}) \omega_{\mathcal{M}} \mathbf{Ab}) \dots) \omega_{\mathcal{M}} \mathbf{Ab}.$

Notice if one shifts to the right all the parentheses in the above corollary one obtains the strictly smaller pseudovariety $\mathbf{Sl} \omega_{\mathcal{M}} \mathbf{G}_{\text{sol}}$ [11]. However it is still open whether an inverse semigroup belongs to $\mathbf{Sl} \omega_{\mathcal{M}} \mathbf{G}_{\text{sol}}$ if and only if its subgroups are solvable.

For $\mathbf{V} = \mathbf{A}$ and $\mathbf{H} = \mathbf{G}$, we obtain:

Corollary 3.9. $\mathbf{EA} = \mathbf{A}^\omega \overset{\circ}{\circlearrowleft} \mathbf{G}$.

Notice that, if $S \notin \mathbf{EA}$, then we cannot in general guarantee that S \mathbf{H} -solvable implies S belongs to $\overline{\mathbf{WH}}$. In fact, for any positive integer n , the semigroup $\langle E(T_{n+1}) \rangle$, where T_{n+1} is the full transformation semigroup on $n + 1$ letters, is \mathbf{H} -solvable, but it contains the symmetric group S_n . It follows that unless $\mathbf{H} = \mathbf{G}$, there exists \mathbf{H} -solvable semigroups whose subgroups do not belong to $\overline{\mathbf{WH}}$.

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