COMBINATORIAL TOPOLOGY AND THE GLOBAL DIMENSION OF ALGEBRAS ARISING IN COMBINATORICS

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ABSTRACT. In a highly influential paper, Bidigare, Hanlon and Rockmore showed that a number of popular Markov chains are random walks on the faces of a hyperplane arrangement. Their analysis of these Markov chains took advantage of the monoid structure on the set of faces. This theory was later extended by Brown to a larger class of monoids called left regular bands. In both cases, the representation theory of these monoids played a prominent role. In particular, it was used to compute the spectrum of the transition operators of the Markov chains and to prove diagonalizability of the transition operators.

In this paper, we establish a close connection between algebraic and combinatorial invariants of a left regular band: we show that certain homological invariants of the algebra of a left regular band coincide with the cohomology of order complexes of posets naturally associated to the left regular band. For instance, we show that the global dimension of these algebras is bounded above by the Leray number of the associated order complex. Conversely, we associate to every flag complex a left regular band whose algebra has global dimension precisely the Leray number of the flag complex.

1. INTRODUCTION

In a highly influential paper [11], Bidigare, Hanlon and Rockmore showed that a number of popular Markov chains, including the Tsetlin library and the riffle shuffle, are random walks on the faces of a hyperplane arrangement (the braid arrangement for these two examples). More importantly, they showed that the representation theory of the monoid of faces, where the monoid structure on the faces of a central hyperplane arrangement is given

by the Tits projections [89], could be used to analyze these Markov chains and, in particular, to compute the spectrum of their transition operators.

Using the topology of arrangements, Brown and Diaconis [25] found resolutions of the simple modules for the face monoid that were later shown by the second author to be the minimal projective resolutions [80]. Brown and Diaconis used these resolutions to prove diagonalizability of the transition operator. Bounds on rates of convergence to stationarity were obtained in [11, 25]. They observed, moreover, that one can replace the faces of a hyperplane arrangement by the covectors of an oriented matroid [16] and the theory carries through. We remark that the original version of Brown's book on buildings [20] makes no mention of the face monoid of a hyperplane arrangement, whereas it plays a prominent role in the new edition [1]. Hyperplane face monoids also have a salient position in the work of Aguiar and Mahajan on combinatorial Hopf algebras [3, 4].

The representation theory of hyperplane face monoids is closely connected to Solomon's descent algebra [84]. Bidigare showed in his thesis [12] (see also [24]) that if W is a finite Coxeter group and $\mathcal{A}_W$ is the associated reflection arrangement, then the descent algebra of W is the algebra of invariants for the action of W on the algebra of the face monoid of $\mathcal{A}_W$. This, together with his study of the representation theory of hyperplane face monoids [80], allowed the second author to compute the quiver of the descent algebra in types A and B [79] (see also [82]).

The face monoid of a hyperplane arrangement satisfies the identities $x^2 = x$ and $xyx = xy$. A semigroup satisfying these identities is known in the literature as a left regular band. Brown developed [23,24] a theory of random walks on finite left regular bands. He gave numerous examples that do not come from hyperplane arrangements, as well as examples of hyperplane walks that could more easily be modeled on simpler left regular bands. For example, Brown considered random walks on bases of matroids. Brown used the representation theory of left regular bands to extend the spectral results of Bidigare, Hanlon and Rockmore [11] and gave an algebraic proof of the diagonalizability of random walks on left regular bands.

Brown's theory has since been used and further developed by numerous authors. Diaconis highlighted hyperplane face monoid and left regular band walks in his 1998 ICM lecture [35]. Björner used it to develop the theory of random walks on complex hyperplane arrangements and interval greedoids [14, 15]. Athanasiadis and Diaconis revisited random walks on hyperplane face monoids and left regular bands in [6]. Chung and Graham considered further left regular band random walks associated to graphs in [29]. Saliola and Thomas proposed a definition of oriented interval greedoids by generalizing the left regular bands associated to oriented matroids and antimatroids [81]. See also the recent work of Reiner, Saliola and Welker on symmetrized random walks on hyperplane face monoids [74]. Left regular bands have also appeared in Lawvere's work in topos theory [55, 56].

Left regular bands have directed quasi-hereditary algebras and hence have acyclic quivers and finite global dimension. The second author computed in [80] the Ext-spaces between simple modules in the case of the algebras of face monoids of hyperplane arrangements using the resolutions of Brown and Diaconis coming from the topology of hyperplane arrangements. Consequently, he computed the global dimension of these algebras. In [78] the second author computed the projective indecomposable modules for arbitrary left regular band algebras and also the quiver. In this setting, one didn't seem to have any topology available to compute the minimal resolutions and so he was unable to compute Ext-spaces between simple modules.

The paper [78] also contained an intriguing unpublished result of Ken Brown stating that the algebra of a free left regular band is hereditary. The proof is via a computation of the quiver and amounts to proving that the dimension of the path algebra is the cardinality of the free left regular band. The right Cayley graph of a free left regular band is a tree (after removing loop edges) and this leads one to suspect that there is a topological explanation for the fact that its algebra has global dimension one. This paper arose in part to give a conceptual explanation to this result of Brown.

There seem to be only a handful of results in the finite dimensional algebra literature that use topological techniques to compute homological invariants of algebras. The primary examples seem to be in the setting of incidence algebras where the order complex of the poset plays a key role [30,44,49]. A more general setting is considered in [26]. In this paper we use topological techniques to compute the Ext-spaces between simple modules of the algebra of a left regular band. In particular, we use order complexes of posets and classifying spaces of small categories (in the sense of Segal [83]) to achieve this. A fundamental role is played by the celebrated Quillen's Theorem A, which gives a sufficient condition for a functor between categories to induce a homotopy equivalence of classifying spaces. Somewhat surprisingly to us, a combinatorial invariant of simplicial complexes, the Leray number [50,51], plays an important part in this paper. The Leray number is tied to the Castelnuovo-Mumford regularity of Stanley-Reisner rings. In particular, the paper gives a new, non-commutative interpretation of the regularity of the Stanley-Reisner ring of a flag complex.

Let us give a more technical overview of the paper. We will assume that all left regular bands are finite. Our goal is to study the algebra $\Bbbk B$ of a left regular band B over a commutative ring with unit $\Bbbk$. The reader should feel free to assume that $\Bbbk$ is a field if he/she likes. The principal goal is to compute $\mathrm{Ext}^n_{\Bbbk B}(\Bbbk_X, \Bbbk_Y)$ for all $n \geq 0$, where $\Bbbk_X$ and $\Bbbk_Y$ are certain $\Bbbk B$-modules; in the case that $\Bbbk$ is a field, these are the simple $\Bbbk B$-modules. Our main result identifies these Ext-spaces with the cohomology of order complexes of posets naturally associated to the left regular band. This establishes a close connection between these algebraic invariants and the combinatorics of these order complexes. For instance, we show that the global dimension of $\Bbbk B$ is bounded above by the Leray number [50,51] of the

associated order complex. Conversely, we associate to every flag complex K a left regular band whose algebra has global dimension precisely the Leray number of K.

The article is outlined as follows. In Section 2 we recall definitions and properties of left regular bands and related constructions. Section 3 surveys several examples of left regular bands. We illustrate how some of the left regular bands that have appeared in the literature are special cases of classical semigroup-theoretic constructions. We also introduce some new examples: *free partially commutative left regular bands*, which are analogues of trace monoids and right-angled Artin groups [10]; *geometric left regular bands*, which include nearly all the left regular bands that have appeared in the algebraic combinatorics literature; and the *left regular band of an acyclic quiver* whose semigroup algebra is the path algebra of the quiver.

Since the proof of our main theorem is rather involved, we decided to discuss its applications before presenting its proof. So Section 4 is devoted to applications of the main theorem. We begin with a new description of the quiver of a left regular band algebra. We show that the algebra's global dimension is bounded above by the Leray number of the order complex of the left regular band. This leads to a characterization of the free partially commutative left regular bands with a hereditary algebra as those constructed from chordal graphs.

The proof of the main theorem is split across two sections. Section 5 reviews the second author's construction of a complete set of orthogonal idempotents, which are then used to identify the Schützenberger representations of the left regular band as projective modules (indecomposable over a field). We construct projective resolutions of the modules $\Bbbk_X$, which recasts the computation of $\mathrm{Ext}_{\Bbbk B}^n(\Bbbk_X, \Bbbk_Y)$ into one involving monoid cohomology and classifying spaces.

Section 6 contains the crux of the proof. Our main tools are classifying spaces and the cohomology of monoids and small categories. Although we are mostly interested in monoid cohomology, which is a natural generalization of group cohomology, we will also need to work with categories that are not monoids; namely, posets and the semidirect product of a monoid with a set (also known as the Grothendieck construction, or category of elements).

2. Left Regular Bands

2.1. **Bands and left regular bands.** A *band* is a monoid B all of whose elements are idempotents. A particularly important class of bands arising in probability theory and in algebraic combinatorics is the class of left regular bands [3, 14, 15, 23–25, 29, 78, 80].

Definition 2.1. A *band* is a monoid B satisfying the identity

$$x^2 = x \text{ for all } x \in B. \tag{2.1}$$

A band is *left regular* if it satisfies the identity

$$xyx = xy \text{ for all } x, y \in B. \tag{2.2}$$

The class of left regular bands is hence a variety of bands. It is known (cf. [76, Proposition 7.3.2]) to be generated by the band $\{0, +, -\}$ where 0 is the identity element and the binary operation $\circ$ is given by

$$+ \circ + = + \circ - = + \quad \text{and} \quad - \circ + = - \circ - = -$$

Important examples of left regular bands arising in combinatorics are real and complex hyperplane face semigroups, oriented matroids, matroids and interval greedoids. Other interesting examples will be seen in Section 3.

It is easy to see that a band B is left regular if and only if each left ideal is two-sided. Indeed, if B is left regular and L is a left ideal, then $a \in L$ implies $ab = aba \in L$ for all $b \in B$. Conversely, if each two-sided ideal is a left ideal, then $ab \in Ba$ and so $aba = ab$ for all $a, b \in B$.

2.2. **Support lattice and the support map.** If B is a band, then it was shown by Clifford [31, 32] that the principal ideals are closed under intersection and hence form a (meet) semilattice $\Lambda(B)$ with maximum. More precisely, he proved that $BaB \cap BbB = BabB$ and hence $\sigma \colon B \longrightarrow \Lambda(B)$ given by $\sigma(a) = BaB$ is a monoid homomorphism. In fact, σ is the universal map from B to a semilattice.

If B is a finite left regular band, then $\Lambda(B)$ is the set of principal left ideals, which is a lattice under inclusion with intersection as the meet. Following the standard convention of lattice theory, we denote by $\widehat{1}$ the top of $\Lambda(B)$ (which is B, itself) and by $\widehat{0}$ the bottom (called the *minimal ideal* of B). Brown calls $\Lambda(B)$ the *support lattice* of B [23, 24] (actually, he uses the opposite ordering). The map $\sigma \colon B \longrightarrow \Lambda(B)$, above, becomes $\sigma(a) = Ba$ and is called the *support map*. It is possible to give a definition of left regular bands in terms of the support map; see e.g. [23, Appendix B] for a proof of the following.

Proposition 2.2. *A finite monoid M is a* left regular band *if and only if there exist a lattice Λ and a surjection $\sigma \colon M \longrightarrow \Lambda$ satisfying the following properties for all $x, y \in M$:*

$$\sigma(xy) = \sigma(x) \wedge \sigma(y) \tag{2.3}$$

$$xy = x \text{ if and only if } \sigma(y) \geq \sigma(x) \tag{2.4}$$

where $\wedge$ denotes the meet operation (greatest lower bound) of the lattice Λ.

In semigroup parlance, this is the well known fact that a left regular band is the same thing as a semilattice of left zero semigroups.

2.3. **Green's $\mathscr{R}$-order.** Let M be a monoid. *Green's $\mathscr{R}$-preorder* is defined on M by $m \leq_{\mathscr{R}} n$ if $mM \subseteq nM$. The associated equivalence relation is denoted $\mathscr{R}$ and is one of *Green's relations* on a monoid. See [32, 46].

If B is a band, one has $aB \subseteq bB$ if and only if $ba = a$. Hence,

$$a \leq_{\mathscr{R}} b \text{ if and only if } ba = a.$$

In a left regular band, if $ba = a$ and $ab = b$, then $a = aba = ab = b$. It follows that a left regular band B is partially ordered with respect to $\leq_{\mathscr{R}}$. (In fact a band is left regular if and only if $\leq_{\mathscr{R}}$ is a partial order.) We call this partial order the $\mathscr{R}$-order on B and denote it simply by $\leq$. Note that the support map $\sigma \colon B \longrightarrow \Lambda(B)$ is order-preserving: that is, if $a \leq b$, then $\sigma(a) \leq \sigma(b)$. Figures 1, 3 and 4 illustrate the $\mathscr{R}$-order on three examples.

The following is a special case of an elementary result of Rhodes [75].

Lemma 2.3. *Let B be a left regular band with support map $\sigma \colon B \longrightarrow \Lambda(B)$. Then we have the following.*

   (1) *If $b_0 < b_1 < \cdots < b_n$ in B, then $\sigma(b_0) < \sigma(b_1) < \cdots < \sigma(b_n)$.*
   (2) *If $X_0 < X_1 < \cdots < X_n$ is a chain in $\Lambda(B)$, then there is a chain $b_0 < b_1 < \cdots < b_n$ in B with $\sigma(b_i) = X_i$ for all $0 \leq i \leq n$.*

Proof. For the first statement, it suffices to observe that if $a \leq b$ and $\sigma(a) = \sigma(b)$, then $a = ba = b$. For the second statement, choose a_i with $\sigma(a_i) = X_i$, for $0 \leq i \leq n$, and define $b_i = a_n a_{n-1} \cdots a_i$, for $0 \leq i \leq n$. $\square$

2.4. Local, induced and interval submonoids. If $X \in \Lambda(B)$, then

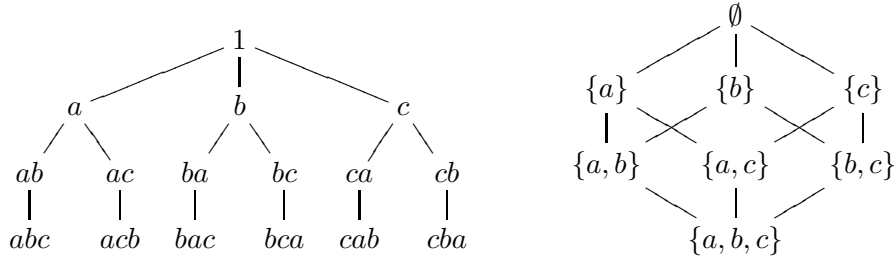$$B_{\geq X} = \{b \in B \mid \sigma(b) \geq X\}$$

is a submonoid of B. The set $B_{\ngeq X} = B \setminus B_{\geq X}$ is a prime ideal of B and all prime ideals of B are obtained in this way. (Recall that an ideal P is *prime* if $ab \in P$ implies $a \in P$ or $b \in P$.)

Note that

$$a^{\uparrow} = \{b \in B \mid b \geq a\} = \{b \in B \mid ba = a\}$$

is a submonoid of B, in fact, it is the left stabilizer of a. Notice that if $a \in B$, then $aB = aBa$ is a left regular band with identity a and the map $b \mapsto ab$ gives a retraction $\tau_a \colon B \longrightarrow aB$. The monoid aB is called the *local submonoid* of B at a. One has $\Lambda(aB) = \Lambda(B)_{\leq \sigma(a)}$, the principal downset of $\Lambda(B)$ generated by $\sigma(a)$. If $\sigma(a) = X = \sigma(b)$, then $aB \cong bB$ via the restriction of τ_a to bB and the restriction of τ_b to aB. The corresponding left regular band (well defined up to isomorphism) will be denoted $B[X]$ and called the *induced submonoid on X*.

If $X \leq Y$ in $\Lambda(B)$, then $B_{\geq X}[Y] \cong B[Y]_{\geq X}$ and this left regular band will be denoted $B[X, Y]$. It has support lattice the interval $[X, Y]$ of $\Lambda(B)$. Hence we shall call $B[X, Y]$ the *interval submonoid* of B associated to $[X, Y]$. Of course, $B[\widehat{0}, X] = B[X]$ and $B[X, \widehat{1}] = B_{\geq X}$. It will be convenient to denote by $B[X, Y)$ the ideal of $B[X, Y]$ obtained by removing the identity.

FIGURE 1. The $\mathscr{R}$-order and the support lattice of $F(\{a, b, c\})$.

3. Examples of Left Regular Bands

This section surveys some examples of left regular bands. We illustrate how several of the examples of left regular bands found in the combinatorics literature are special cases of certain semigroup-theoretic constructions; and we introduce a new class of examples: *free partially commutative left regular bands*.

3.1. **Free left regular bands.** The *free left regular band* on a set A is denoted $F(A)$ and the free left regular band on n-generators will be written F_n. The word problem for $F(A)$ is quite elegant. Let A^* denote the free monoid on A. One can view $F(A)$ as consisting of all injective words over A, i.e., those elements of A^* with no repeated letters. Multiplication is given by concatenation followed by removal of repetitions (reading from left to right). In particular, if A is finite, then so is $F(A)$. Hence any finitely generated left regular band is finite (actually any finitely generated band is finite). The support lattice of $F(A)$ can be identified with the power set $P(A)$ with the operation of union. The support map σ takes an injective word to its content (or alphabet). If $C \subsetneq B \subseteq A$, then $F(A)[B, C] \cong F(B \setminus C)$.

3.2. **Hyperplane face monoids and oriented matroids.** The set of faces of a hyperplane arrangement, and more generally the set of covectors of an oriented matroid, is endowed with a natural associative product providing an important source of examples of left regular bands. These turn out to be submonoids of $\{0, +, -\}^n$, where $\{0, +, -\}$ is as defined in Section 2.1.

We recall the construction and properties of these left regular bands referring the reader to [23, Appendix A] for details. A *central hyperplane arrangement* in $V = \mathbb{R}^n$ is a finite collection $\mathcal{A}$ of hyperplanes of V passing through the origin. For each hyperplane $H \in \mathcal{A}$, fix a labelling H^+ and H^- of the two open half spaces of V determined by H; the choice of labels H^+ and H^- is arbitrary, but fixed throughout. For convenience, let $H^0 = H$.

A *face* x of $\mathcal{A}$ is a non-empty intersection of the form $x = \bigcap_{H \in \mathcal{A}} H^{\epsilon_H}$ with $\epsilon_H \in \{0, +, -\}$. Consequently, for every hyperplane $H \in \mathcal{A}$, a face of $\mathcal{A}$ is contained in exactly one of H^+, H^- or H^0. If y is a face of $\mathcal{A}$ and

$H \in \mathcal{A}$, let $\varepsilon_H(y) \in \{0, +, -\}$ be such that $y \subseteq H^{\varepsilon_H(y)}$. The sequence $\varepsilon(y) = (\varepsilon_H(y))_{H \in \mathcal{A}}$ is called the *sign sequence* of y and it completely determines y.
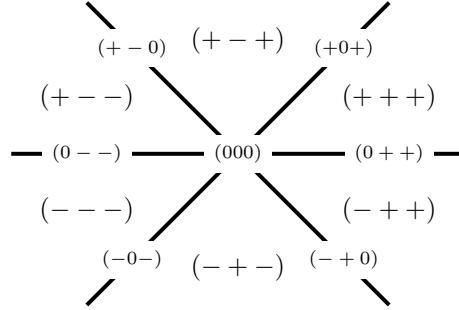

FIGURE 2. The sign sequences of the faces of the hyperplane arrangement in $\mathbb{R}^2$ consisting of three distinct lines.
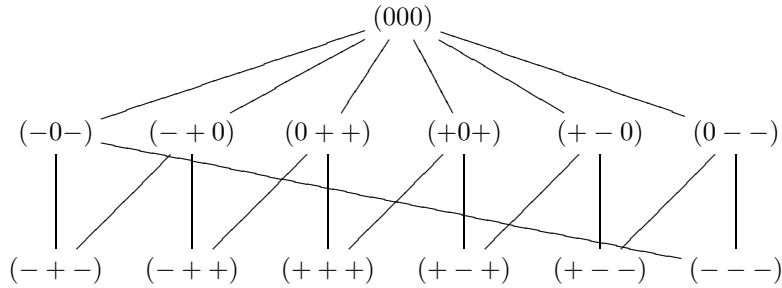

FIGURE 3. The $\mathscr{R}$-order on the face monoid of Figure 2.

Let $\mathcal{F}$ denote the set of faces of $\mathcal{A}$. The image of ε identifies $\mathcal{F}$ with a submonoid of $\{0, +, -\}^{\mathcal{A}}$ and so we obtain a monoid structure on $\mathcal{F}$ by defining the product of $x, y \in \mathcal{F}$ to be the face with sign sequence $\varepsilon(x) \circ \varepsilon(y)$. In other words, xy is defined by the property that xy lies: on the same side of H as x if $x \not\subseteq H$; on the same side of H as y if $x \subseteq H$, but $y \not\subseteq H$; and inside H if $x, y \subseteq H$. This product admits an alternative geometric description: xy is the unique face—possibly x itself—containing the point obtained by moving a small positive distance along a straight line from a point in x toward a point in y.

The left regular band $\mathcal{F}$ is called the *face monoid* of $\mathcal{A}$. The lattice $\Lambda(\mathcal{F})$ of principal left ideals of $\mathcal{F}$ is isomorphic to the *intersection lattice* $\mathcal{L}$ of $\mathcal{A}$; it is the set of subspaces of V that can be expressed as an intersection of hyperplanes from $\mathcal{A}$ ordered by reverse inclusion. Under this isomorphism, the universal map $\sigma : \mathcal{F} \longrightarrow \Lambda(\mathcal{F})$ corresponds to the map from $\mathcal{F}$ to $\mathcal{L}$ that sends a face x to the smallest subspace $\bigcap_{\{H \in \mathcal{A} \mid x \subseteq H\}} H \in \mathcal{L}$ that contains x. Observe that for $X < Y$ in $\mathcal{L}$, $\mathcal{F}[X, Y]$ is the face monoid of the hyperplane

arrangement in X obtained by intersecting X with the hyperplanes $H \in \mathcal{A}$ containing Y but not X.

3.2.1. *Oriented matroids.* An *oriented matroid* $\mathscr{X}$ is an abstraction of the properties enjoyed by a configuration of vectors in a vector space, or what amounts to the same thing by working instead with their orthogonal complements, a hyperplane arrangement. They are also submonoids of the left regular band $\{0, +, -\}^n$ [16, §4.1], but not all submonoids of $\{0, +, -\}^n$ are oriented matroids (*cf.* §3.7). Much of the monoid structure of $\mathscr{X}$, as well as the structure of its monoid algebra, parallels the theory for the face monoid of a hyperplane arrangement. See [25, §6] and [80, §11] for details.

3.3. **Complex hyperplane arrangements.** In this section we describe a left regular band associated to a complex hyperplane arrangement following [14] and [17]. All unproved assertions can be found in these references.

Define a left regular band structure on $\mathcal{S} = \{0, +, -, i, j\}$ via the multiplication table in Figure 4. The Hasse diagram of the $\mathscr{R}$-order of $\mathcal{S}$ is also depicted in Figure 4.

	0	+	−	i	j
0	0	+	−	i	j
+	+	+	+	i	j
−	−	−	−	i	j
i	i	i	i	i	i
j	j	j	j	j	j

FIGURE 4. The multiplication table and $\mathscr{R}$-order of $\mathcal{S}$.

Define a function $\mathbf{s} \colon \mathbb{C} \longrightarrow \mathcal{S}$ by

$$\mathbf{s}(x + iy) = \begin{cases} i, & \text{if } y > 0, \\ j, & \text{if } y < 0, \\ +, & \text{if } y = 0, x > 0, \\ -, & \text{if } y = 0, x < 0, \\ 0, & \text{if } x = 0 = y. \end{cases}$$

A *complex hyperplane arrangement* is a set $\mathcal{A} = \{H_1, \ldots, H_n\}$ where H_i is the zero set of a complex linear form f_i on $\mathbb{C}^d$, for $1 \leq i \leq n$. We always assume that $H_1 \cap \cdots \cap H_n = \{0\}$. The position of a point $z \in \mathbb{C}^d$ relative to $\mathcal{A}$ can be described by the map $\tau \colon \mathbb{C}^d \longrightarrow \mathcal{S}^n$ given by

$$\tau(z_1, \ldots, z_d) = (\mathbf{s}(f_1(z_1)), \ldots, \mathbf{s}(f_d(z_d))).$$

The image $\mathcal{F} = \tau(\mathbb{C}^d)$ is a submonoid of $\mathcal{S}^n$. Moreover, for each $F \in \mathcal{F}$, one has that $\tau^{-1}(F)$ is a relative-open convex cone. The intersections $\tau^{-1}(F) \cap S^{2d-1}$ are the open cells of a regular CW-decomposition of S^{2d-1}

and the face poset of this decomposition is the opposite of the $\mathscr{R}$-order on $\mathcal{F}$ (where the identity corresponds to the empty face). See [17, Theorem 2.5] for details. For this reason, elements of $\mathcal{F}$ will be called *faces* and we will call $\mathcal{F}$ the *face monoid* of $\mathcal{A}$. The minimal ideal of $\mathcal{F}$ consists of all elements of $\mathcal{F} \cap \{i, j\}^n$, cf. [14, Proposition 3.1]. It is shown in [17, Theorem 3.5] that the $\mathscr{R}$-order on the ideal $\mathcal{F} \cap (\mathcal{S} \setminus \{0\})^n$ of $\mathcal{F}$ is the face poset of a regular CW complex that is homotopy equivalent to the complement $\mathbb{C}^d \setminus (H_1 \cup \cdots \cup H_n)$.

The *augmented intersection lattice* $\mathcal{L}$ of $\mathcal{A}$ is the collection of all intersections of elements of

$$\mathcal{A}_{\mathrm{aug}} = \{H_1, \ldots, H_n, H_1^{\mathbb{R}}, \ldots, H_n^{\mathbb{R}}\}$$

ordered by reverse inclusion. Here $H_i^{\mathbb{R}} = \{z \in \mathbb{C}^d \mid \mathbf{s}(f_i(z)) \in \{0, +, -\}\}$, which is a real hyperplane defined by $\Im(f_i(z)) = 0$. One has that $\mathcal{L}$ is the support lattice of $\mathcal{F}$ and the support map takes $F \in \mathcal{F}$ to the intersection of all elements of $\mathcal{A}_{\mathrm{aug}}$ containing $\tau^{-1}(F)$ [14, Proposition 3.3]. The lattice $\mathcal{L}$ is a semimodular lattice of length $2d$ [14, Proposition 3.2]. More generally, if $X < Y$ in $\mathcal{L}$ then the length of the longest chain from X to Y in $\mathcal{L}$ is $\dim X - \dim Y$.

3.4. The Karnofsky-Rhodes expansion.

If L is a lattice generated (under meet) by a finite set A, then there is a universal A-generated left regular band with support lattice L, known as the Karnofsky-Rhodes expansion of L. Let us first describe the construction for monoids in general.

Let M be a monoid with generating set A; we do not assume $A \subseteq M$ although we treat it this way notationally. If $w \in A^*$, then $[w]_M$ will denote the image of w under the canonical projection $A^* \longrightarrow M$. Let $\Gamma_A(M)$ be the *right Cayley graph* of M with respect to the generators A; so $\Gamma_A(M)$ is the digraph (or quiver, if you like) with vertex set M and edge set $M \times A$ where the edge (m, a) goes from m to ma. We usually think of this edge as being labeled by a and draw it

$$m \xrightarrow{\ a\ } ma.$$

Given any vertex $m \in \Gamma_A(M)$ and word $w \in A^*$, there is a unique path labeled by w with initial vertex m (the terminal vertex will be $m[w]_M$). We call this the path *read* by w from m. Let us say that an edge e of $\Gamma_A(M)$ is a *transition edge*, if its initial and terminal vertices are in different strongly connected components of $\Gamma_A(M)$.

Define an equivalence relation on A^* by putting $u \equiv v$ if and only if:

- $[u]_M = [v]_M$;
- the sets of transition edges visited by the paths read from 1 in $\Gamma_A(M)$ by u and v coincide.

It is known that $\equiv$ is a congruence [40]; clearly it is contained in the kernel congruence of the projection $A^* \longrightarrow M$. The *Karnofsky-Rhodes expansion* of M with respect to generators A is given by $\widehat{M}_A = A^*/\equiv$. This construction is an endofunctor of the category of A-generated monoids (with morphisms

preserving generators). Moreover, the collection of canonical projections $\eta_M \colon \widehat{M}_A \longrightarrow M$ constitute a natural transformation to the identity functor.

Suppose now that L is an A-generated meet semilattice with identity. Since each strong component of the Cayley graph of L has a unique vertex, the word problem for $\widehat{L}_A$ is much simpler. Let $w = a_1 \cdots a_n$ be in A^* with the $a_i \in A$. We say that a_i is a *transition* of w if $[a_1 \cdots a_{i-1}]_L > [a_1 \cdots a_i]_L$. The empty string has no transitions. Notice that a_1 is a transition if and only if $[a_1]_L \neq 1$. The transitions of w are exactly the labels of the transition edges visited in $\Gamma_A(L)$ by the path read from 1 by w.

We say that w is *reduced* if either it is empty, or all its letters are transitions. In other words, $w = a_1 \cdots a_n$ is reduced if and only if

$$1 > [a_1]_L > [a_1 a_2]_L > \cdots > [a_1 \cdots a_n]_L.$$

Define the reduction $\mathrm{reduce}(w)$ to be the word obtained from w by erasing all its letters that are not transitions. Notice that w is reduced if and only if $w = \mathrm{reduce}(w)$. It is easy to see from the definition of the Karnofsky-Rhodes expansion that w and $\mathrm{reduce}(w)$ represent the same element of $\widehat{L}_A$ and that distinct reduced words represent distinct elements of $\widehat{L}_A$. Thus $\widehat{L}_A$ can be viewed as the set of reduced words with product $vw = \mathrm{reduce}(vw)$. It is routine to verify that $\widehat{L}_A$ is a left regular band and $\eta_L \colon \widehat{L}_A \longrightarrow L$ is the support map. It follows from the universal property of the Karnofsky-Rhodes expansion given by Elston [40] that if B is any A-generated left regular band with support lattice L (with the support map σ the identity on A), then there is a unique surjective homomorphism $\varphi \colon \widehat{L}_A \longrightarrow B$ such that the diagram

$$\begin{array}{ccc} \widehat{L}_A & \xrightarrow{\ \varphi\ } & B \\ & \searrow{\scriptstyle \eta_L} \quad \swarrow{\scriptstyle \sigma} & \\ & L & \end{array}$$

commutes.

Remark 3.1. Ken Brown considers reduced words in [23] as part of his proof of the diagonalizability of random walks on left regular bands. His proof essentially boils down to showing that a random walk on a Karnofsky-Rhodes expansion of a semilattice is diagonalizable and then deducing the result for arbitrary left regular bands from this case.

We now consider some examples. Nearly all of these examples can be found in Brown's paper [23].

Example 3.2 (Free left regular band). Consider the free semilattice on a finite set A; it is the power set $P(A)$ ordered by reverse inclusion (and so the operation is union). Then a letter a_i of $w = a_1 \cdots a_n$ is a transition if and only if $a_i \notin \{a_1, \ldots, a_{i-1}\}$. Thus the reduced words are precisely the injective words. If w is an arbitrary word, then $\mathrm{reduce}(w)$ is obtained

from w by removing repeated letters (reading from left to right). Thus $\widehat{P(A)}_A = F(A)$. One can also easily deduce this from the universal property.

Example 3.3. The following example is from [23, Section 5.1]. Let $\overline{F}_n$ be the quotient of F_n that identifies an injective word w of length $n-1$ with the unique injective word of length n with w as a prefix. Let $L(n)$ be the quotient of the free semilattice on n-generators that identifies all subsets of size $n-1$ with the subset of size n. Then $\overline{F}_n = \widehat{L(n)}_A$ where $A = \{1, \ldots, n\}$.

Example 3.4 (q-analogue). Next we consider the example from [23, Section 1.4]. Let q be a prime power and let $\mathbb{F}_q$ denote the field of q elements. Define $F_{n,q}$ to consist of all ordered linearly independent subsets $(x_1, \ldots, x_s)$ of $\mathbb{F}_q^n$ with product

$$(x_1, \ldots, x_s)(y_1, \ldots, y_t) = (x_1, \ldots, x_s, y_1, \ldots, y_t)^\wedge$$

where $\wedge$ means delete any vector that is linearly dependent on the preceding vectors. Set $L(n, q)$ to be the lattice of subspaces of $\mathbb{F}_q^n$ ordered by reverse inclusion and put $A = \mathbb{F}_q^n \setminus \{0\}$. Define $\sigma \colon A \longrightarrow L(n, q)$ by sending a vector $v \neq 0$ to the one-dimensional subspace it spans. Then it is easy to see that $\widehat{L(n, q)}_A = F_{n,q}$.

Example 3.5 (Matroids). The following example is from [23, Section 6.2]. Let $\mathcal{M}$ be a matroid with underlying set E; see [70] for the basic definitions. The associated monoid M consists of all ordered independent subsets $(x_1, \ldots, x_s)$ of E. The product is given by

$$(x_1, \ldots, x_s)(y_1, \ldots, y_t) = (x_1, \ldots, x_s, y_1, \ldots, y_t)^\wedge$$

where $\wedge$ means delete any element that depends on earlier elements. Let A be the set of non-loops of E and let L be the lattice of flats of $\mathcal{M}$ ordered by reverse inclusion. Let $\sigma \colon A \longrightarrow L$ be given by sending $a \in A$ to its closure. Then it is not hard to check that $M = \widehat{L}_A$.

Example 3.6 (Interval greedoids). Let L be a semimodular lattice and let E be the set of join-irreducible elements of L and make L a monoid via the join operation. Then the dual lattice $L^\vee$ is a meet semilattice. Björner associates an interval greedoid and a left regular band to the pair (L, E) [14]. It is easy to see that his left regular band is the submonoid of $\widehat{L^\vee}_E$ consisting of those reduced words whose associated chain consists of covers in the order.

Notice that if L is an A-generated lattice, $X \in L$ and $A_{\geq X}$ denotes the set of elements of A which are above X, then $(\widehat{L}_A)_{\geq X} = (\widehat{L_{\geq X}})_{A_{\geq X}}$.

3.5. **The Rhodes expansion.** Next we consider the Rhodes expansion of a lattice. The notion is defined more generally for monoids, cf. [88]. Let L be a finite lattice with top $\widehat{1}$, which we view as a monoid via its meet. If $X \subseteq L$ and $e \in L$, then put $eX = \{ex \mid x \in X\}$. If $X \subseteq L$ is a chain, then $\min X$

denotes the minimum element of X. The (monoidal right) *Rhodes expansion* of L is the set $\widehat{L}$ of all chains of L containing $\widehat{1}$ with the multiplication

$$X \cdot Y = X \cup (\min X)Y.$$

More explicitly, the product is given by

$$(\widehat{1} > x_1 > \cdots > x_n) \cdot (\widehat{1} > y_1 > \cdots > y_m) =$$
$$\mathrm{reduce}(\widehat{1} > x_1 > \cdots > x_n \geq x_n y_1 \geq \cdots \geq x_n y_m)$$

where 'reduce' means to remove repetitions. The identity is the chain consisting only of $\widehat{1}$.

It is not hard to see that the monoid discussed in [23, Section 5.2] and the monoid $\overline{S}$ associated to a matroid in [23, Section 6.2] are submonoids of Rhodes expansions of lattices.

3.6. **Free partially commutative left regular bands.** If $\Gamma = (V, E)$ is a simple graph, then the *free partially commutative left regular band* associated to Γ is the left regular band $B(\Gamma)$ with presentation

$$\big\langle V \mid xy = yx \text{ for all } (x, y) \in E \big\rangle.$$

This is the left regular band analogue of free partially commutative monoids (also called *trace monoids* or *graph monoids* [28, 36]) and of free partially commutative groups (also called *right-angled Artin groups* or *graph groups* [10]). For example, if Γ is a complete graph, then $B(\Gamma)$ is the free semilattice on the vertex set of Γ, whereas if Γ has no edges, then $B(\Gamma)$ is the free left regular band on the vertex set.

The Tsetlin library Markov chain can be modelled as a hyperplane random walk, but is most naturally a random walk on the free left regular band, see [23, 24]. Similarly, the random walk on acyclic orientations of a graph considered by Athanasiadis and Diaconis as a function of a hyperplane walk [6] is most naturally a random walk on a free partially commutative left regular band.

Let us first solve the word problem for $B(\Gamma)$. We need to determine when two elements of $F(V)$ yield the same element of $B(\Gamma)$. It turns out two injective words represent the same element of Γ if and only if they represent the same element of the trace monoid associated to Γ. We will not assume, however, prior knowledge of trace theory.

It will be convenient to denote by $\overline{\Gamma}$ the complementary graph of $\Gamma = (V, E)$. If $W \subseteq V$, then $\Gamma[W]$ denotes the induced subgraph of Γ with vertex set W and $\overline{\Gamma}[W]$ the induced subgraph of $\overline{\Gamma}$ with vertex set W. Note that $\overline{\Gamma[W]} = \overline{\Gamma}[W]$.

If $w \in F(V)$ with support $\sigma(w) \subseteq V$, then define an acyclic orientation $\mathcal{O}(w)$ of $\overline{\Gamma}[\sigma(w)]$ by directing the edge (x, y) from x to y if x comes before y in w. The following theorem is inspired by [36, Theorem 2.36].

Theorem 3.7. *Let $\Gamma = (V, E)$ be a graph. Two elements $v, w \in F(V)$ are equal in $B(\Gamma)$ if and only if $\sigma(v) = \sigma(w)$ and $\mathcal{O}(v) = \mathcal{O}(w)$. Moreover, if*

$W \subseteq V$ and $\mathcal{O}$ is an acyclic orientation of $\overline{\Gamma}[W]$, then $\mathcal{O} = \mathcal{O}(w)$ if and only if $\sigma(w) = W$ and w is a topological sorting of the directed graph $(\overline{\Gamma}[W], \mathcal{O})$.

Proof. Suppose first $\sigma(v) = \sigma(w)$ and $\mathcal{O}(v) = \mathcal{O}(w)$, call this orientation $\mathcal{O}$. By construction, it follows that v and w are topological sortings of $(\overline{\Gamma}[W], \mathcal{O})$. But it is well known that any two topological sortings of an acyclic digraph can be obtained from each other by repeatedly transposing consecutive vertices which are not connected by an edge, cf. [36, Lemma 2.3.5]. Thus the defining relations of $B(\Gamma)$ let us transform v to w (since vertices not connected by an edge of $\overline{\Gamma}$ commute in $B(\Gamma)$). We conclude that v and w are equal in $B(\Gamma)$.

For the converse, first note that the support map $\sigma \colon F(V) \longrightarrow P(V)$ factors through $B(\Gamma)$ and so if v and w are equal in $B(\Gamma)$, then $\sigma(v) = \sigma(w)$; call this common support W. To verify that $\mathcal{O}(v) = \mathcal{O}(w)$, it suffices to show that if (x, y) is an edge of $\overline{\Gamma}[W]$, then x and y appear in the same order in both v and w. Define a mapping $\tau \colon V \longrightarrow \{0, +, -\}$ by

$$\tau(z) = \begin{cases} + & \text{if } z = x \\ - & \text{if } z = y \\ 0 & \text{else.} \end{cases}$$

Then since (x, y) is not an edge of Γ, it follows that if $(s, t) \in E$, then at least one of s and t maps to 0. Thus τ extends to a homomorphism $\tau \colon B(\Gamma) \longrightarrow \{0, +, -\}$ and so $\tau(v) = \tau(w)$. But clearly $\tau(v)$ is $+$ if x appears before y and $-$ if y appears before x, and similarly for w. Thus $\mathcal{O}(v) = \mathcal{O}(w)$. $\qquad\square$

It follows from the theorem that we can identify $B(\Gamma)$ with the set of pairs $(W, \mathcal{O})$ where $W \subseteq V$ and $\mathcal{O}$ is an acyclic orientation of $\overline{\Gamma}[W]$. A vertex $v \in V$ is identified with the trivial acyclic orientation on the induced subgraph $\overline{\Gamma}[\{v\}]$, which contains no arrows. The product is then given by

$$(W_1, \mathcal{O}_1)(W_2, \mathcal{O}_2) = (W_1 \cup W_2, \mathcal{O})$$

where $\mathcal{O}$ is the orientation satisfying $(x, y) \in \mathcal{O}$ if $(x, y) \in \mathcal{O}_1$, or if $x \in W_1$ and $y \in W_2 \setminus W_1$, or if $x, y \in W_2 \setminus W_1$ and $(x, y) \in \mathcal{O}_2$. The support lattice is $P(V)$ ordered by reverse inclusion. The minimal ideal consists of the acyclic orientations of $\overline{\Gamma}$.

If $v \in V$ and $(V, \mathcal{O})$ is an acyclic orientation of $\overline{\Gamma}$, then $v \cdot (V, \mathcal{O})$ is the orientation of $\overline{\Gamma}$ that orients all edges containing v away from v and which otherwise agrees with $\mathcal{O}$. Thus the random walk on the minimal ideal of $B(\Gamma)$ driven by a probability supported on V is exactly the random walk on acyclic orientations of $\overline{\Gamma}$ considered by Athanasiadis and Diaconis [6]. The computation of the spectrum and the bounds on the rate of convergence to stationary can be more easily computed using this monoid. Actually, as in the case of the Tsetlin library (which corresponds to when Γ has no edges) to obtain the best bounds on the rate of convergence, one should use the

quotient of $B(\Gamma)$ that identifies the image of a word in $F(V)$ of length $|V|-1$ with the unique word of length $|V|$ containing that word as a prefix.

If $U \subsetneq W \subseteq V$, then one easily verifies $B(\Gamma)[W, U] \cong B(\Gamma[W \setminus U])$. Thus the interval submonoids of $B(\Gamma)$ are precisely the free partially commutative monoids on induced subgraphs of Γ.

3.7. Geometric and right hereditary left regular bands.

We say that a left regular band B is *geometric* if $a^{\uparrow} = \{b \in B \mid b \geq a\}$ is commutative (and hence a lattice under the order $\leq$ with the meet given by the product) for all $a \in B$. For example, $\{0, +, -\}$ is geometric. The class of geometric left regular bands is closed under taking direct products and submonoids. Therefore, the left regular bands associated to hyperplane arrangements and oriented matroids are geometric, whence the name. If B is geometric, then so is any local submonoid and interval submonoid of B. An example of a non-geometric left regular band is obtained by taking any left regular band which is not a lattice and adjoining a multiplicative zero. However, almost all the left regular bands appearing so far in the algebraic combinatorics literature are geometric. The main exception is the class of complex hyperplane arrangement face monoids discussed above.

Let us say that B is *right hereditary*, if the Hasse diagram of the order $\leq$ is a tree. For example, the free left regular band is right hereditary, as are the Rhodes and Karnofsky-Rhodes expansions of a lattice. In particular, the left regular bands associated by Brown to matroids (Example 3.5) and by Björner to interval greedoids (Example 3.6) are right hereditary.

Notice that if B is right hereditary, then so are its submonoids, as well as its local submonoids and interval submonoids. Clearly, a left regular band B which is right hereditary is geometric since each poset of the form $a^{\uparrow}$ is a chain. The reason for the terminology "right hereditary" is that a left regular band B is right hereditary if and only if each right ideal of B is projective in the category of right B-sets [39]. This is an amusing coincidence of terminology since we shall see later that the algebra of a right hereditary, left regular band over a field is hereditary in the ring theoretic sense!

Free partially commutative left regular bands are also geometric. This follows from the proof of Theorem 3.7, which shows that $B(\Gamma)$ embeds in $\{0, 1\}^V \times \{0, +, -\}^{\overline{E}}$, where $\overline{E}$ is the edge set of $\overline{\Gamma}$. In fact, a well-known result from trace theory [36, Proposition 5.5.1] implies that each subset $a^{\uparrow}$ of $B(\Gamma)$ is a distributive lattice.

3.8. Left regular bands associated to acyclic quivers.

Here we construct from any finite acyclic quiver Q a left regular band $B_Q^{\prec}$ whose monoid algebra is isomorphic to the path algebra of Q. For quivers and their path algebras, we adhere to the notation and conventions of [5].

Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver with vertex set Q_0 and set of arrows Q_1. If $\alpha \in Q_1$, let $s(\alpha)$ denote its source and $t(\alpha)$ denote its target. A *path* in Q of *length l* from v_0 to v_l is a sequence $(v_0 \mid \alpha_1, \ldots, \alpha_l \mid v_l)$, or

$(\alpha_1 \cdots \alpha_l)$ for brevity, satisfying: $\alpha_i \in Q_1$ for all $1 \le i \le l$; $s(\alpha_1) = v_0$; $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \le i < l$; and $t(\alpha_l) = v_l$. If $v \in Q_0$, we denote by ε_v the *stationary* (or empty) path $(v \parallel v)$ of length 0 with source and target equal to v.

The *path algebra* $\Bbbk Q$ of Q with coefficients in a commutative ring with unit $\Bbbk$ consists of formal $\Bbbk$-linear combinations of paths in Q; the product of two paths is

$$(\alpha_1 \cdots \alpha_r)(\beta_1 \cdots \beta_s) = \begin{cases} (\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s), & \text{if } t(\alpha_r) = s(\beta_1), \\ 0, & \text{if } t(\alpha_r) \ne s(\beta_1). \end{cases}$$

In particular,

$$\varepsilon_v \varepsilon_u = \delta_{v,u} \varepsilon_v \tag{3.1}$$

where $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ if $u \ne v$.

Definition 3.8. Fix a total order $\prec$ on Q_0 with the property that $s(\alpha) \prec t(\alpha)$ for every arrow α in Q_1. For a path $(\alpha_1 \cdots \alpha_n)$ in Q, define

$$\ell(\alpha_1 \cdots \alpha_n) = \sum_{u \succeq t(\alpha_n)} \varepsilon_u + \sum_{i=1}^{n} (\alpha_i \cdots \alpha_n)$$

and let $B_Q^{\prec}$ denote the image of the paths of Q under the map ℓ:

$$B_Q^{\prec} = \{\ell(\alpha_1 \cdots \alpha_n) \mid (\alpha_1 \cdots \alpha_n) \text{ is a path of } Q\}.$$

That such a total order exists follows from the assumption that Q is acyclic. Of course, the definition depends on the choice of total order.

Proposition 3.9. *Suppose* $(\alpha_1 \cdots \alpha_n)$ *and* $(\beta_1 \cdots \beta_m)$ *are two paths in* Q.

(1) *If* $t(\alpha_n) \succeq t(\beta_m)$, *then*

$$\ell(\alpha_1 \cdots \alpha_n) \cdot \ell(\beta_1 \cdots \beta_m) = \ell(\alpha_1 \cdots \alpha_n).$$

(2) *If there exists* r *such that* $t(\alpha_n) = s(\beta_r)$, *then*

$$\ell(\alpha_1 \cdots \alpha_n) \cdot \ell(\beta_1 \cdots \beta_m) = \ell(\alpha_1 \cdots \alpha_n \beta_r \cdots \beta_m).$$

(3) *If there exists* r *such that* $s(\beta_{r-1}) \prec t(\alpha_n) \prec s(\beta_r)$, *then*

$$\ell(\alpha_1 \cdots \alpha_n) \cdot \ell(\beta_1 \cdots \beta_m) = \ell(\beta_r \cdots \beta_m).$$

where $s(\beta_0)$ *is understood as smaller than all vertices.*

Proof. The product $\ell(\alpha_1 \cdots \alpha_n) \cdot \ell(\beta_1 \cdots \beta_m)$ expands as

$$\sum_{u \succeq t(\alpha_n)} \sum_{v \succeq t(\beta_m)} \varepsilon_u \varepsilon_v + \sum_{i=1}^{n} \sum_{v \succeq t(\beta_m)} (\alpha_i \cdots \alpha_n) \varepsilon_v$$

$$+ \sum_{j=1}^{m} \sum_{u \succeq t(\alpha_n)} \varepsilon_u (\beta_j \cdots \beta_m) + \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \cdots \alpha_n)(\beta_j \cdots \beta_m).$$

Since $\varepsilon_u \varepsilon_v = \delta_{u,v} \varepsilon_u$, it follows that

$$\sum_{u \succeq t(\alpha_n)} \sum_{v \succeq t(\beta_m)} \varepsilon_u \varepsilon_v = \begin{cases} \sum_{u \succeq t(\alpha_n)} \varepsilon_u, & \text{if } t(\alpha_n) \succeq t(\beta_m), \\ \sum_{v \succeq t(\beta_m)} \varepsilon_v, & \text{if } t(\beta_m) \succeq t(\alpha_n). \end{cases}$$

Since $(\alpha_i \cdots \alpha_n) \varepsilon_v = \delta_{v,t(\alpha_n)} (\alpha_i \cdots \alpha_n)$,

$$\sum_{i=1}^{n} \sum_{v \succeq t(\beta_m)} (\alpha_i \cdots \alpha_n) \varepsilon_v = \begin{cases} \sum_{i=1}^{n} (\alpha_i \cdots \alpha_n), & \text{if } t(\alpha_n) \succeq t(\beta_m), \\ 0, & \text{if } t(\alpha_n) \not\succeq t(\beta_m). \end{cases}$$

Since $\varepsilon_u (\beta_j \cdots \beta_m) = \delta_{u,s(\beta_j)} (\beta_j \cdots \beta_m)$,

$$\sum_{j=1}^{m} \sum_{u \succeq t(\alpha_n)} \varepsilon_u (\beta_j \cdots \beta_m) = \begin{cases} 0, & \text{if } s(\beta_j) \not\succeq t(\alpha_n) \text{ for all } j, \\ \sum_{j=r}^{m} (\beta_j \cdots \beta_m), & \text{if } s(\beta_{r-1}) \prec t(\alpha_n) \preceq s(\beta_r). \end{cases}$$

Since $(\alpha_i \cdots \alpha_n)(\beta_j \cdots \beta_m) = 0$ unless $t(\alpha_n) = s(\beta_j)$ for some $1 \leq j \leq m$,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \cdots \alpha_n)(\beta_j \cdots \beta_m)$$

$$= \begin{cases} 0, & \text{if } t(\alpha_n) \neq s(\beta_j) \text{ for all } j, \\ \sum_{i=1}^{n} (\alpha_i \cdots \alpha_n \beta_r \cdots \beta_m), & \text{if } t(\alpha_n) = s(\beta_r) \text{ for some } r. \end{cases}$$

The result now follows by combining these identities. $\qquad \square$

Theorem 3.10. $B_Q^{\prec}$ *is a left regular band and* $\Bbbk B_Q^{\prec} = \Bbbk Q$.

Proof. We prove $B_Q^{\prec}$ is a left regular band using Proposition 2.2. Let $\Lambda = (Q_0, \prec)^{op}$ denote the lattice with elements Q_0 ordered by the opposite of $\prec$. Define a map $\sigma \colon B_Q^{\prec} \longrightarrow \Lambda$ by $\sigma(\alpha_1 \cdots \alpha_n) = t(\alpha_n)$. By Proposition 3.9, $\sigma(xy) = \sigma(x) \wedge \sigma(y)$, where $\wedge$ denotes the meet operator of Λ, and $xy = x$ iff $\sigma(x) \succeq \sigma(y)$. Hence, the conditions of Proposition 2.2 are satisfied. Note that $B_Q^{\prec}$ is a monoid with identity the stationary path at the smallest object of Q_0.

It is easy to see that $B_Q^{\prec}$ is a basis for $\Bbbk Q$. Indeed, if we order the basis of paths so that each suffix of a path is larger than the path itself and so that stationary paths of vertices are ordered using $\prec$ in the obvious way, then the map sending a path $(\alpha_1 \cdots \alpha_n)$ to $\ell(\alpha_1 \cdots \alpha_n)$ has an upper triangular matrix with ones along the diagonal. $\qquad \square$

Note that $B_Q^{\prec}$ is almost never a geometric left regular band.

3.9. **Idempotent inner derivations.** Let A be an associative algebra over a field $\Bbbk$. For an element $a \in A$, let $\partial_a(x) = xa - ax$ denote the associated inner derivation. The following was noticed by Lawvere in his work on graphical toposes (see below).

Proposition 3.11 (Lawvere [57])**.** *Let A be an associative algebra over a field $\Bbbk$ of characteristic different from* 2*. The set of idempotent elements of A for which the associated inner derivation is also idempotent*

$$\left\{ a \in A \mid a^2 = a \text{ and } \partial_a^2 = \partial_a \right\}$$

is a left regular band.

3.10. **Graphical topos.** Left regular bands have also appeared in topos theory. In [55,56], Lawvere introduced a class of toposes called graphical. A *graphical topos* is a topos which is generated by objects whose endomorphism monoid is a finite left regular band. The topos of B-sets, where B is a finite left regular band, is a graphical topos. He used a different name for left regular bands: he called them *graphical monoids* instead; and he named the identity $xyx = xy$ the *Schützenberger-Kimura identity*. The reason for the terminology 'graphical' is that if $B = \{0, +, -\}$, then the category of B-sets can be identified with the category of directed graphs with morphisms that are allowed to collapse an edge to a vertex.

4. Applications of the Main Result

Since the proof of the main result is rather involved, we defer it to Sections 5 and 6. This section is devoted to applications of the main result. We begin by stating the main result. Fix a finite left regular band B and a commutative ring with unit $\Bbbk$.

4.1. **Statement of the main result.** Let $X \in \Lambda(B)$. Let $\Bbbk_X$ denote the ring $\Bbbk$ viewed as a $\Bbbk B$-module via the action

$$b \cdot \alpha = \begin{cases} \alpha, & \text{if } \sigma(b) \geq X, \\ 0, & \text{otherwise,} \end{cases}$$

for all $b \in B$ and $\alpha \in \Bbbk$. These are the simple $\Bbbk B$-modules when $\Bbbk$ is a field (Section 5.3). The main result of this paper is a computation of $\operatorname{Ext}^n_{\Bbbk B}(\Bbbk_X, \Bbbk_Y)$ for all $n \geq 0$ and all $X, Y \in \Lambda(B)$. It turns out that this coincides with the cohomology of a simplicial complex associated to the interval subsemigroup $B[X, Y]$ of B (Section 2.4). Recall that $B[X, Y]$ is a poset with respect to Green's $\mathscr{R}$-order: $a \leq b$ if $ba = a$ (see Section 2.3). Let $\Delta(X, Y)$ be its order complex, which is the simplical complex whose vertex set is $B[X, Y]$ and whose simplices are the finite chains in $B[X, Y]$. Our main result is the following theorem.

Theorem 4.1. *Let B be a finite left regular band and let $\Bbbk$ be a commutative ring with unit. Let $X, Y \in \Lambda(B)$. Then*

$$\operatorname{Ext}^n_{\Bbbk B}(\Bbbk_X, \Bbbk_Y) = \begin{cases} \widetilde{H}^{n-1}(\Delta(X, Y), \Bbbk), & \text{if } X < Y, \ n \geq 1, \\ \Bbbk, & \text{if } X = Y, \ n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

4.2. **The quiver of a left regular band.** Assume for the moment that $\Bbbk$ is a field. Recall that a finite dimensional $\Bbbk$-algebra A is *split basic* if each of its simple modules is one-dimensonal. It is well known that $\sigma \colon \Bbbk B \longrightarrow \Bbbk \Lambda(B)$ is the semisimple quotient and $\Bbbk \Lambda(B) \cong \Bbbk^{\Lambda(B)}$, *cf.* [23]. Thus $\Bbbk B$ is a split basic $\Bbbk$-algebra. The (Gabriel) *quiver* $Q(A)$ of a unital split basic $\Bbbk$-algebra A is the directed graph with vertices the isomorphism classes of simple A-modules and with the number of edges from S_1 to S_2 equal to $\dim_{\Bbbk} \operatorname{Ext}^1_A(S_1, S_2)$.

The second author computed the quiver of a left regular band algebra in [78]. We give a new description here, using Theorem 4.1, which is more conceptual and therefore sometimes easier to apply.

Theorem 4.2. *Let B be a finite left regular band and $\Bbbk$ a field. Then the quiver $Q(\Bbbk B)$ has vertex set $\Lambda(B)$. The number of arrows from X to Y is zero unless $X < Y$, in which case it is one less than the number of connected components of $\Delta(X, Y)$.*

It is not hard to see that the number of connected components of the order complex $\Delta(P)$ of a poset P coincides with the number of equivalence classes of the equivalence relation on P generated by the partial order, or equivalently, with the number of components of the Hasse diagram of P. From this observation, it is straightforward to verify that our description of the quiver $Q(\Bbbk B)$ coincides with the one in [78].

Let A be a split basic algebra with an acyclic quiver Q. A result of Bongartz [18] says that if S_1, S_2 are the simple A-modules corresponding to vertices v_1 and v_2 of Q, then the number of relations involving paths from v_1 to v_2 in a minimal quiver presentation of A is the dimension of $\operatorname{Ext}^2_A(S_1, S_2)$. Thus Theorem 4.1 admits the following corollary.

Corollary 4.3. *If $X < Y$ in $\Lambda(B)$, then the number of relations involving paths from X to Y in a minimal quiver presentation of $\Bbbk B$ is given by $\dim_{\Bbbk} \widetilde{H}^1(\Delta(X, Y), \Bbbk)$.*

4.3. **Global dimension of left regular band algebras.** Let us next apply Theorem 4.1 to global dimension. For a detailed discussion of global dimension the reader is referred to [5, 9, 27].

The notion of global dimension can be formulated in terms of either left modules or right modules, but it is well known that these two formulations coincide for a finite dimensional algebra A over a field $\Bbbk$. Thus, it suffices to define the notion for left modules.

The *global dimension* $\operatorname{gl.dim} A$ of a finite dimensional algebra A over a field $\Bbbk$ is the least n such that $\operatorname{Ext}^{n+1}_A(V, W) = 0$ for all finite dimensional A-modules V, W. A simple induction on the length of a composition series and application of the long exact sequence for Ext shows that $\operatorname{gl.dim} A$ is the least n such that $\operatorname{Ext}^{n+1}_A(S_1, S_2) = 0$ for all simple A-modules S_1, S_2, or equivalently the least n such that $\operatorname{Ext}^m_A(S_1, S_2) = 0$ for all $m > n$ and all simple A-modules S_1, S_2 [5, 7, 9].

An algebra A has global dimension zero if and only if it is semisimple. A finite dimensional algebra A is said to be *hereditary* if gl. dim $A \leq 1$. This is equivalent to the property that each submodule of a projective A-module is projective, as well as to the property that each left ideal of A is a projective module. It follows from Gabriel's theorem that a split basic $\Bbbk$-algebra (such as the algebra of a finite left regular band) is hereditary if and only if its quiver is acyclic and it is isomorphic to the path algebra of its quiver [5,7,9].

Let B be a finite left regular band. Nico's results [66, 68] on global dimension of the algebra of a von Neumann regular semigroup imply that the global dimension of $\Bbbk B$ is finite and bounded by the length of the longest chain in $\Lambda(B)$ where the length of a chain C in a poset is $|C| - 1$. This can also be deduced from the theory of quasi-hereditary algebras [33] because $\Bbbk B$ is a directed quasi-hereditary algebra with weight poset the opposite of $\Lambda(B)$ [62, 72]. We easily recover Nico's result, restricted to left regular bands, using Theorem 4.1.

Theorem 4.4. *Let B be a finite left regular band and $\Bbbk$ a field. Then*

$$\mathrm{gl. \, dim} \, \Bbbk B = \min\{n \mid \widetilde{H}^n(\Delta(X,Y), \Bbbk) = 0, \textit{ for all } X < Y \in \Lambda(B)\}.$$

In particular, one has

$$\mathrm{gl. \, dim} \, \Bbbk B \leq m$$

where m is the length of the longest chain in $\Lambda(B)$.

Proof. The first statement on global dimension is immediate from Theorem 4.1. If P is a finite poset, then the dimension of $\Delta(P)$ is the length of the longest chain in P. By Lemma 2.3 the longest chain in $\Delta(\widehat{0}, \widehat{1})$ has length $m - 1$. Therefore, for $X < Y$, one has $\dim \Delta(X,Y) \leq \dim \Delta(\widehat{0}, \widehat{1}) = m - 1$ and so $\widetilde{H}^m(\Delta(X,Y), \Bbbk) = 0$. Thus gl. dim $\Bbbk B \leq m$. $\square$

As an immediate corollary, we obtain a characterization of those algebras $\Bbbk B$ that are hereditary in terms of the order complex of B.

Corollary 4.5. *$\Bbbk B$ is hereditary if and only if each connected component of each simplicial complex $\Delta(X,Y)$, for $X < Y \in \Lambda(B)$, is acyclic.*

It is well known that, for a finite dimensional algebra A, the projective dimension of a finite dimensional A-module M is the least d such that $\mathrm{Ext}_A^{d+1}(M, S) = 0$ for all simple A-modules S. The global dimension A coincides with the maximum projective dimension of a simple A-module. Theorem 4.1 thus yields the following refinement of Theorem 4.4.

Corollary 4.6. *Let B be a finite left regular band and let $\Bbbk$ be a field. Let $X \in \Lambda(B)$. Then the projective dimension of $\Bbbk_X$ is given by*

$$\mathrm{proj. \, dim} \, \Bbbk_X = \min\{n \mid \widetilde{H}^n(\Delta(X,Y), \Bbbk) = 0, \ \forall Y > X\}.$$

4.4. **Leray numbers and an improved upper bound.** We can improve greatly on Nico's upper bound on the global dimension in the case of left regular bands. First we recall the notion of the Leray number of a simplicial complex. If K is a simplicial complex with vertex set V and $W \subseteq V$, then the induced subcomplex $K[W]$ is the subcomplex consisting of all simplices whose vertices belong to W.

The $\Bbbk$-*Leray number* of a finite simplicial complex K with vertex set V is defined by

$$L_{\Bbbk}(K) = \min\{d \mid \widetilde{H}^i(K[W], \Bbbk) = 0, \forall i \geq d, \forall W \subseteq V\}.$$

See for instance [50, 51]. Originally, interest in Leray numbers came about because of connections with Helly-type theorems [51, 58, 91]: the Leray number of a simplicial complex provides an obstruction for realizing the complex as the nerve of a collection of compact convex subsets of $\mathbb{R}^d$.

Leray numbers also play a role in combinatorial commutative algebra. Let K be a simplicial complex with vertex set $\{x_1, \ldots, x_n\}$. Recall that the *face ideal* I_K of K is the ideal of the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_m}$ with $\{x_{i_1}, \ldots, x_{i_m}\}$ not a face of K. The *Stanley-Reisner ring* of K over $\Bbbk$ is $\Bbbk[x_1, \ldots, x_n]/I_K$ [87]. The $\Bbbk$-Leray number of a simplicial complex K turns out to be the Castelnuovo-Mumford regularity of the Stanley-Reisner ring of K over $\Bbbk$ [38, 51, 93]. Equivalently, the regularity of the ideal I_K is $L_{\Bbbk}(K) + 1$ [50].

Theorem 4.4 implies the following upper bound on the global dimension of a left regular band algebra.

Theorem 4.7. *Let B be a finite left regular band and $\Bbbk$ a field. Then* $\mathrm{gl.dim}\,\Bbbk B$ *is bounded above by the Leray number* $L_{\Bbbk}(\Delta(B))$ *of the order complex of B.*

4.5. **Right hereditary left regular bands have hereditary algebras.** To apply the bound of Theorem 4.7, we need the well-known notion of the *clique complex* or *flag complex* of a graph. If $\Gamma = (V, E)$ is a simple graph, then the clique complex $\mathsf{Cliq}(\Gamma)$ is the simplicial complex whose vertex set is V and whose simplices are the finite cliques (subsets of vertices inducing a complete subgraph). Notice that Γ is the 1-skeleton of $\mathsf{Cliq}(\Gamma)$ and that $\mathsf{Cliq}(\Gamma)$ is obtained by 'filling in' the 1-skeleton of every q-simplex found in Γ. If $W \subseteq V$, then $\mathsf{Cliq}(\Gamma)[W] = \mathsf{Cliq}(\Gamma[W])$.

As an example, let P be a poset. The *comparability graph* $\Gamma(P)$ is the graph with vertex set P and edge set all pairs (p, q) with $p < q$ or $q < p$. It is immediate from the definition that $\mathsf{Cliq}(\Gamma(P)) = \Delta(P)$.

It is known that $L_{\Bbbk}(K) = 0$ if and only if K is a simplex and $L_{\Bbbk}(K) \leq 1$ if and only if K is the clique complex of a chordal graph, cf. [58, 91]. Recall that a graph Γ is *chordal* if it contains no induced cycle of length greater than or equal to 4. Of course, any induced subgraph of a chordal graph is chordal. Let us sketch a proof.

Proposition 4.8 (Folklore). *Let K be a finite simplicial complex. Then $L_{\Bbbk}(K) \leq 1$ if and only if K is the clique complex of a chordal graph. Moreover, $L_{\Bbbk}(K) = 0$ if and only if K is a simplex.*

Proof. Since every induced subcomplex of a simplex is a simplex, clearly the Leray number of a simplex is 0.

Suppose that $L_{\Bbbk}(K) \leq 1$. Then K is the clique complex of its 1-skeleton Γ. Otherwise, there is a clique W in Γ which is not a simplex. Necessarily $|W| \geq 3$. If we choose W to be of minimal size, then we have the induced subcomplex $K[W]$ is topologically a sphere of dimension $|W| - 2$. Thus $L_{\Bbbk}(K) \geq |W| - 1 \geq 2$. Thus K is the clique complex of Γ. If K is not a simplex, then there is a pair of vertices $\{v, w\}$ which do not form an edge. The induced subcomplex $K[\{v, w\}]$ is not connected and so $L_{\Bbbk}(K) > 0$. Therefore, $L_{\Bbbk}(K) = 0$ implies K is a simplex. Next suppose that C_n is an induced n-cycle in Γ with $n \geq 4$. Then since $\mathsf{Cliq}(C_n) = C_n$ and $\widetilde{H}^1(C_n, \Bbbk) \cong \Bbbk$, it follows that $L_{\Bbbk}(\mathsf{Cliq}(\Gamma)) \geq 1$. Thus $L_{\Bbbk}(\mathsf{Cliq}(\Gamma)) \leq 1$ implies Γ is chordal.

For the converse, it suffices to show that every connected chordal graph $\Gamma = (V, E)$ has a contractible clique complex, cf. [38, Lemma 3.1]. This is done by induction on the number of vertices of Γ. The proof relies on a classical result of Dirac [37] and of Boland and Lekkerkerker [58] that every chordal graph has a simplicial vertex v, that is, a vertex v whose neighbors form a clique. By induction $\mathsf{Cliq}(\Gamma[V \setminus \{v\}])$ is contractible. Let X be the set of neighbors of v. Then X is a simplex of $\mathsf{Cliq}(\Gamma[V \setminus \{v\}])$ and $\mathsf{Cliq}(\Gamma)$ is obtained from $\mathsf{Cliq}(\Gamma[V \setminus \{v\}])$ by attaching the simplex $Y = X \cup \{v\}$ along the facet X. Thus we can collapse Y into X, yielding a simple homotopy equivalence of $\mathsf{Cliq}(\Gamma)$ and $\mathsf{Cliq}(\Gamma[V \setminus \{v\}])$. This completes the proof. $\square$

From Proposition 4.8, it follows that the bound in Theorem 4.4 is not tight. Indeed, if L is a finite lattice, viewed as a left regular band via the meet operation, then $\mathrm{gl.\,dim}\,\Bbbk L = 0$, but unless L is a chain, its order complex is not a simplex. However, Theorem 4.4 is tight for right hereditary left regular bands.

Theorem 4.9. *Let B be a finite left regular band and $\Bbbk$ a field. Suppose that the Hasse diagram of B is a rooted tree, i.e., B is right hereditary. Then $\Bbbk B$ is hereditary.*

Proof. It is well known that the comparability graph of a poset whose Hasse diagram is a rooted tree is chordal. In fact, it is known that a finite graph has no induced simple path on four nodes and no induced cycle on four nodes if and only if it is the comparability graph of a finite poset whose Hasse diagram is a disjoint union of rooted trees [92]. The idea is that any simple path of length 3 of the form $a < b < c$ or $a > b > c$ has a chord in the induced subgraph. In a disjoint union of rooted trees, there are no simple paths of the form $a > b < c$. $\square$

Since the free left regular band is right hereditary, this provides a conceptual proof that the algebra of a free left regular band is hereditary, a result

first proved by K. Brown using quivers and a counting argument [78, Theorem 13.1]. We shall momentarily give another, more transparent proof of Theorem 4.9, which also makes it simple to compute the quiver of a right hereditary left regular band.

4.6. **Geometric left regular bands and commutation graphs.** Suppose now that B is a finite geometric left regular band. Since this class is closed under taking interval submonoids, we can restrict our attention to $\Delta(\widehat{0}, \widehat{1})$. We provide a simplicial complex homotopy equivalent to $\Delta(\widehat{0}, \widehat{1})$, using the following special case of Rota's cross-cut theorem, see the survey paper of Björner [13]. A proof is given for completeness in Corollary 6.8 below.

Theorem 4.10 (Rota). *Let P be a finite poset such that any subset of P with a common lower bound has a meet. Define a simplicial complex K with vertex set the set $\mathscr{M}(P)$ of maximal elements of P and with simplices those subsets of $\mathscr{M}(P)$ with a common lower bound. Then K is homotopy equivalent to the order complex $\Delta(P)$.*

Let $\mathscr{M}(B)$ be the set of maximal elements of $B \setminus \{1\}$ and let $\Gamma(\mathscr{M}(B))$ be the *commutation graph* of $\mathscr{M}(B)$, that is, the graph whose vertex set is $\mathscr{M}(B)$ and whose edges are pairs (a, b) such that $ab = ba$.

Theorem 4.11. *Let B be a finite geometric left regular band. Then $\Delta(\widehat{0}, \widehat{1})$ is homotopy equivalent to the clique complex $\mathsf{Cliq}(\Gamma(\mathscr{M}(B)))$ of the commutation graph of the set $\mathscr{M}(B)$ of maximal elements of $B \setminus \{1\}$.*

Proof. In a finite geometric left regular band, a subset $A \subseteq B$ has a lower bound if and only if the elements of A all mutually commute, in which case A has a meet, namely the product of all elements of A. The result now follows from Rota's cross-cut theorem (Theorem 4.10), and the definition of the commutation graph. $\square$

Theorem 4.11 can be used to give another proof that if a left regular band B is right hereditary, then $\Bbbk B$ is hereditary. This proof also leads to an easy computation of the quiver.

Theorem 4.12. *Let B be a finite left regular band that is right hereditary and let $\Bbbk$ be a field. Then $\Bbbk B$ is hereditary. The quiver $Q(\Bbbk B)$ has vertex set $\Lambda(B)$. The number of edges from X to Y is zero unless $X < Y$. If $X < Y$, choose $e_Y \in B$ with $\sigma(e_Y) = Y$. Then the number of edges from X to Y is one less than the number of children of e_Y with support greater than or equal to X.*

Proof. If $X < Y$, then $B[X, Y]$ is also right hereditary and $\mathscr{M}(B[X, Y])$ consists of the children of e_Y with support greater than or equal to X. Since the Hasse diagram of $B[X, Y]$ is a tree, no two elements of $\mathscr{M}(B[X, Y])$ have a common lower bound and hence $\Gamma(\mathscr{M}(B[X, Y]))$ has no edges, and so in particular is its own clique complex. Thus $\widetilde{H}^0(\Delta(X, Y)) = |\mathscr{M}(B[X, Y])| - 1$

and $\widetilde{H}^n(\Delta(X,Y)) = 0$ for all $n \geq 1$ by Theorem 4.11. The result now follows immediately from Theorem 4.4. $\square$

Theorem 4.12 covers the algebras of nearly all the left regular bands considered by K. Brown in [23], except the hyperplane semigroups. It also covers the interval greedoid left regular bands of Björner [14].

Corollary 4.13. *The algebra $\Bbbk F_n$ is hereditary over any field $\Bbbk$. The quiver $Q(\Bbbk F_n)$ has vertex set the subsets of $\{1, \ldots, n\}$. If $X \supsetneq Y$, then there are $|X \setminus Y| - 1$ edges from X to Y. There are no other edges.*

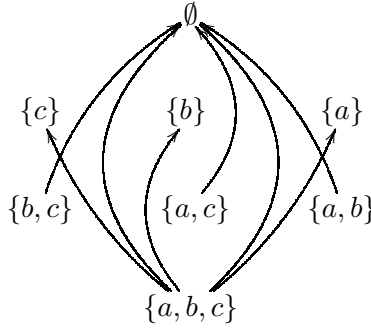The quiver of $\Bbbk F_3$ is depicted in Figure 5.

FIGURE 5. The quiver of $\Bbbk F(\{a,b,c\})$, cf. Figure 1.

Proof. The free left regular band is right hereditary. If $X \supsetneq Y$ and w is a word with support Y, then the children of w with support greater than or equal to X are the words wx with $x \in X \setminus Y$. This completes the proof. $\square$

Let us generalize the above result to Karnofsky-Rhodes expansions.

Corollary 4.14. *Let L be a finite lattice with monoid generating set A and let $\Bbbk$ be a field. Let $\widehat{L}_A$ be the Karnofsky-Rhodes expansion of L with respect to A. Then $\Bbbk \widehat{L}_A$ is hereditary. The quiver of $\Bbbk \widehat{L}_A$ has vertex set L. The number of edges from X to Y is zero unless $X < Y$, in which case it is one less than the number of elements $a \in A$ such that $X \leq Y \wedge a < Y$.*

Proof. Again, we have $\widehat{L}_A$ is right hereditary. If w is a reduced word with support Y, then the children of w with support greater than or equal to X are the words wa such that $Y \wedge a < Y$ and $Y \wedge a \geq X$. $\square$

One similarly has that the Rhodes expansion of a lattice is right hereditary and one can explicitly write down its quiver. The number of edges from X to Y when $X < Y$ is one less than the number of elements $Z \geq X$, which are covered by Y.

Remark 4.15. One can alternatively prove Theorem 4.12 via a counting argument using the description above of the quiver and Gabriel's theorem.

4.7. **Free partially commutative left regular bands.** We prove that the global dimension of a free partially commutative left regular band is the Leray number of the clique complex of the corresponding graph. This gives a new interpretation of the Leray number of a clique complex in terms of non-commutative algebra.

Theorem 4.16. *Let $\Gamma = (V, E)$ be a finite graph and $\Bbbk$ a commutative ring with unit. Then, for $W \subsetneq U \subseteq V$, we have*

$$\mathrm{Ext}^n_{\Bbbk B(\Gamma)}(\Bbbk_U, \Bbbk_W) = \widetilde{H}^{n-1}(\mathsf{Cliq}(\Gamma[U \setminus W]), \Bbbk)$$

for $n \geq 1$.

Proof. One has that $B(\Gamma)[U, W] = B(\Gamma[U \setminus W])$. Thus we may assume without loss of generality that $W = \emptyset$ and $U = V$. The maximal elements of $B(\Gamma) \setminus \{1\}$ are the elements of V. The commutation graph for this set is exactly Γ. Since $B(\Gamma)$ is a geometric left regular band, we conclude $\Delta(\widehat{0}, \widehat{1})$ is homotopy equivalent to $\mathsf{Cliq}(\Gamma)$ by Theorem 4.11. The theorem now follows from Theorem 4.1. $\square$

We present two immediate corollaries. The first characterizes the free partially commutative left regular bands with a hereditary $\Bbbk$-algebra.

Corollary 4.17. *If $\Bbbk$ is a field and Γ a finite graph, then the global dimension of $\Bbbk B(\Gamma)$ is the $\Bbbk$-Leray number $L_{\Bbbk}(\mathsf{Cliq}(\Gamma))$. In particular, $\Bbbk B(\Gamma)$ is hereditary if and only if Γ is a chordal graph.*

Our next corollary computes the quiver of the algebra of a free partially commutative left regular band.

Corollary 4.18. *Let $\Gamma = (V, E)$ be a finite graph. The quiver of $\Bbbk B(\Gamma)$ has vertex set the power set of V. If $U \supsetneq W$, then the number of edges from U to W is one less than the number of connected components of $\Gamma[U \setminus W]$. There are no other edges.*

Corollary 4.13 can be recovered from these results by specializing to the case that Γ has no edges.

It is easy to see that if Γ is *triangle-free*, that is, has no 3-element cliques, then $\mathsf{Cliq}(\Gamma) = \Gamma$ and so $L_{\Bbbk}(\mathsf{Cliq}(\Gamma)) = 2$ unless Γ is a forest (in which case Γ is chordal). This provides a natural infinite family of finite dimensional algebras of global dimension 2.

It is known that $L_{\Bbbk}(\mathsf{Cliq}(\Gamma))$ is bounded by the minimal number of chordal graphs needed to cover Γ [93, Theorem 13]. If C_n is the cycle with n nodes and P_n is the path with n nodes, then it is known that

$$L_{\Bbbk}(\mathsf{Cliq}(\overline{C}_n)) = L_{\Bbbk}(\mathsf{Cliq}(\overline{P}_n)) = \left\lfloor \frac{n-2}{3} \right\rfloor + 1$$

for $n \geq 3$ [93, Proposition 9]. If Γ_1, Γ_2 are two graphs, then

$$L_{\Bbbk}(\mathsf{Cliq}(\Gamma_1 * \Gamma_2)) = L_{\Bbbk}(\mathsf{Cliq}(\Gamma_1)) + L_{\Bbbk}(\mathsf{Cliq}(\Gamma_2))$$

where $*$ denotes the join of graphs; see [93, Lemma 8]. If $\overline{\Gamma}$ is chordal, then $L_{\Bbbk}(\mathsf{Cliq}(\Gamma))$ is the maximal size of an induced matching in $\overline{\Gamma}$ [93, Corollary 18]. If Γ is planar, $L_{\Bbbk}(\mathsf{Cliq}(\Gamma)) \leq 3$ [93, Proposition 23].

4.8. **Hyperplane face monoids.** We return to the setting of Section 3.2. Recall that $\mathcal{A}$ denotes a central hyperplane arrangement in a d-dimensional real vector space, $\mathcal{L}$ its intersection lattice, and $\mathcal{F}$ its monoid of faces. Without loss of generality, we can suppose that the intersection of all the hyperplanes in $\mathcal{A}$ is the origin: otherwise quotient the vector space by this intersection; the resulting monoid of faces is isomorphic to $\mathcal{F}$.

We argue that $\Delta(\widehat{0}, \widehat{1})$ is a $(d-1)$-sphere. Note that the $\mathscr{R}$-order on $\mathcal{F}$ can be described geometrically as $y \leq x$ if and only if $x \subseteq \overline{y}$, where $\overline{y}$ denotes the set-theoretic closure of y. This establishes an order-reversing bijection between the faces $\mathcal{F}$ and the cells of the regular cell decomposition Σ obtained by intersecting the hyperplane arrangement with a sphere centered at the origin. The dual of Σ is the boundary of a polytope Z (a zonotope, actually), and so the poset of faces of Z is isomorphic to $\mathcal{F}$ [25, Section 2E]. Since the order complex of the poset of faces of a polytope is the barycentric subdivision of the polytope, it follows that $\Delta(\widehat{0}, \widehat{1})$ is a $(d-1)$-sphere.

This argument also applies to $\Delta(X, Y)$ for $X \leq Y$ in $\mathcal{L}$. Indeed, $\mathcal{F}[X, Y]$ is the face monoid of the hyperplane arrangement in X obtained by intersecting X with the hyperplanes $H \in \mathcal{A}$ containing Y but not X. It follows that $\Delta(X, Y)$ is a sphere of dimension $\dim(X) - \dim(Y) - 1$. Consequently, we recover Lemma 8.3 of [80].

Proposition 4.19. *For $X, Y \in \mathcal{L}$ and $n \geq 0$,*

$$\mathrm{Ext}^n_{\Bbbk\mathcal{F}}(\Bbbk_X, \Bbbk_Y) \cong \begin{cases} \Bbbk, & \text{if } Y \subseteq X \text{ and } \dim(X) - \dim(Y) = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We apply Theorem 4.1. Since $\Delta(X, Y)$ is a sphere of dimension $\dim(X) - \dim(Y) - 1$, it follows that $\widetilde{H}^{n-1}(\Delta(X, Y), \Bbbk)$ is 0 unless $\dim(X) - \dim(Y) = n$, in which case it is $\Bbbk$. $\square$

It follows that the quiver of $\Bbbk\mathcal{F}$ coincides with the Hasse diagram of $\mathcal{L}$ ordered by reverse inclusion.

Corollary 4.20 (Saliola [80, Corollary 8.4]). *The quiver of $\Bbbk\mathcal{F}$ has vertex set $\mathcal{L}$. The number of arrows from X to Y is zero unless $Y \subsetneq X$ and $\dim(X) - \dim(Y) = 1$, in which case there is exactly one arrow.*

In [80] a set of quiver relations for $\Bbbk\mathcal{F}$ was described: for each interval of length two in $\mathcal{L}$ take the sum of all paths of length two in the interval. It was also shown that $\Bbbk\mathcal{F}$ is a Koszul algebra and that its Koszul dual algebra is isomorphic to the incidence algebra of the intersection lattice $\mathcal{L}$.

4.9. **Complex hyperplane face monoids.** We show that the situation for complex hyperplane arrangements is similar to that of real hyperplane arrangements. Things are slightly more complicated in this setting because interval submonoids of complex hyperplane face monoids are not again complex hyperplane face monoids. So we have to exploit much more the PL structure of the cell complex associated to the arrangement.

Fix a complex hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^d$ with augmented intersection lattice $\mathcal{L}$. The main technical result we shall need is the following.

Proposition 4.21. *Let $X, Y \in \mathcal{L}$ with $X < Y$. Then $\Delta(X, Y)$ is a sphere of dimension $\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} Y - 1$.*

Assuming for the moment the proposition, we obtain the following analogues of the results from the real case.

Proposition 4.22. *For $X, Y \in \mathcal{L}$ and $n \geq 0$,*

$$\mathrm{Ext}^n_{\Bbbk \mathcal{F}}(\Bbbk_X, \Bbbk_Y) \cong \begin{cases} \Bbbk, & if \ Y \subseteq X \ and \ \dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(Y) = n, \\ 0, & otherwise. \end{cases}$$

Proof. This is an application of Theorem 4.1. Since $\Delta(X, Y)$ is a sphere of dimension $\dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(Y) - 1$ by Proposition 4.21, it follows that $\widetilde{H}^{n-1}(\Delta(X, Y), \Bbbk)$ is 0 except for when $\dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(Y) = n$, in which case it is $\Bbbk$. $\qquad\square$

An immediate consequence is that the quiver of $\Bbbk \mathcal{F}$ coincides with the Hasse diagram of $\mathcal{L}$ ordered by reverse inclusion, as was the case for real hyperplane arrangements.

Corollary 4.23. *The quiver of $\Bbbk \mathcal{F}$ has vertex set $\mathcal{L}$. The number of arrows from X to Y is zero unless $Y \subsetneq X$ and $\dim(X) - \dim(Y) = 1$, in which case there is exactly one arrow.*

We now prove Proposition 4.21.

Proof of Proposition 4.21. Viewing $\mathcal{F}^{op}$ as the face poset of a regular CW decomposition of S^{2d-1}, one has that $\mathcal{F}^{op}_{\geq X}$ is the subcomplex of $\mathcal{F}^{op}$ corresponding to the intersection $X \cap S^{2d-1}$ (see [14, Theorem 3.5] and the discussion preceding it). In [17, Theorem 2.6], it is shown that there is a real hyperplane arrangement $\mathcal{A}'$ in $\mathbb{R}^{2d} = \mathbb{C}^d$ such that the regular CW complex decomposition of S^{2d-1} induced by $\mathcal{A}'$ is a subdivision of the CW decomposition associated to $\mathcal{A}$. Recalling that X is a real subspace of $\mathbb{C}^d$, it follows that the subcomplex $X \cap S^{2d-1}$ has a subdivision coming from the real hyperplane arrangement $\mathcal{A}'' = \{H \cap X \mid H \in \mathcal{A}' \ \text{and} \ X \nsubseteq H\}$ in X. Thus $\mathcal{F}^{op}_{\geq X}$ is the face poset of a PL regular CW decomposition of $S^{\dim_{\mathbb{R}} X - 1}$, cf. [16, Theorem 2.2.2]. Therefore, the order complex $\Delta(\mathcal{F}^{op}_{\geq X})$ is a PL sphere of dimension $\dim_{\mathbb{R}} X - 1$, cf. [16, Lemma 4.7.25].

Fix $F \in \mathcal{F}$ with support Y. Let c be a maximal chain in $\mathcal{F}^{op}_{\geq X}$ from the identity to F. It follows from Lemma 2.3 that the length of c is the same

as the length of the interval $[Y, \widehat{1}]$ of $\mathcal{L}$, which is $\dim_{\mathbb{R}} Y$. Next we observe that $\Delta(\mathcal{F}[X, Y]^{op})$ is the link of c in $\Delta(\mathcal{F}_{\geq X}^{op})$ and thus is a PL sphere of dimension $\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} Y - 1$ by [16, Theorem 4.7.21(iv)]. Since the order complex of a poset and its opposite are the same, we have $\Delta(X, Y) = \Delta(\mathcal{F}[X, Y]) \cong S^{\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} Y - 1}$. $\hfill\square$

5. Proof of the Main Result: the Algebraic Component

We begin the proof of the main result. In this section, we develop the tools that will allow us to recast the computation of $\mathrm{Ext}_{\mathbb{k}B}^{n}(\mathbb{k}_X, \mathbb{k}_Y)$ into one involving monoid cohomology and classifying spaces. No knowledge of monoid cohomology or classifying spaces is required to read this section.

Let $\mathbb{k}$ be a commutative ring with unit and fix a finite left regular band B. We begin by describing certain projective modules for a left regular band algebra. The most important case is when $\mathbb{k}$ is a field, but it is conceivable that the case $\mathbb{k} = \mathbb{Z}$ may be of interest in the future and so we do not restrict ourselves here. We use module unqualified to mean left module.

5.1. **Orthogonal idempotents.** Many of the results of the second author [78] from the case where $\mathbb{k}$ is a field generalize to any commutative ring with unit. In particular, he constructed a complete set of orthogonal idempotents that are defined over $\mathbb{Z}B$. In the case $\mathbb{k}$ is a field, they are the primitive idempotents. But they are useful in general since they give us a decomposition of $\mathbb{k}B$ into a direct sum of projective modules.

Fix for each $X \in \Lambda(B)$ an element f_X with $X = Bf_X$. Define e_X recursively by $e_{\widehat{0}} = f_{\widehat{0}}$ and, for $X > \widehat{0}$,

$$ e_X = f_X \left(1 - \sum_{Y < X} e_Y\right). \tag{5.1} $$

Notice that, by induction, one can write

$$ e_X = \sum_{b \in B} c_b b $$

with the c_b integers such that $f_X \geq b$ for all b with $c_b \neq 0$ and the coefficient of f_X in e_X is 1.

The following results are proved in Lemma 4.1 and Theorem 4.2 of [78] when $\mathbb{k}$ is a field, but the proofs make no use of the assumption that $\mathbb{k}$ is a field.

Theorem 5.1. *The elements $\{e_X\}_{X \in \Lambda(B)}$ enjoy the following properties:*
  (1) *if $b \in B$ and $X \in \Lambda(B)$ are such that $b \in B_{\not\geq X}$, then $be_X = 0$;*
  (2) *$\{e_X\}_{X \in \Lambda(B)}$ is a complete set of orthogonal idempotents, that is, one has $e_X e_Y = \delta_{X,Y} e_X$ and*

$$ \sum_{X \in \Lambda(B)} e_X = 1; $$

(3) *if $\Bbbk$ is a field, then e_X is a primitive idempotent.*

The following is [78, Corollary 4.4], but we must adapt the proof since we are not assuming $\Bbbk$ is a field.

Corollary 5.2. *The set $\{be_{\sigma(b)} \mid b \in B\}$ is a basis of idempotents for $\Bbbk B$.*

Proof. Since the map $a \mapsto ba$ is a homomorphism $B \longrightarrow bB$, it follows that left multiplication by b is a ring homomorphism $\Bbbk B \longrightarrow \Bbbk bB$ and hence $be_{\sigma(b)}$ is an idempotent. Consider the module homomorphism $\varphi \colon \Bbbk B \longrightarrow \Bbbk B$ given by $b \mapsto be_{\sigma(b)}$. By the remarks before Theorem 5.1 and the fact that $bf_{\sigma(b)} = b$, it follows that $be_{\sigma(b)}$ is an integral linear combination of elements of bB and that the coefficient of b itself is 1. Thus if we order B in a way compatible with the $\mathscr{R}$-order, then the matrix of φ is unitriangular and hence invertible over any commutative ring with unit. This establishes the corollary. $\qquad\square$

5.2. **Schützenberger representations.** Next we recall the classical (left) Schützenberger representation associated to an element $X \in \Lambda(B)$. Let $L_X = \sigma^{-1}(X)$; it is called an $\mathscr{L}$-class in the semigroup theory literature [32]. Define a $\Bbbk B$-module structure on $\Bbbk L_X$ by putting, for $a \in B$ and $b \in L_X$,

$$a \cdot b = \begin{cases} ab & \text{if } \sigma(a) \geq X \\ 0 & \text{else.} \end{cases}$$

It is proved by the second author [78] that $\Bbbk Be_X \cong \Bbbk L_X$ when $\Bbbk$ is a field. The argument is easily adapted to the general case. Namely, using Corollary 5.2, the proof of [78, Lemma 5.1] goes through to show that a $\Bbbk$-basis for $\Bbbk Be_X$ is the set $\{be_X \mid \sigma(b) = X\}$. It is then shown in [78, Proposition 5.2] that the map $\varphi \colon \Bbbk L_X \longrightarrow \Bbbk Be_X$ given by $b \mapsto be_X$ for $b \in L_X$ is a $\Bbbk B$-module homomorphism. Since it sends a basis to a basis, it is an isomorphism. Putting everything together we have the following theorem.

Theorem 5.3. *Let B be a left regular band and $\Bbbk$ a commutative ring with unit. Then the Schützenberger representations $\Bbbk L_X$ with $X \in \Lambda(X)$ are projective and*

$$\Bbbk B \cong \bigoplus_{X \in \Lambda(B)} \Bbbk L_X.$$

If $\Bbbk$ is a field, then this is the decomposition of $\Bbbk B$ into projective indecomposables.

If $X \in \Lambda(B)$, there is a $\Bbbk$-algebra homomorphism $\rho_X \colon \Bbbk B \longrightarrow \Bbbk B_{\geq X}$ given by

$$\rho_X(b) = \begin{cases} b & \text{if } \sigma(b) \geq X \\ 0 & \text{else.} \end{cases}$$

This homomorphism allows us to consider any $\Bbbk B_{\geq X}$-module M as a $\Bbbk B$-module via the action $b \cdot m = \rho_X(b)m$ for all $b \in B$ and $m \in M$.

The kernel of ρ_X is the ideal $\Bbbk B_{\not\geq X}$. If $Y \geq X$, then $\Bbbk L_Y$ is a $\Bbbk B_{\geq X}$-module and is the corresponding projective module for $\Bbbk B_{\geq X}$. We thus obtain the following corollary of Theorem 5.3.

Corollary 5.4. *If $X \in \Lambda(B)$, then $\Bbbk B_{\geq X}$ is a projective $\Bbbk B$-module and the decomposition*

$$\Bbbk B = \Bbbk B_{\geq X} \oplus \bigoplus_{Y \not\geq X} \Bbbk L_Y$$

holds. Consequently, any projective $\Bbbk B_{\geq X}$-module is a projective $\Bbbk B$-module (via ρ_X).

As a corollary, it follows that we can compute Ext in either $\Bbbk B$ or $\Bbbk B_{\geq X}$ for $\Bbbk B_{\geq X}$-modules.

Corollary 5.5. *Let $X \in \Lambda(B)$ and let M, N be $\Bbbk B_{\geq X}$-modules. Then*

$$\mathrm{Ext}^n_{\Bbbk B}(M, N) \cong \mathrm{Ext}^n_{\Bbbk B_{\geq X}}(M, N)$$

for all $n \geq 0$.

Proof. Corollary 5.4 implies any projective resolution of M over $\Bbbk B_{\geq X}$ is also a projective resolution over $\Bbbk B$. The result is now immediate. $\square$

5.3. **Computation of** $\mathrm{Ext}^n_{\Bbbk B}(\Bbbk_X, \Bbbk_Y)$. If $X \in \Lambda(B)$, let $\Bbbk_X$ be the trivial $\Bbbk B_{\geq X}$-module. It is then a $\Bbbk B$-module via ρ_X. There is a surjective homomorphism $\varepsilon_X \colon \Bbbk L_X \longrightarrow \Bbbk_X$ sending each element of L_X to 1. If $\Bbbk$ is a field, then the modules $\Bbbk_X$ form a complete set of non-isomorphic simple $\Bbbk B$-modules and $\varepsilon_X \colon \Bbbk L_X \longrightarrow \Bbbk_X$ is the projective cover [23, 62, 78]. In general, the simple $\Bbbk B$-modules are the modules of the form $(\Bbbk/\mathfrak{m})_X$ where $\mathfrak{m}$ is a maximal ideal of $\Bbbk$, as follows from the results of [43].

We begin with a construction of a projective resolution of the modules $\Bbbk_X$ with $X \in \Lambda(B)$. Recall that these are the simple $\Bbbk B$-modules when $\Bbbk$ is a field.

Proposition 5.6. *Let B be a finite left regular band and let $X \in \Lambda(B)$. Define a chain complex $C_\bullet(B, X)$ by putting $C_n(B, X)$ to be the free $\Bbbk B_{\geq X}$-module on the set $B[X, \widehat{1})^n = (B_{\geq X} \setminus \{1\})^n$. Denote a basis element by $[s_0|\cdots|s_{n-1}]$. When $n = 0$, the unique basis element is denoted $[\,]$. The augmentation $\varepsilon_X \colon C_0(B, X) \longrightarrow \Bbbk_X$ sends $[\,]$ to 1. Define*

$$d_n \colon C_n(B, X) \longrightarrow C_{n-1}(B, X),$$

for $n \geq 1$, by

$$d_n([s_0|\cdots|s_{n-1}]) = s_0[s_1|\cdots|s_{n-1}]$$

$$+ \sum_{i=1}^{n-1} (-1)^i [s_0|\cdots|s_{i-1}s_i|\cdots|s_{n-1}]$$

$$+ (-1)^n [s_0|\cdots|s_{n-2}]$$

Then $C_\bullet(B, X) \longrightarrow \Bbbk_X$ is a projective resolution of $\Bbbk_X$ as a $\Bbbk B$-module.

Proof. Observe that $C_\bullet(B, X) \longrightarrow \Bbbk_X$ is the normalized bar resolution of $\Bbbk_X$ as a $\Bbbk B_{\geq X}$-module (see [60, Chapter X]). The result now follows from Corollary 5.4. $\square$

As a consequence, we can show that $\mathrm{Ext}_{\Bbbk B}^n(\Bbbk_X, \Bbbk_Y)$ vanishes when $Y \not\geq X$.

Proposition 5.7. *Let B be a finite left regular band and $\Bbbk$ a commutative ring with unit. Let $X, Y \in \Lambda(B)$ and assume $Y \not\geq X$. Then*

$$\mathrm{Ext}_{\Bbbk B}^n(\Bbbk_X, \Bbbk_Y) = 0$$

for all $n \geq 0$.

Proof. Let $a \in B$ with $\sigma(a) = Y$. Then a annihilates $\Bbbk B_{\geq X}$ and acts as the identity on $\Bbbk_Y$. Thus $\mathrm{Hom}_{\Bbbk B}(\Bbbk B_{\geq X}, \Bbbk_Y) = 0$. The result now follows by using the resolution in Proposition 5.6 to compute $\mathrm{Ext}_{\Bbbk B}^n(\Bbbk_X, \Bbbk_Y)$. $\square$

We are left studying the case that $X \leq Y$. This can be recast in terms of monoid cohomology, which we do in the next section.

6. Proof of the Main Result: Classifying Spaces and Cohomology

We now turn to classifying spaces and the cohomology of monoids and categories. The cohomology of monoids, which is a natural generalization of group cohomology [22], is both a special case of the cohomology of augmented algebras [27] and of the cohomology of small categories [90]. Although we are mostly interested in monoid cohomology, we will also need the cohomology of small categories. The main example of a small category that we need that is neither a monoid nor a poset is the semidirect product of a monoid with a set.

The approach we take is inspired by the paper of Nunes [69]. In what follows, M will denote a monoid and the set of idempotents of M will be denoted by $E(M)$. Of course, if M is a band, then $M = E(M)$.

6.1. **Cohomology of a small category.** Let $\mathscr{C}$ be a small category and fix a commutative ring $\Bbbk$ with unit. By a *left $\mathscr{C}$-module (over $\Bbbk$)* we mean a (covariant) functor $F: \mathscr{C} \longrightarrow \Bbbk\text{-}\mathbf{mod}$. For instance, if we view a monoid M as a one-object category, then a left M-module (over $\Bbbk$) is the same thing as a left $\Bbbk M$-module. The category $\Bbbk\text{-}\mathbf{mod}^{\mathscr{C}}$ of left $\mathscr{C}$-modules is well known to be an abelian category with enough projectives and injectives [64].

The morphisms between $\mathscr{C}$-modules F, G are natural transformations and we write $\mathrm{Hom}_{\Bbbk\mathscr{C}}(F, G)$ for the morphism set.

There is a functor $\Delta\colon \Bbbk\text{-}\mathbf{mod} \longrightarrow \Bbbk\text{-}\mathbf{mod}^{\mathscr{C}}$ that sends a $\Bbbk$-module V to the constant functor $\Delta(V)\colon \mathscr{C} \longrightarrow \Bbbk\text{-}\mathbf{mod}$ that sends all objects to V and all arrows to the identity 1_V. For example, if M is a monoid, then $\Delta(V)$ is V with the trivial $\Bbbk M$-module structure. The functor Δ has a right adjoint $\varprojlim$ [61]. For instance, if V is a $\Bbbk M$-module, then $\varprojlim V$ is the $\Bbbk$-module of M-invariants. One has a natural isomorphism $\varprojlim F \cong \mathrm{Hom}_{\Bbbk\mathscr{C}}(\Delta(\Bbbk), F)$ for any left $\mathscr{C}$-module F. Thus the right derived functors $R^n \varprojlim$ can be identified with $\mathrm{Ext}^n_{\Bbbk\mathscr{C}}(\Delta(\Bbbk), -)$ (where the subscript "$\Bbbk\mathscr{C}$" is to indicate we are considering left $\mathscr{C}$-modules over $\Bbbk$). One defines the *cohomology* of $\mathscr{C}$ with coefficients in F by

$$H^n(\mathscr{C}, F) = \mathrm{Ext}^n_{\Bbbk\mathscr{C}}(\Delta(\Bbbk), F) = R^n \varprojlim F$$

where R^n denotes the nth-right derived functor. When $\mathscr{C}$ is a monoid or a group, this agrees with usual monoid and group cohomology. See [8,42,73,90] for more details.

We are primarily interested in monoid cohomology. In this case, we will write $\Bbbk$ instead of $\Delta(\Bbbk)$, as is customary. In particular, if M is a monoid and V is a $\Bbbk M$-module, then $H^n(M, V) = \mathrm{Ext}^n_{\Bbbk M}(\Bbbk, V)$. See [60, Chapter X] for associated bar resolutions.

There was some work on monoid cohomology in the late sixties and early seventies [2, 65, 67]. Since the nineties, there has been a surge in papers on monoid homology and cohomology, in part due to connections with string rewriting systems [21, 34, 45, 47, 48, 52–54, 69, 71, 85].

The following proposition follows immediately from Corollary 5.5 and the definition of monoid cohomology.

Proposition 6.1. *Let B be a finite left regular band and $\Bbbk$ a commutative ring with unit. Let $X \in \Lambda(B)$. There is a natural isomorphism of functors*

$$\mathrm{Ext}^n_{\Bbbk B}(\Bbbk_X, -) \cong H^n(B_{\geq X}, -)$$

from the category of $\Bbbk B_{\geq X}$-modules to the category of $\Bbbk$-modules. In particular, if $Y \geq X$, then

$$\mathrm{Ext}^n_{\Bbbk B}(\Bbbk_X, \Bbbk_Y) \cong H^n(B_{\geq X}, \Bbbk_Y) \tag{6.1}$$

for all $n \geq 0$.

6.2. **The Eckmann-Shapiro lemma.** In what follows we shall need a variant of the Eckmann-Shapiro lemma for monoids. It will be convenient to use a very general Eckmann-Shapiro type lemma in the context of abelian categories that was proved by Adams and Rieffel in their study of semigroup cohomology [2, Theorem 1].

Theorem 6.2. *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be abelian categories such that $\mathscr{A}$ has enough injectives and suppose that one has a diagram of additive functors*

$$\mathscr{A} \underset{T}{\overset{S}{\rightleftarrows}} \mathscr{B} \overset{F}{\longrightarrow} \mathscr{C}$$

such that S is right adjoint to T and S, T are exact. Then there is a natural isomorphism of functors $R^n(F \circ S) \cong (R^n F) \circ S$ for all $n \geq 0$.

Let $\varphi \colon M \longrightarrow N$ be a semigroup homomorphism of monoids. That is, $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)$ for all $m_1, m_2 \in M$, but we do not assume $\varphi(1) = 1$. Let $e = \varphi(1)$; it is an idempotent. Then $\Bbbk Ne$ is a $\Bbbk N$-$\Bbbk M$-bimodule and $\Bbbk eN$ is a $\Bbbk M$-$\Bbbk N$-bimodule. If V is a $\Bbbk N$-module, we have that eV is a $\Bbbk M$-module via the action $mv = \varphi(m)v$. Notice that $1v = ev = v$ since $v \in eV$. The functor $V \mapsto eV$ is exact and has left adjoint $W \mapsto \Bbbk Ne \otimes_{\Bbbk M} W$ and right adjoint $W \mapsto \mathrm{Hom}_M(eN, W)$ where $\mathrm{Hom}_M(eN, W)$ is the set of all left M-set morphisms $f \colon eN \longrightarrow W$ with pointwise $\Bbbk$-module structure and N-action $nf(n') = f(n'n)$. Notice that $\mathrm{Hom}_M(eN, W) \cong \mathrm{Hom}_{\Bbbk M}(\Bbbk eN, W)$ via restriction to the basis eN. Thus $\mathrm{Hom}_M(eN, -)$ is exact whenever $\Bbbk eN$ is a projective $\Bbbk M$-module. Theorem 6.2 specializes in this context as follows.

Lemma 6.3. *Let $\varphi \colon M \longrightarrow N$ be a semigroup homomorphism of monoids and let $e = \varphi(1)$. Let $\Bbbk$ be a commutative ring with unit and suppose that $\Bbbk eN$ is a projective $\Bbbk M$-module. Then, for any $\Bbbk M$-module W and $\Bbbk N$-module V, one has a natural isomorphism*

$$\mathrm{Ext}^n_{\Bbbk M}(eV, W) \cong \mathrm{Ext}^n_{\Bbbk N}(V, \mathrm{Hom}_M(eN, W))$$

for all $n \geq 0$. In particular,

$$H^n(M, W) \cong H^n(N, \mathrm{Hom}_M(eN, W))$$

for all $n \geq 0$.

Proof. In Theorem 6.2, take $\mathscr{A} = \Bbbk M$-**mod**, $\mathscr{B} = \Bbbk N$-**mod** and $\mathscr{C} = \Bbbk$-**mod**, take $T(V) = eV$, $S(W) = \mathrm{Hom}_M(eN, W)$ and $F = \mathrm{Hom}_{\Bbbk N}(V, -)$ and use that $F \circ S = \mathrm{Hom}_{\Bbbk N}(V, \mathrm{Hom}_M(eN, -)) \cong \mathrm{Hom}_{\Bbbk M}(eV, -)$. The final statement follows because $e\Bbbk = \Bbbk$. $\qquad\square$

A corollary we shall use later for a dimension-shifting argument is the following. Let $e \in E(M)$ be an idempotent and suppose that V is any $\Bbbk$-module. Then V^{eM} with the natural $\Bbbk M$-module structure given by $mf(m') = f(m'm)$ is acyclic for cohomology.

Corollary 6.4. *Let V be a $\Bbbk$-module and $e \in E(M)$. Then*

$$H^n(M, V^{eM}) \cong \begin{cases} V & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

Proof. Let $\{1\}$ denote the trivial monoid and consider the semigroup homomorphism $\varphi\colon \{1\} \longrightarrow M$ with $\varphi(1) = e$. The algebra of the trivial monoid is $\Bbbk$ and $\Bbbk eM$ is a free $\Bbbk$-module. Thus Lemma 6.3 applies. Observe that $\mathrm{Hom}_1(eM, V) = V^{eM}$ and so $H^n(M, V^{eM}) \cong H^n(\{1\}, V)$. The result follows. $\square$

6.3. **Classifying space of a small category.** To each small category $\mathscr{C}$, there is naturally associated a CW complex $\mathcal{B}\mathscr{C}$ called the *classifying space of* $\mathscr{C}$. Usually, $\mathcal{B}\mathscr{C}$ is defined as the geometric realization of a certain simplicial set called the nerve of $\mathscr{C}$ [86], but we follow instead the construction in [77, Definition 5.3.15].

There is a 0-cell of $\mathcal{B}\mathscr{C}$ for each object of the category $\mathscr{C}$. For $q \geq 1$, there is a q-cell for each diagram

$$c_0 \xrightarrow{\,f_0\,} c_1 \xrightarrow{\,f_1\,} \cdots \xrightarrow{\,f_{q-2}\,} c_{q-1} \xrightarrow{\,f_{q-1}\,} c_q \tag{6.2}$$

with no f_i an identity arrow. This q-cell (which should be thought of as a q-simplex) is attached in the obvious way to any cell of smaller dimension that can be obtained by deleting some c_i and, if $i \notin \{0, q\}$, replacing f_{i-1} and f_i by $f_i f_{i-1}$ (where if this is an identity morphism then we delete this arrow as well). Notice that $\mathcal{B}\mathscr{C}$ is homeomorphic to $\mathcal{B}\mathscr{C}^{op}$ [73].

A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a cellular map $\mathcal{B}F\colon \mathcal{B}\mathscr{C} \longrightarrow \mathcal{B}\mathscr{D}$ by applying F to (6.2) and deleting identity morphisms. Thus $\mathcal{B}$ is a functor from the category of small categories to the category of CW complexes.

Let us give some examples. If G is a group, viewed as a one-object category, then $\mathcal{B}G$ is Milnor's classifying space of G and is an Eilenberg-Mac Lane $K(G, 1)$-space. When M is a monoid, viewed as a one-object category, one has that $\mathcal{B}M$ is the classical classifying space of M. It is known that every CW complex is homotopy equivalent to the classifying space of a monoid [63]. It is easy to see that if V is a $\Bbbk$-module, viewed as a $\Bbbk M$-module via the trivial action, then the co-chain complex associated to the cellular cohomology of $\mathcal{B}M$ with coefficients in V is precisely the co-chain complex used to compute $H^n(M, V)$ if one uses the standard bar resolution [60, Chapter X] for the trivial $\Bbbk M$-module. Thus $H^\bullet(M, V) = H^\bullet(\mathcal{B}M, V)$. More generally, one has, the following theorem, see for instance [73], [90, Theorem 5.3] or [42, Appendix II, 3.3].

Theorem 6.5. *For a small category $\mathscr{C}$, one has $H^\bullet(\mathscr{C}, \Delta(V)) \cong H^\bullet(\mathcal{B}\mathscr{C}, V)$ for any $\Bbbk$-module V.*

Next we recall that a poset P can be viewed as a category. The object set of P is P itself. The arrow set is $\{(p, q) \in P \times P \mid p \leq q\}$. The arrow (p, q) goes from p to q and composition is given by $(q, r)(p, q) = (p, r)$. The identity at p is (p, p). Functors between posets are precisely order preserving maps. Galois connections are adjunctions between posets. Notice the q-cells of $\mathcal{B}P$ are precisely the chains of length q in P and the gluing is by homeomorphisms of the faces. Thus $\mathcal{B}P$ is the *order complex* $\Delta(P)$; see [73].

A key result of Segal [83] is the following (cf. [77, Lemma 5.3.17]).

Lemma 6.6 (Segal)**.** *If $F, G\colon \mathscr{C} \longrightarrow \mathscr{D}$ are functors between small categories and there is a natural transformation $F \Rightarrow G$, then $\mathcal{B}F$ and $\mathcal{B}G$ are homotopic.*

In particular, if $\mathscr{C}$ and $\mathscr{D}$ are naturally equivalent, then $\mathcal{B}\mathscr{C}$ and $\mathcal{B}\mathscr{D}$ are homotopy equivalent. More generally, if a functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ has an adjoint, then $\mathcal{B}\mathscr{C}$ and $\mathcal{B}\mathscr{D}$ are homotopy equivalent [86, Corollary 3.7]. Actually, we have the following more general consequence.

Proposition 6.7. *Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ be functors such that there exist natural transformations between GF and $1_{\mathscr{C}}$ and between FG and $1_{\mathscr{D}}$, in either direction. Then $\mathcal{B}\mathscr{C}$ is homotopy equivalent to $\mathcal{B}\mathscr{D}$.*

An important special case is the well-known fact that a Galois connection between posets yields a homotopy equivalence of order complexes [13]. The following corollary is the special case of Rota's cross-cut theorem [13] that we used earlier (Theorem 4.10).

Corollary 6.8 (Rota)**.** *Let P be a finite poset such that any subset of P with a common lower bound has a meet. Define a simplicial complex K with vertex set the set $\mathscr{M}(P)$ of maximal elements of P and with simplices those subsets of $\mathscr{M}(P)$ with a common lower bound. Then K is homotopy equivalent to the order complex $\Delta(P)$.*

Proof. Let $\mathscr{F}$ be the face poset of K. Then it is well known that $\Delta(\mathscr{F})$ is the barycentric subdivision of K and hence homeomorphic to it. Thus it suffices, by Proposition 6.7, to establish a contravariant Galois connection between P and $\mathscr{F}$. Define $F\colon P \longrightarrow \mathscr{F}$ by $F(p) = \{m \in \mathscr{M}(P) \mid m \geq p\}$ and $G\colon \mathscr{F} \longrightarrow P$ by $G(X) = \bigwedge X$ for a simplex X of K. Then $X \leq F(p)$ if and only if $p \leq G(X)$ and so F and G form a contravariant Galois connection. $\square$

Another well-known corollary is that a category with a terminal object has a contractible classifying space [86, Corollary 3.7].

Corollary 6.9. *If $\mathscr{C}$ is a category with a terminal object, then $\mathcal{B}\mathscr{C}$ is contractible.*

Proof. Let $\{1\}$ denote the trivial monoid and let t be the terminal object of $\mathscr{C}$. Then the unique functor $F\colon \mathscr{C} \longrightarrow \{1\}$ and the functor $G\colon \{1\} \longrightarrow \mathscr{C}$ sending the unique object of $\{1\}$ to t form an adjoint pair. $\square$

Let us prove the folklore result that a monoid with a left zero element has a contractible classifying space. Our proof is easier than the one in [63].

Proposition 6.10 (Folklore)**.** *Let M be a monoid with a left zero element. Then $\mathcal{B}M$ is contractible.*

Proof. Let $\varphi\colon M \longrightarrow \{1\}$ be the trivial homomorphism and $\psi\colon \{1\} \longrightarrow M$ be the inclusion. Trivially, $\varphi\psi = 1_{\{1\}}$. Next we define a natural transformation $\eta\colon 1_M \Rightarrow \psi\varphi$. Let z be the left zero. The component of η at the unique

object of M is z. Then one has, for $m \in M$, that $zm = z = \psi\varphi(m)z$, i.e., η is a natural transformation. Thus $\mathcal{B}M$ is contractible. $\square$

6.4. **Quillen's Theorem A.** An important tool for determining whether a functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a homotopy equivalence of classifying spaces is the famous Quillen's Theorem A [73, 86]. If d is an object of $\mathscr{D}$, then the *left fiber category* F/d has object set all morphisms $f\colon F(c) \longrightarrow d$ with c an object of $\mathscr{C}$. A morphism from $f\colon F(c) \longrightarrow d$ to $f'\colon F(c') \longrightarrow d$ in F/d is a morphism $g\colon c \longrightarrow c'$ such that

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\ \ F(g)\ \ } & F(c') \\
 & {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle f'} & \\
 & d &
\end{array}
$$

commutes.

Remark 6.11. If $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a functor with $\mathscr{D}$ a poset, then there is at most one arrow $F(x) \longrightarrow d$. Thus F/d can be identified with the full subcategory of $\mathscr{C}$ with object set $F^{-1}(D_{\leq d})$ where $D_{\leq d}$ consists of all objects of $\mathscr{D}$ less than or equal to d in the order.

We now state Quillen's Theorem A; see [73, 86] for a proof.

Theorem 6.12 (Quillen's Theorem A). *Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor such that the left fiber categories F/d have contractible classifying spaces for all objects d of $\mathscr{D}$. Then F induces a homotopy equivalence of classifying spaces.*

6.5. **Semidirect product of a monoid with a set.** The main example of a small category which is not a monoid or a poset that we shall need is the semidirect product of a monoid with a set. Let M be a monoid and let X be a right M-set. The *semidirect product* (also known as the *category of elements*, or the *Grothendieck construction*) $X \rtimes M$ has object set X and arrow set $X \times M$. An arrow (x, m) has domain xm and range x, and we draw it

$$
(x, m) = \ xm \xrightarrow{\ \ m\ \ } x \ .
$$

Composition is given by $(x, m)(xm, n) = (x, mn)$, or in pictures

$$
xmn \xrightarrow{\ \ n\ \ } xm \xrightarrow{\ \ m\ \ } x \ .
$$
$$
\underbrace{\phantom{xmn \xrightarrow{\ \ n\ \ } xm \xrightarrow{\ \ m\ \ } x}}_{mn}
$$

The identity at x is $(x, 1)$. The assignment $X \mapsto X \rtimes M$ is a functor from the category of right M-sets to the category of small categories.

There is an exact pair of adjoint functors between the categories $\Bbbk M\text{-}\mathbf{mod}$ and $\Bbbk\text{-}\mathbf{mod}^{X \rtimes M}$ to which we can apply the Adams-Rieffel theorem. Namely, if V is a $\Bbbk M$-module, we can define a left $X \rtimes M$-module P_V on objects by $P_V(x) = V$ for all $x \in X$. We define $P_V(x, m)\colon V \longrightarrow V$ on a morphism (x, m) by $v \mapsto mv$. The functor $P\colon \Bbbk M\text{-}\mathbf{mod} \longrightarrow \Bbbk\text{-}\mathbf{mod}^{X \rtimes M}$ given by

$P(V) = P_V$ on objects (with the obvious effect on morphisms) is clearly exact. It has right adjoint G which sends a left $X \rtimes M$-module Q to the direct product $\prod_{x \in X} Q(x)$ with action given by $(mf)(x) = Q(x, m)f(xm)$ where we view an element of the direct product as a mapping $f \colon X \longrightarrow \coprod_{x \in X} Q(x)$ with $f(x) \in Q(x)$ for all $x \in X$. Plainly G is an exact functor. The isomorphism

$$\mathrm{Hom}_{\Bbbk(X \rtimes M)}(P(V), Q) \longrightarrow \mathrm{Hom}_{\Bbbk M}(V, G(Q))$$

sends a natural transformation $\eta \colon P_V \Rightarrow Q$ to the mapping $\prod_{x \in X} \eta_x$ where $\eta_x \colon V \longrightarrow Q(x)$ is the component of η at the object x. Conversely, any homomorphism $\eta \colon V \longrightarrow G(Q)$ gives rise to a natural transformation $P_V \Rightarrow Q$ whose component at x is the composition of η with the projection to the factor $Q(x)$.

Theorem 6.13. *Let M be a monoid and X a right M-set. Then one has a natural isomorphism*

$$\mathrm{Ext}^{\bullet}_{\Bbbk(X \rtimes M)}(P(V), Q) \cong \mathrm{Ext}^{\bullet}_{\Bbbk M}(V, G(Q))$$

where P and G are the adjoint functors considered above. In particular, one has

$$H^{\bullet}(X \rtimes M, Q) \cong H^{\bullet}(M, G(Q)).$$

Proof. Apply Theorem 6.2 taking $\mathscr{A} = \Bbbk\text{-}\mathbf{mod}^{X \rtimes M}$, $\mathscr{B} = \Bbbk M\text{-}\mathbf{mod}$, $\mathscr{C} = \Bbbk\text{-}\mathbf{mod}$, $S = G$, $T = P$ and $F = \mathrm{Hom}_{\Bbbk M}(V, -)$. One uses that $F \circ S = \mathrm{Hom}_{\Bbbk M}(V, G(-)) \cong \mathrm{Hom}_{\Bbbk(X \rtimes M)}(P(V), -)$. The final statement follows because $P(\Bbbk) = \Delta(\Bbbk)$. $\square$

Let V be a $\Bbbk$-module. Then one computes readily that $G(\Delta(V)) = V^X$ with the $\Bbbk M$-module structure given via the left action $mf(x) = f(xm)$. Thus we have the following corollary to Theorem 6.13. It is the dual of a result of Nunes proved for the case of homology [69]; see also the appendix of [59]. Our proof though is more conceptual.

Corollary 6.14. *Let M be a monoid and $\Bbbk$ a commutative ring with unit. Suppose that X is a right M-set and V is a $\Bbbk$-module. Then*

$$H^n(M, V^X) \cong H^n(X \rtimes M, \Delta(V)) \cong H^n(\mathcal{B}(X \rtimes M), V)$$

for $n \geq 0$.

From now on, we say that a right M-set X is *contractible* if $\mathcal{B}(X \rtimes M)$ is contractible. Let X be a right M-set and denote by $\Omega(X)$ the poset of all cyclic M-subsets xM with $x \in X$ ordered by inclusion. There is a natural functor $\Phi_X \colon X \rtimes M \longrightarrow \Omega(X)$ given by $x \mapsto xM$ on objects and by sending the arrow $(x, m) \colon xm \longrightarrow x$ to the unique arrow $xmM \longrightarrow xM$. The following result is a key tool in computing the global dimension of left regular band algebras.

Theorem 6.15. *Suppose that X is a right M-set such that each cyclic M-subset xM with $x \in X$ is contractible. Then $\Phi_X \colon X \rtimes M \longrightarrow \Omega(X)$ induces a homotopy equivalence of classifying spaces.*

Proof. By Quillen's Theorem A it suffices to show $\mathcal{B}(\Phi_X/xM)$ is contractible for each $x \in X$. But by Remark 6.11, one has that Φ_X/xM is the full subcategory of $X \rtimes M$ on those objects $y \in X$ with $yM \subseteq xM$. But this subcategory is precisely $xM \rtimes M$, which has contractible classifying space by assumption. Thus Φ_X is a homotopy equivalence. $\square$

The key example of a monoid M and a right M-set to which the theorem will be applied is a right ideal X of a von Neumann regular (or more generally, right PP, defined below) monoid M.

Proposition 6.16. *The right M-set M is contractible.*

Proof. Notice that 1 is a terminal object of $M \rtimes M$ since $(1, m) \colon m \longrightarrow 1$ is the unique arrow from m to 1. Therefore, $\mathcal{B}(M \rtimes M)$ is contractible by Corollary 6.9. $\square$

Let $\mathbf{Set}^{M^{op}}$ be the category of right M-sets. This is a special case of a presheaf category and so the statements below can be obtained from the more general statements in this context that one can find in a standard text in category theory, e.g., [19]. The epimorphisms in $\mathbf{Set}^{M^{op}}$ are precisely the surjective maps. Projective objects in this category are defined in the usual way. An M-set is *indecomposable* if it cannot be expressed as a coproduct (equals disjoint union) of two M-sets. Up to isomorphism the projective indecomposable M-sets are those of the form eM with e an idempotent of M. These are also the cyclic projective M-sets up to isomorphism.

Proposition 6.17. *A projective indecomposable right M-set is contractible.*

Proof. Let P be a projective indecomposable right M-set. Then P is cyclic, being isomorphic to eM for some idempotent e, and so there is an epimorphism $\varphi \colon M \longrightarrow P$. Then φ splits and so P is a retract of M. Hence, by functoriality, $P \rtimes M$ is a retract of $M \rtimes M$ and so $\mathcal{B}(P \rtimes M)$ is a retract of $\mathcal{B}(M \rtimes M)$. Since retracts of contractible spaces are contractible, we are done by Proposition 6.16. $\square$

6.6. **Application to right PP monoids.** A monoid M is called a *right PP monoid* if each principal right ideal mM is projective in $\mathbf{Set}^{M^{op}}$. For example, every band is a right PP monoid, as is every von Neumann regular monoid. Right PP monoids were first characterized in [39]. The following description of right PP monoids is due to Fountain [41]. Two elements m, n of M are said to be $\mathscr{L}^*$-*equivalent* if $mx = my$ if and only if $nx = ny$ for all $x, y \in M$. Note that $m \mathscr{L}^* n$ if and only if there is an isomorphism mM to nM taking m to n. A monoid is right PP if and only if each $\mathscr{L}^*$-class of M contains an idempotent [39, 41].

Recall that $\mathscr{R}$ denotes the relation associated to Green's $\mathscr{R}$-preorder on M (see Section 2.3). If R is a right ideal of M, then the poset $R/\mathscr{R}$ can be identified with the poset of principal right ideals of M contained in R.

Corollary 6.18. *Let M be a right PP monoid, for example a left regular band, and let R be a right ideal of M. Then $\mathcal{B}(R \rtimes M)$ is homotopy equivalent to $\Delta(R/\mathscr{R})$.*

Proof. In this case, $\Omega(R) = R/\mathscr{R}$. Since M is right PP, each principal ideal mM is projective and hence contractible by Proposition 6.17. Theorem 6.15 now provides the desired result. $\square$

We continue to denote by $\Bbbk$ a commutative ring with unit. The following results are related to results of Nunes [69], proved using spectral sequences, in the context of monoid homology. Let M be a monoid and $e \in E(M)$ an idempotent. Suppose that $\emptyset \neq R \subsetneq eM$ is a right ideal. Let V be a $\Bbbk$-module. The exact sequence of right $\Bbbk M$-modules

$$0 \longrightarrow \Bbbk R \longrightarrow \Bbbk eM \longrightarrow \Bbbk eM/\Bbbk R \longrightarrow 0$$

gives rise to an exact sequence of left $\Bbbk M$-modules

$$0 \longrightarrow W \longrightarrow V^{eM} \longrightarrow V^R \longrightarrow 0 \qquad (6.3)$$

where $W = \{f \in V^{eM} \mid f(R) = 0\}$ and the map $V^{eM} \longrightarrow V^R$ is given by restriction.

Theorem 6.19. *Let M be a monoid and $e \in E(M)$. Let $\emptyset \neq R \subsetneq eM$ be a right ideal and V a $\Bbbk$-module. Let $W = \{f \in V^{eM} \mid f(R) = 0\}$. Then*

$$H^{n+1}(M, W) \cong \widetilde{H}^n(\mathcal{B}(R \rtimes M), V)$$

for all $n \geq 0$. Moreover, if M is a right PP monoid then

$$H^{n+1}(M, W) \cong \widetilde{H}^n(\Delta(R/\mathscr{R}), V)$$

where $R/\mathscr{R}$ is the poset of principal right ideals contained in R.

Proof. Corollary 6.4 implies that $H^n(M, V^{eM}) = 0$ for $n \geq 1$. Thus the long exact sequence for cohomology applied to (6.3) yields a short exact sequence

$$0 \longrightarrow H^n(M, V^R) \longrightarrow H^{n+1}(M, W) \longrightarrow 0$$

for each $n \geq 1$. In light of Corollary 6.14, this proves the first statement of the theorem for $n \geq 1$. Let us examine the initial terms of the long exact sequence. Note that $H^0(M, W) = 0$ because if $f \in W$ is fixed by M and $r \in R, m \in M$, then $f(em) = mf(e) = mrf(e) = f(erm) = 0$. Also $H^1(M, V^{eM}) = 0$ so we have a short exact sequence

$$0 \longrightarrow H^0(M, V^{eM}) \xrightarrow{\pi_*} H^0(M, V^R) \longrightarrow H^1(M, W) \longrightarrow 0 \ .$$

If X is a right M-set, the isomorphism $H^0(M, V^X) \cong H^0(X \rtimes M, \Delta(V)) \cong H^0(\mathcal{B}(X \rtimes M), V)$ in Corollary 6.14 allows us to identify $H^0(M, V^X)$ with

those functions $f \colon X \longrightarrow V$ which are constant on connected components of $\mathcal{B}(X \rtimes M)$. Since $\mathcal{B}(eM \rtimes M)$ is contractible, hence connected, $H^0(M, V^{eM})$ consists of the constant functions $eM \longrightarrow V$ and so the image of π_* is the set of functions which are constant on R. Thus $\operatorname{coker} \pi_* = \widetilde{H}^0(\mathcal{B}(R \rtimes M), V)$. This completes the proof of the first statement. The final statement follows from Corollary 6.18. □

Suppose now that B is a finite left regular band and let $\widehat{0} \neq Y \in \Lambda(B)$. Suppose that $Y = Be$. Then $R = eB \setminus \{e\}$ is a right ideal and

$$\Bbbk_Y \cong \{f \in \Bbbk^{eB} \mid f(R) = 0\}$$

via the map $f \mapsto f(e)$. Applying Theorem 6.19 we obtain the following corollary.

Corollary 6.20. *Let B be a left regular band and let $\widehat{0} \neq Y \in \Lambda(B)$. Then $H^0(B, \Bbbk_Y) = 0$ and*

$$H^{n+1}(B, \Bbbk_Y) \cong \widetilde{H}^n(\Delta(B[\widehat{0}, Y)), \Bbbk)$$

for all $n \geq 0$ where $\Delta(B[\widehat{0}, Y))$ is the order complex of $B[\widehat{0}, Y)$.

6.7. **The proof of Theorem 4.1.** At this point, we can give the proof of our main result, Theorem 4.1. Let B be a finite left regular band. Suppose that $X < Y$ in $\Lambda(B)$ and let $\Delta(X, Y)$ be the order complex of $B[X, Y]$.

Theorem 4.1. *Let B be a finite left regular band and let $\Bbbk$ be a commutative ring with unit. Let $X, Y \in \Lambda(B)$. Then*

$$\operatorname{Ext}_{\Bbbk B}^n(\Bbbk_X, \Bbbk_Y) = \begin{cases} \widetilde{H}^{n-1}(\Delta(X, Y), \Bbbk) & \text{if } X < Y, \ n \geq 1 \\ \Bbbk & \text{if } X = Y, \ n = 0 \\ 0 & \text{else.} \end{cases}$$

Proof. If $Y \not\geq X$, the result is Proposition 5.7. For $X \leq Y$, we have $\operatorname{Ext}_{\Bbbk B}^n(\Bbbk_X, \Bbbk_Y) = H^n(B_{\geq X}, \Bbbk_Y)$ by Proposition 6.1. Next suppose that $X = Y$. Then $H^n(B_{\geq X}, \Bbbk_X) = H^n(\mathcal{B}B_{\geq X}, \Bbbk)$. Any element of $B_{\geq X}$ with support X is a left zero and hence $\mathcal{B}B_{\geq X}$ is contractible by Proposition 6.10. Thus we are left with the case $X < Y$. But this is handled by Corollary 6.20. □

REFERENCES

[1] P. Abramenko and K. S. Brown. *Buildings*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008. Theory and applications.

[2] W. W. Adams and M. A. Rieffel. Adjoint functors and derived functors with an application to the cohomology of semigroups. *J. Algebra*, 7:25–34, 1967.

[3] M. Aguiar and S. Mahajan. *Coxeter groups and Hopf algebras*, volume 23 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2006. With a foreword by Nantel Bergeron.

[4] M. Aguiar and S. Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[5] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[6] C. A. Athanasiadis and P. Diaconis. Functions of random walks on hyperplane arrangements. *Adv. in Appl. Math.*, 45(3):410–437, 2010.

[7] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[8] H. J. Baues. *Algebraic homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989.

[9] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Basic representation theory of finite groups and associative algebras.

[10] M. Bestvina and N. Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[11] P. Bidigare, P. Hanlon, and D. Rockmore. A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements. *Duke Math. J.*, 99(1):135–174, 1999.

[12] T. P. Bidigare. *Hyperplane arrangement face algebras and their associated Markov chains*. ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)–University of Michigan.

[13] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier, Amsterdam, 1995.

[14] A. Björner. Random walks, arrangements, cell complexes, greedoids, and self-organizing libraries. In *Building bridges*, volume 19 of *Bolyai Soc. Math. Stud.*, pages 165–203. Springer, Berlin, 2008.

[15] A. Björner. Note: Random-to-front shuffles on trees. *Electron. Commun. Probab.*, 14:36–41, 2009.

[16] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[17] A. Björner and G. Ziegler. Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.*, 5(1):105–149, 1992.

[18] K. Bongartz. Algebras and quadratic forms. *J. London Math. Soc. (2)*, 28(3):461–469, 1983.

[19] F. Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[20] K. S. Brown. *Buildings*. Springer-Verlag, New York, 1989.

[21] K. S. Brown. The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem. In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, volume 23 of *Math. Sci. Res. Inst. Publ.*, pages 137–163. Springer, New York, 1992.

[22] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[23] K. S. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[24] K. S. Brown. Semigroup and ring theoretical methods in probability. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 3–26. Amer. Math. Soc., Providence, RI, 2004.

[25] K. S. Brown and P. Diaconis. Random walks and hyperplane arrangements. *Ann. Probab.*, 26(4):1813–1854, 1998.

[26] J. C. Bustamante. The classifying space of a bound quiver. *J. Algebra*, 277(2):431–455, 2004.

[27] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum. Reprint of the 1956 original.

[28] P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements*. Lecture Notes in Mathematics, No. 85. Springer-Verlag, Berlin, 1969.

[29] F. Chung and R. Graham. Edge flipping in graphs. *Adv. in Appl. Math.*, 48(1):37–63, 2012.

[30] C. Cibils. Cohomology of incidence algebras and simplicial complexes. *J. Pure Appl. Algebra*, 56(3):221–232, 1989.

[31] A. H. Clifford. Semigroups admitting relative inverses. *Ann. of Math. (2)*, 42:1037–1049, 1941.

[32] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.

[33] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[34] D. E. Cohen. A monoid which is right FP_∞ but not left FP_1. *Bull. London Math. Soc.*, 24(4):340–342, 1992.

[35] P. Diaconis. From shuffling cards to walking around the building: an introduction to modern Markov chain theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, Extra Vol. I, pages 187–204, 1998.

[36] V. Diekert and G. Rozenberg, editors. *The book of traces*. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.

[37] G. A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, 25:71–76, 1961.

[38] A. Dochtermann and A. Engström. Algebraic properties of edge ideals via combinatorial topology. *Electron. J. Combin.*, 16(2, Special volume in honor of Anders Bjorner):Research Paper 2, 24, 2009.

[39] M. P. Dorofeeva. Hereditary and semi-hereditary monoids. *Semigroup Forum*, 4:301–311, 1972.

[40] G. Z. Elston. Semigroup expansions using the derived category, kernel, and Malcev products. *J. Pure Appl. Algebra*, 136(3):231–265, 1999.

[41] J. Fountain. A class of right PP monoids. *Quart. J. Math. Oxford Ser. (2)*, 28(111):285–300, 1977.

[42] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[43] O. Ganyushkin, V. Mazorchuk, and B. Steinberg. On the irreducible representations of a finite semigroup. *Proc. Amer. Math. Soc.*, 137(11):3585–3592, 2009.

[44] M. Gerstenhaber and S. D. Schack. Simplicial cohomology is Hochschild cohomology. *J. Pure Appl. Algebra*, 30(2):143–156, 1983.

[45] R. Gray and S. J. Pride. Homological finiteness properties of monoids, their ideals and maximal subgroups. *J. Pure Appl. Algebra*, 215(12):3005–3024, 2011.

[46] J. A. Green. On the structure of semigroups. *Ann. of Math. (2)*, 54:163–172, 1951.

[47] V. S. Guba and S. J. Pride. Low-dimensional (co)homology of free Burnside monoids. *J. Pure Appl. Algebra*, 108(1):61–79, 1996.

[48] V. S. Guba and S. J. Pride. On the left and right cohomological dimension of monoids. *Bull. London Math. Soc.*, 30(4):391–396, 1998.

[49] K. Igusa and D. Zacharia. On the cohomology of incidence algebras of partially ordered sets. *Comm. Algebra*, 18(3):873–887, 1990.

[50] G. Kalai and R. Meshulam. Intersections of Leray complexes and regularity of monomial ideals. *J. Combin. Theory Ser. A*, 113(7):1586–1592, 2006.

[51] G. Kalai and R. Meshulam. Leray numbers of projections and a topological Helly-type theorem. *J. Topol.*, 1(3):551–556, 2008.

[52] Y. Kobayashi. The homological finiteness property FP$_1$ and finite generation of monoids. *Internat. J. Algebra Comput.*, 17(3):593–605, 2007.

[53] Y. Kobayashi and F. Otto. For finitely presented monoids the homological finiteness conditions FHT and bi-FP$_3$ coincide. *J. Algebra*, 264(2):327–341, 2003.

[54] Y. Lafont. A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier). *J. Pure Appl. Algebra*, 98(3):229–244, 1995.

[55] F. W. Lawvere. Qualitative distinctions between some toposes of generalized graphs. In *Categories in computer science and logic (Boulder, CO, 1987)*, volume 92 of *Contemp. Math.*, pages 261–299. Amer. Math. Soc., Providence, RI, 1989.

[56] F. W. Lawvere. More on graphic toposes. *Cahiers Topologie Géom. Différentielle Catég.*, 32(1):5–10, 1991. International Category Theory Meeting (Bangor, 1989 and Cambridge, 1990).

[57] F. W. Lawvere. Linearization of graphic toposes via Coxeter groups. *J. Pure Appl. Algebra*, 168(2-3):425–436, 2002. Category theory 1999 (Coimbra).

[58] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962/1963.

[59] J.-L. Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[60] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[61] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[62] S. Margolis and B. Steinberg. The quiver of an algebra associated to the Mantaci-Reutenauer descent algebra and the homology of regular semigroups. *Algebr. Represent. Theory*, 14(1):131–159, 2011.

[63] D. McDuff. On the classifying spaces of discrete monoids. *Topology*, 18(4):313–320, 1979.

[64] B. Mitchell. Rings with several objects. *Advances in Math.*, 8:1–161, 1972.

[65] W. R. Nico. On the cohomology of finite semigroups. *J. Algebra*, 11:598–612, 1969.

[66] W. R. Nico. Homological dimension in semigroup algebras. *J. Algebra*, 18:404–413, 1971.

[67] W. R. Nico. A counterexample in the cohomology of monoids. *Semigroup Forum*, 4:93–94, 1972.

[68] W. R. Nico. An improved upper bound for global dimension of semigroup algebras. *Proc. Amer. Math. Soc.*, 35:34–36, 1972.

[69] M. Nunes. Cohomological results in monoid and category theory via classifying spaces. *J. Pure Appl. Algebra*, 101(2):213–244, 1995.

[70] J. G. Oxley. *Matroid theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992.

[71] S. J. Pride. Homological finiteness conditions for groups, monoids, and algebras. *Comm. Algebra*, 34(10):3525–3536, 2006.

[72] M. S. Putcha. Complex representations of finite monoids. II. Highest weight categories and quivers. *J. Algebra*, 205(1):53–76, 1998.

[73] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[74] V. Reiner, F. Saliola, and V. Welker. Spectra of symmetrized shuffling operators, April 2011. http://arxiv.org/abs/1102.2460.

[75] J. Rhodes. Some results on finite semigroups. *J. Algebra*, 4:471–504, 1966.

[76] J. Rhodes and B. Steinberg. *The q-theory of finite semigroups*. Springer Monographs in Mathematics. Springer, New York, 2009.

[77] J. Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[78] F. V. Saliola. The quiver of the semigroup algebra of a left regular band. *Internat. J. Algebra Comput.*, 17(8):1593–1610, 2007.

[79] F. V. Saliola. On the quiver of the descent algebra. *J. Algebra*, 320(11):3866–3894, 2008.

[80] F. V. Saliola. The face semigroup algebra of a hyperplane arrangement. *Canad. J. Math.*, 61(4):904–929, 2009.

[81] F. Saliola and H. Thomas, Oriented interval greedoids. *Discrete Comput. Geom.*, 47(1):64–105, 2012.

[82] M. Schocker. The module structure of the Solomon-Tits algebra of the symmetric group. *J. Algebra*, 301(2):554–586, 2006.

[83] G. Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[84] L. Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.

[85] C. C. Squier. Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, 49(1-2):201–217, 1987.

[86] V. Srinivas. *Algebraic K-theory*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, second edition, 2008.

[87] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[88] B. Tilson. *Complexity of semigroups and morphisms*, chapter XII, pages 313–384. Academic Press, New York, 1976.

[89] J. Tits. Two properties of Coxeter complexes. *J. Algebra*, 41(2):265–268, 1976. Appendix to "A Mackey formula in the group ring of a Coxeter group" (J. Algebra 41 (1976), no. 2, 255–264) by Louis Solomon.
2007.

[90] P. Webb. An introduction to the representations and cohomology of categories. In *Group representation theory*, pages 149–173. EPFL Press, Lausanne, 2007.

[91] G. Wegner. *d*-collapsing and nerves of families of convex sets. *Arch. Math. (Basel)*, 26:317–321, 1975.

[92] E. S. Wolk. A note on "The comparability graph of a tree". *Proc. Amer. Math. Soc.*, 16:17–20, 1965.

[93] R. Woodroofe. Matchings, coverings, and Castelnuovo-Mumford regularity. arXiv:1009.2756, 2011.

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