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CHARACTERIZATION OF GROUP RADICALS WITH AN APPLICATION TO MAL'CEV PRODUCTS

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ABSTRACT. Radicals for Fitting pseudovarieties of groups are investigated from a profinite viewpoint in order to describe Mal'cev products on the left by the corresponding local pseudovariety of semigroups.

1. Introduction

The study of radicals in group theory emerged in the early 1960s following earlier work on radicals in rings. In recent years, there has been a surge of interest in obtaining simple characterizations of finite solvable groups and the solvable radical of finite groups modeled on classical results concerning the nilpotent case [8, 10, 12, 18, 23, 24, 39].

Extending earlier work of Rhodes and Tilson [33, 38], radical congruences have also been studied in the context of finite semigroup theory [5, 25, 29]. In the authors' recent paper [3], some relationships between radicals associated with specific pseudovarieties of groups and semigroup radical congruences have been explored via representation theory, generalizing and clarifying earlier work of Rhodes [32].

One of the aims of that paper is to describe Mal'cev products of the form $\mathsf{LH} \circledm \mathsf{V}$, where LH is the pseudovariety consisting of all finite semigroups

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whose local submonoids belong to a given pseudovariety H of groups and V is a pseudovariety of semigroups. For the purpose of applying representation theory, only the cases of the trivial pseudovariety and pseudovarieties of p-groups are considered there. Yet, as shown in the present paper, the same original argument of Rhodes and Tilson applies to pseudovarieties of groups possessing a radical, which are named *Fitting pseudovarieties* since they are pseudovarieties of groups which are simultaneously Fitting classes [14]. We further investigate how to obtain bases of pseudoidentities for $\mathsf{LH} \textcircled{m} \mathsf{V}$ from a given Fitting pseudovariety H and a basis of pseudoidentities for V (cf. Section 6). For this purpose, we need to obtain the type of characterization of H-radicals that is available since the 1950s for nilpotent groups and p-groups and for which group theorists have been searching in the solvable case. This leads us to set up a general profinite framework for studying radicals for Fitting pseudovarieties, in particular for extension-closed pseudovarieties.

2. Preliminaries

Given finite semigroups S and T, we write $S \prec T$ if S is a homomorphic image of a subsemigroup of T, in which case we also say that S *divides* T. A nonempty class of finite semigroups closed under taking divisors and finite direct products is called a *pseudovariety*. We denote respectively, by S and G the pseudovarieties consisting of all finite semigroups and all finite groups.

The following definition is a special case of a more general definition of radical class which is classical in group theory [36]. A *radical class of finite groups* is a subclass $\mathcal{X} \subseteq \mathsf{G}$ with the following properties:

(1) $\mathcal{X}$ is closed under taking homomorphic images;
(2) if G is a finite group and N_1 and N_2 are normal subgroups of G which belong to $\mathcal{X}$, then so does their product N_1N_2; we then denote by $G_{\mathcal{X}}$ the product of all normal subgroups of G which belong to $\mathcal{X}$ and we call it the $\mathcal{X}$*-radical* of G;
(3) for every finite group G, the subgroup $(G/G_{\mathcal{X}})_{\mathcal{X}}$ is trivial.

It is well known and easy to see that, in the presence of the other two conditions, condition (3) is equivalent to $\mathcal{X}$ being extension-closed. On the other hand, since an extension-closed pseudovariety H of groups satisfies condition (3) by the second isomorphism theorem, H is a radical class of finite groups. Hence, a pseudovariety of groups is radical if and only if it is extension closed. The radical pseudovarieties of groups are therefore in bijection with the division-closed sets of finite simple groups.

For a class of specific examples, given a set π of prime integers, consider the class G_π of all finite groups G such that all primes dividing $|G|$ belong to π. Note that G_π is an extension-closed pseudovariety of groups and, therefore, radical. Here are some particular cases of interest:

- if π is the set of all primes, then $\mathsf{G}_\pi = \mathsf{G}$;

- if $\pi = \emptyset$, then G_π is the trivial pseudovariety I;
- if $\pi = \{p\}$ is a singleton, then G_π is the pseudovariety G_p of all finite p-groups;
- if $\pi = 2'$ consists of all primes different from 2, then G_π is the pseudovariety of all finite groups of odd order.

The class $\mathsf{G}_{\mathrm{sol}}$, of all finite solvable groups, is also a radical pseudovariety.

In case a pseudovariety H of groups satisfies (2) but not necessarily (3), we will also say that H is a *Fitting pseudovariety of groups*. The identification of Fitting pseudovarieties is apparently harder.

For example, the class $\mathsf{G}_{\mathrm{nil}}$, of all finite nilpotent groups, is a Fitting but not a radical pseudovariety. The $\mathsf{G}_{\mathrm{nil}}$-radical of a finite group G is also known as its *Fitting subgroup*, and is denoted $\mathrm{Fit}(G)$. Note that the intersection of any nonempty family of Fitting (respectively, radical) pseudovarieties has again the same property. In particular, for every set π of primes, $\mathsf{G}_{\pi,\mathrm{nil}} = \mathsf{G}_\pi \cap \mathsf{G}_{\mathrm{nil}}$ is a Fitting pseudovariety while $\mathsf{G}_{\pi,\mathrm{sol}} = \mathsf{G}_\pi \cap \mathsf{G}_{\mathrm{sol}}$ is a radical pseudovariety. These pseudovarieties are, respectively, the smallest pseudovariety and the smallest extension-closed pseudovariety containing all G_p with $p \in \pi$. By the Feit–Thompson theorem [15], we have $\mathsf{G}_{2'} = \mathsf{G}_{2',\mathrm{sol}}$.

A useful remark about the radical $G_{\mathcal{X}}$ of a group G for a Fitting class $\mathcal{X}$ is that it is a characteristic subgroup of G. More generally, in view of property (1), if $\varphi: G \to H$ is an onto homomorphism of finite groups then $\varphi(G_{\mathcal{X}}) \subseteq H_{\mathcal{X}}$. The following result presents some further elementary observations about radicals.

LEMMA 2.1. *Let $(\mathsf{H}_i)_{i\in I}$ be a family of Fitting pseudovarieties and let G be a finite group. Then $\mathsf{H} = \bigcap_{i\in I} \mathsf{H}_i$ is also a Fitting pseudovariety and $G_{\mathsf{H}} = \bigcap_{i\in I} G_{\mathsf{H}_i}$.*

As a consequence, we conclude that the Fitting pseudovarieties form a complete lattice under inclusion.

Given two pseudovarieties of groups $\mathsf{H}_1, \mathsf{H}_2$, we denote by $\mathsf{H}_1\mathsf{H}_2$ the *product pseudovariety* consisting of all extensions of a group in H_1 by a group in H_2. We remind the reader that this multiplication is associative and distributes on the left over pseudovariety joins and meets. We write H^n to denote the n-fold product of copies of H. The following elementary result connects our study of Fitting pseudovarieties with the classical theory of Fitting classes.

LEMMA 2.2. *Let H_1 and H_2 be Fitting pseudovarieties of groups and let G be a finite group.*

(1) *We have $G \in \mathsf{H}_1\mathsf{H}_2$ if and only if $G/G_{\mathsf{H}_1} \in \mathsf{H}_2$.*
(2) *The product $\mathsf{H}_1\mathsf{H}_2$ is also a Fitting pseudovariety.*
(3) *The formula $G_{\mathsf{H}_1\mathsf{H}_2}/G_{\mathsf{H}_1} = (G/G_{\mathsf{H}_1})_{\mathsf{H}_2}$ holds.*

Proof. (1) By definition of $\mathsf{H}_1\mathsf{H}_2$, each $G \in \mathsf{H}_1\mathsf{H}_2$ must have a normal subgroup K such that $K \in \mathsf{H}_1$ and $G/K \in \mathsf{H}_2$. By definition of the radical, it follows that $K \subseteq G_{\mathsf{H}_1}$ and so also $G/G_{\mathsf{H}_1} \in \mathsf{H}_2$. The converse is obvious.

(2) Suppose that N_1 and N_2 are two normal subgroups of G which belong to $\mathsf{H}_1\mathsf{H}_2$. Let $R_i = (N_i)_{\mathsf{H}_1}$ $(i = 1,2)$. By (1), both quotients N_i/R_i belong to H_2. Since R_i is a characteristic subgroup of N_i, R_i is also a normal subgroup of G. Since H_1 is a Fitting pseudovariety, we deduce that $R_1R_2 \in \mathsf{H}_1$. Thus, to conclude that $N_1N_2 \in \mathsf{H}_1\mathsf{H}_2$, it suffices to show that $N_1N_2/R_1R_2 \in \mathsf{H}_2$. Note that

$$N_1N_2/R_1R_2 = (N_1R_2/R_1R_2) \cdot (N_2R_1/R_1R_2).$$

Moreover, N_1R_2/R_1R_2 is a normal subgroup of N_1N_2/R_1R_2 and a homomorphic image of N_1/R_1, which therefore belongs to H_2, and similarly for the other factor. Hence, the quotient N_1N_2/R_1R_2 belongs to H_2 since this pseudovariety is a Fitting class.

(3) Let now $R = G_{\mathsf{H}_1\mathsf{H}_2}$ and $K = G_{\mathsf{H}_1}$. Note that the H_1-radical of R coincides with K: as R is a normal subgroup of G, its characteristic subgroup R_{H_1} is also a normal subgroup of G and, since it belongs to H_1, $R_{\mathsf{H}_1} \subseteq K$; conversely, K is a normal subgroup of R, since it is contained in R, and therefore $K \subseteq R_{\mathsf{H}_1}$. Since $R \in \mathsf{H}_1\mathsf{H}_2$, we obtain $R/K \in \mathsf{H}_2$ by (1). Hence, $R/K \subseteq (G/K)_{\mathsf{H}_2}$ since R/K is a normal subgroup of G/K. For the reverse inclusion, let N be the union of all cosets of K which belong to $(G/K)_{\mathsf{H}_2}$. Then N is a normal subgroup of G such that $N/K = (G/K)_{\mathsf{H}_2} \in \mathsf{H}_2$, with $K \in \mathsf{H}_1$, and so $N \in \mathsf{H}_1\mathsf{H}_2$. This shows that $N \subseteq R$ and establishes the equality $R/K = (G/K)_{\mathsf{H}_2}$. □

Part (1) of Lemma 2.2 states that the product of Fitting pseudovarieties coincides with their product as Fitting classes (cf. [13]). Thus, parts (2) and (3) are well-known facts in the theory of Fitting classes. Proofs are being provided for the sake of completeness.

3. Pseudoidentities for exclusion pseudovarieties

We say that a finite group P is *prime for direct products* or $\times$*-prime* if, whenever $P \prec H_1 \times H_2$, for finite groups H_1 and H_2, $P \prec H_1$ or $P \prec H_2$. This is precisely the condition that guarantees that the following class of finite groups is a pseudovariety:

$$\mathrm{Excl}_{\mathsf{G}}(P) = \{H \in \mathsf{G} : P \not\prec H\}.$$

Note that cyclic groups of prime power and finite simple groups are $\times$-prime. But there are many other $\times$-prime groups (see [28, Theorem 53.31]).

For the sequel, we recall some background on the profinite approach to the theory of pseudovarieties. See [1, 2] for further details.

A *profinite semigroup* is a compact zero-dimensional semigroup or, equivalently, a compact semigroup which is residually finite as a topological semigroup [2]. We denote by $\overline{\Omega}_n\mathsf{S}$ the free profinite semigroup on a free generating $\{x_1,\dots,x_n\}$ set with n elements (often called *variables*). It may be described as the completion of the free semigroup $\{x_1,\dots,x_n\}^+$ with respect to metric d such that $d(u,v)\le 2^{-r}$ if and only if the identity $u=v$ is verified in all semigroups with at most r elements. We view $\overline{\Omega}_n\mathsf{S}$ as naturally embedded in $\overline{\Omega}_{n+1}\mathsf{S}$ by sending each free generator x_i of $\overline{\Omega}_n\mathsf{S}$ to the corresponding free generator x_i of $\overline{\Omega}_{n+1}\mathsf{S}$.

Elements of $\overline{\Omega}_n\mathsf{S}$ may be viewed as *n-ary implicit operations* on S: families $(u_S)_{S\in\mathsf{S}}$ of n-ary operations such that, for every homomorphism $\varphi: S\to T$ between finite semigroups and for all $s_1,\dots,s_n\in S$, $\varphi(u_S(s_1,\dots,s_n)) = u_T(\varphi(s_1),\dots,\varphi(s_n))$. Given $u\in\overline{\Omega}_n\mathsf{S}$, the corresponding operation $u_S: S^n\to S$ maps the n-tuple $(s_1,\dots,s_n)$ to $f(u)$, where $f:\overline{\Omega}_n\mathsf{S}\to S$ is the unique continuous homomorphism that maps the ith variable x_i to s_i $(i=1,\dots,n)$. For simplicity, we may write $u(s_1,\dots,s_n)$ instead of $u_S(s_1,\dots,s_n)$. Also, we may refer to the implicit operation $u(x_1,\dots,x_n)$.

A formal equality $u=v$ of elements of some $\overline{\Omega}_n\mathsf{S}$ is called a *pseudoidentity*. We say that a finite semigroup S *satisfies* the pseudoidentity $u=v$ and we write $S\models u=v$ if $\varphi(u)=\varphi(v)$ for every continuous homomorphism $\varphi:\overline{\Omega}_n\mathsf{S}\to S$. We use $u=1$ to abbreviate the pseudoidentities $ux=xu=x$, where x is a variable that is not a factor of u. For a set Σ of pseudoidentities, $[\![\Sigma]\!]$ stands for the class of all finite semigroups that satisfy all pseudoidentities from Σ. It is easy to see that $[\![\Sigma]\!]$ is a pseudovariety and by Reiterman's theorem [31] every pseudovariety V can be so described by a set Σ of pseudoidentities, which is called a *basis of pseudoidentities* for V.

PROPOSITION 3.1. *Suppose that P is an n-generated $\times$-prime finite group. Then there is some $u_P\in\overline{\Omega}_n\mathsf{S}$ such that* $\mathrm{Excl}_{\mathsf{G}}(P)=[\![u_P=1]\!]$.

Proof. We make the collection n-$\mathrm{Excl}_{\mathsf{G}}(P)$ of all n-generated groups in $\mathrm{Excl}_{\mathsf{G}}(P)$ (up to isomorphism respecting the choice of generators) be an ordered set by letting a group K be greater than or equal to a group H if there is a homomorphism $K\to H$ which respects the choice of generators. (Observe that such a homomorphism is automatically onto.) It is easy to see that this ordered set is upwards directed—indeed, if G with the generators $g_1,\dots,g_n$ and H with the generators $h_1,\dots,h_n$ are two groups in n-$\mathrm{Excl}_{\mathsf{G}}(P)$, then the subgroup of $G\times H$ generated by the pairs $(g_1,h_1),\dots,(g_n,h_n)$ belongs to n-$\mathrm{Excl}_{\mathsf{G}}(P)$ (since P is $\times$-prime) and is greater than or equal to both G and H. Since n-$\mathrm{Excl}_{\mathsf{G}}(P)$ is countable, it implies that this ordered set has a cofinal sequence. Let $(H_k)_k$ be such a sequence. Since each $H_k\in\mathrm{Excl}_{\mathsf{G}}(P)$ and P is $\times$-prime, P does not belong to the pseudovariety generated by H_k. By Reiterman's theorem, there is a pseudoidentity of the form $u_k=1$ which is valid in H_k but not in P. Since P is n-generated, we may assume that

$u_k \in \overline{\Omega}_n\mathsf{S}$. Let u be the limit of a subsequence of $(u_k)_k$ in the compact metric space $\overline{\Omega}_n\mathsf{S}$.

We first note that P fails the pseudoidentity $u=1$. Indeed, there is k such that $P \models u = u_k$ and, by construction, $P \not\models u_k = 1$. On the other hand, every H_k satisfies $u=1$. Indeed, given k, there is $\ell \geq k$ such that $H_k \models u = u_\ell$ and, by construction, $H_\ell \models u_\ell = 1$; hence $H_k \models u = u_\ell = 1$ since H_k is a homomorphic image of H_ℓ.

Next, we claim that $\mathrm{Excl}_{\mathsf{G}}(P) = [\![u=1]\!]$. Let H be a finite group. If H is divisible by P, then it cannot satisfy the pseudoidentity $u=1$ since P does not satisfy it, as was shown above. Conversely, if H is not divisible by P, to show that $H \models u = 1$, it suffices to assume that H is n-generated. Then H is a homomorphic image of some H_k, so that $H_k \models u = 1$ by the above. Hence, $H \models u = 1$. This proves the claim and establishes the proposition. □

The above proof can be easily put in the more general setting of pseudovarieties of finite algebraic structures. A more efficient proof and its setting in the context of the theory of continuous lattices can be found in [35] (cf. Section 7.1 and, in particular, Proposition 7.1.9).

It is well known that the classification of finite simple groups implies that all finite simple groups are 2-generated. Combining with Proposition 3.1, we obtain the following result.

THEOREM 3.2. *Let* V *be an extension-closed pseudovariety of groups. Then there is* $w \in \overline{\Omega}_2\mathsf{S}$ *such that* $\mathsf{V} = [\![w=1]\!]$.

Proof. Let $\mathcal{S}$ be the set of all division-minimal simple groups, up to isomorphism, which do not belong to V. Note that

$$\mathsf{V} = \bigcap_{P\in\mathcal{S}} \mathrm{Excl}_{\mathsf{G}}(P).$$

Let $\mathcal{S} = \{P_1, P_2, \ldots\}$ be an enumeration of the elements of $\mathcal{S}$. For each index i, let $u_i \in \overline{\Omega}_2\mathsf{S}$ be such that $\mathrm{Excl}_{\mathsf{G}}(P) = [\![u_i = 1]\!]$, as given by Proposition 3.1. Let w be the limit in $\overline{\Omega}_2\mathsf{S}$ of a subsequence of the (possibly finite) sequence of products $(u_1 \cdots u_k)_k$. We claim that $\mathsf{V} = [\![w=1]\!]$.

Let G be a finite group. Then, for arbitrarily large k, we have $G \models w = u_1 \cdots u_k$. Suppose first that $G \in \mathsf{V}$. Then $G \in \mathrm{Excl}_{\mathsf{G}}(P_i)$ for all $i \geq 1$, which implies that G satisfies each of the pseudoidentities $u_i = 1$. Hence, $G \models w = 1$. Conversely, assume that $G \models w = 1$. Suppose furthermore that G does not belong to a certain $\mathrm{Excl}_{\mathsf{G}}(P_i)$, that is $P_i \prec G$. Since the elements of $\mathcal{S}$ are incomparable under division, P_i belongs to $\mathrm{Excl}_{\mathsf{G}}(P_j)$ for all $j \neq i$, and so $P_i \models u_j = 1$ whenever $j \neq i$. In particular, if we choose k above so that $k \geq i$ then G, and therefore also P_i, satisfies the pseudoidentities $u_1 \cdots u_k = w = 1$. Since P_i also satisfies $u_j = 1$ for $j \neq i$, we conclude that $P_i \models u_i = 1$, which contradicts the choice of u_i. Hence, G belongs to all $\mathrm{Excl}_{\mathsf{G}}(P_i)$, and so it belongs to V. □

Note that the proofs of Proposition 3.1 and Theorem 3.2 are based on existence compactness arguments. It is another problem to exhibit pseudoidentities defining the pseudovarieties in question. One may wish, for instance, that the implicit operations appearing in them be (efficiently) computable. Of course, since there are uncountably many extension-closed pseudovarieties of groups, not all of them are decidable, and so it is certainly not always possible to obtain such pseudoidentities.

An important example is the pseudovariety of solvable groups. Bases consisting of a single 2-variable pseudoidentity for $\mathsf{G}_{\mathrm{sol}}$ can be drawn from recent work in group theory [9, 12]. The mere existence of such bases had previously been established in [11, 27] while the existence of bases consisting of some set of 2-variable pseudoidentities follows from [37]. The original proofs of all these results depend on part of the classification of finite simple groups. A direct elementary but intricate proof of the existence of 2-variable bases has also been obtained [16]. Theorem 3.2 is a much more general result with a rather straightforward proof but which is again highly dependent on the classification of finite simple groups.

Some extension-closed pseudovarieties of groups may be even defined by a single-variable pseudoidentity. The following result is actually a special case of Proposition 3.1 but its proof provides a more direct "construction" of a defining pseudoidentity.

Proposition 3.3. *Let π be a set of primes. Then G_π is defined by a pseudoidentity in one variable.*

Proof. Since $\mathsf{G}_\emptyset$ is defined by the pseudoidentity $x = 1$, we may assume that π is nonempty. Let $p_1, p_2, \ldots$ be an enumeration of the elements of π, allowing repetitions. Define

$$x^\nu = \lim_{n\to\infty} x^{(p_1\cdots p_n)^{n!}}. \tag{3.1}$$

We will prove that this limit exists and is independent of the enumeration of π. Denote by π' the complementary set of primes to π. Let $S = \langle s\rangle$ be a finite monogenic semigroup with minimal ideal the cyclic group $K = \langle s^{\omega+1}\rangle$. We show that s^ν is the π'-component of $s^{\omega+1}$. Assume $s^{\omega+1} = s_1 s_2$ where s_1 is the π-component and s_2 is the π'-component of $s^{\omega+1}$. Set $i_n = (p_1\cdots p_n)^{n!}$. We need to show that, for n sufficiently large, $s^{i_n} = s_2$. Suppose that S has order ℓ. For $n \geq \ell$, clearly $i_n \geq \ell$ and so s^{i_n} is in K. Next, we compute

$$(s^{\omega+1})^{i_n} = (s^{i_n})^{\omega+1} = (s^{i_n})^\omega s^{i_n} = s^\omega s^{i_n} = s^{i_n}$$

where the last equality follows because s^{i_n} is in the minimal ideal of S, which is a group with identity element s^ω. Thus, without loss of generality, we may assume that $s = s^{\omega+1}$ generates a cyclic group of order ℓ.

Suppose s_1 has order j and s_2 has order k; so j is divisible only by primes in π and k by primes in π' and also $\ell = jk$. Let r be the smallest index such that all prime divisors of j occur among $p_1, \ldots, p_r$. Choose $N = \max\{j, r, \varphi(k)\}$ where φ is the Euler totient function. We claim that, for $n \geq N$, the equality $s^{i_n} = s_2$ holds. Because $n \geq \max\{j, r, \varphi(k)\}$ the following hold:

$$j \mid (p_1 \cdots p_n)^{n!} = i_n \quad \text{and} \quad \varphi(k) \mid n!.$$

Indeed, if p is a prime dividing j, then certainly p is among the list $p_1, \ldots, p_n$ as $n \geq r$; if p^u is the largest power of p dividing j, then evidently $u \leq j!$ and so $j \mid (p_1 \cdots p_n)^{n!}$ as claimed. Because $p_1 \cdots p_n$ is prime to k, Euler's theorem (or the fact that the group of units of $\mathbb{Z}_k$ has order $\varphi(k)$) yields

$$i_n = (p_1 \cdots p_n)^{n!} \equiv 1 \bmod k.$$

Therefore, $s^{i_n} = s_1^{i_n} s_2^{i_n} = s_2$. This completes the proof that s^ν is the π'-component of s. It follows that G_π is defined by the pseudoidentity $x^\nu = 1$. □

A simpler basis of pseudoidentities may be given for the pseudovariety $\mathsf{G}_{2'}$ of all finite groups of odd order, namely

$$\mathsf{G}_{2'} = [\![x^{2^\omega - 1} = 1]\!],$$

where $x^{2^\omega - 1} = \lim_{n\to\infty} x^{2^{n!}-1}$. If a finite group G satisfies the pseudoidentity $x^{2^\omega - 1} = 1$, then G has odd order. Conversely, if m is odd, then 2 is invertible in the ring $\mathbb{Z}/m\mathbb{Z}$ and so $2^\omega = 1$ in this ring. It follows that every finite group of odd order satisfies the pseudoidentity $x^{2^\omega} = x$. Note also that the proof of Proposition 3.3 establishes the equality $\mathsf{G}_{\{p_1,\ldots,p_n\}} = [\![x^{(p_1 \cdots p_n)^\omega} = 1]\!]$ for all primes $p_1, \ldots, p_n$.

4. Characterizations of the radical

Recall the standard notation in group theory for iterated commutators: $[x, {}_1y] = [x, y] = x^{-1}y^{-1}xy$ and $[x, {}_{n+1}y] = [[x, {}_ny], y]$. For a group G, $L(G)$ denotes the set of all *left Engel elements* of G consisting of those $x \in G$ such that, for every $y \in G$, there exists $r \geq 1$ such that $[y, {}_rx] = 1$.

For a subset X of a group G, denote by $\langle X \rangle$ the subgroup generated by X. The following result has been recently established [23].

Theorem 4.1. *An element a of a finite group G belongs to its solvable radical if and only if, for every $b \in G$, the subgroup $\langle a, b\rangle$ is solvable.*

On the other hand, Bandman, Borovoi, Grunewald, Kunyavskiĭ, and Plotkin [8] have formulated and investigated a general conjecture which would lead to a description of the solvable radical similar to Baer's description of the nilpotent radical in terms of left Engel elements. They established the analog of the conjecture for finite-dimensional Lie algebras and reduced the conjecture to a slight strengthening of the case of finite direct products of isomorphic non-Abelian finite simple groups. Although they also proposed constructions of specific candidates, their conjecture amounts to the existence of $w \in \overline{\Omega}_2\mathsf{S}$ such that, for every finite group G and every $a \in G$, a belongs to the solvable radical if and only if, for every $b \in G$, $w(a,b) = 1$.

More generally, let V be a Fitting pseudovariety of groups. We say that the V-radical is *characterized* by a subset $W \subseteq \overline{\Omega}_{r+1}\mathsf{S}$ if, for every finite group G,

$$G_{\mathsf{V}} = \{a \in G : \forall b_1, \ldots, b_r \in G\ \forall w \in W, w(a, b_1, \ldots, b_r) = 1\}. \tag{4.1}$$

We then say that $r+1$ is the *arity* of the characterization. In case the equation holds for all G in a given class $\mathcal{C}$ of finite groups, then we say that the V-radical is *characterized by W over $\mathcal{C}$*. Note that every such characterization contains a countable one.

In this language, the above conjecture is equivalent to the statement that the solvable radical admits a singleton binary characterization $\{w\}$.

For example, as a consequence of a theorem of Baer [7], the nilpotent radical is characterized by the ω-iterated commutator

$$u(x_1, x_2) = [x_2, {}_\omega x_1]$$

which is defined as the limit of $[x_2, {}_{n!}x_1]$ as $n \to \infty$, where $[x_2, x_1] = x_2^{\omega-1} \times x_1^{\omega-1} x_2 x_1$ and, recursively, $[x_2, {}_{n+1}x_1] = [[x_2, {}_n x_1], x_1]$.[1] Moreover, also by Baer's theorem, the p-group radical of a finite group G consists of the elements of $L(G)$ which have order a power of p. Thus, the G_p-radical is characterized by the set $\{[x_2, {}_\omega x_1], x_1^{p^\omega}\}$, where x^{p^ω} denotes the limit $\lim_{n\to\infty} x^{p^{n!}}$. A singleton characterization is given by

$$[x_2, {}_\omega x_1] x_1^{p^\omega}. \tag{4.2}$$

Indeed, for a finite group G, if $[h, {}_\omega g] g^{p^\omega} = 1$ for all $h \in G$ then, in particular, taking $h = 1$, we obtain $g^{p^\omega} = 1$. Hence, the equality $[h, {}_\omega g] g^{p^\omega} = 1$ holds for all $h \in G$ if and only if the equalities $[h, {}_\omega g] = g^{p^\omega} = 1$ hold for all $h \in G$.

The following is a tool to build up characterizations of radicals, although it creates the technical difficulty of the simultaneous build up of the number of variables.

PROPOSITION 4.2. *Suppose that $v_1 \in \overline{\Omega}_{n+1}\mathsf{S}$ and $v_2 \in \overline{\Omega}_{m+1}\mathsf{S}$ characterize the radicals of the Fitting pseudovarieties H_1 and H_2, respectively. Then the $(\mathsf{H}_1\mathsf{H}_2)$-radical is characterized by the $(m+n+1)$-ary implicit operation*

$$v = v_1(v_2(x_1, x_2, \ldots, x_{m+1}), x_{m+2}, \ldots, x_{m+n+1}). \tag{4.3}$$

Proof. Let G be a finite group and let $g \in G$. If $g \in G_{\mathsf{H}_1\mathsf{H}_2}$ then, by Lemma 2.2, $gG_{\mathsf{H}_1} \in (G/G_{\mathsf{H}_1})_{\mathsf{H}_2}$ and so, for all $a_1, \ldots, a_m \in G$, we have $v_2(g, a_1, \ldots, a_m) \in G_{\mathsf{H}_1}$, which implies that

$$v_1(v_2(g, a_1, \ldots, a_m), b_1, \ldots, b_n) = 1 \tag{4.4}$$

[1] By $x^{\omega-1}$ we denote the limit of $x^{n!-1}$ as $n \to \infty$.

for all $b_1, \dots, b_n \in G$. Conversely, suppose that $g \in G$ is such that the equality (4.4) holds for all $a_i, b_j \in G$. Since v_1 characterizes the H_1-radical, $v_2(g, a_1, \dots, a_m)$ is an element of G_{H_1} for all $a_i \in G$. Since v_2 characterizes the H_2-radical, we deduce that $gG_{\mathsf{H}_1} \in (G/G_{\mathsf{H}_1})_{\mathsf{H}_2}$. By Lemma 2.2, it follows that $gG_{\mathsf{H}_1} \in G_{\mathsf{H}_1\mathsf{H}_2}/G_{\mathsf{H}_1}$, which implies that $g \in G_{\mathsf{H}_1\mathsf{H}_2}$. Hence, v characterizes the $(\mathsf{H}_1\mathsf{H}_2)$-radical. □

Denote by Ab the pseudovariety of all finite Abelian groups. The following easy observation already intervenes in the proof of Theorem 4.1.

LEMMA 4.3. *Let* V *be an extension-closed pseudovariety of groups containing* Ab. *If* G *is a finite group,* $a \in G_{\mathsf{V}}$*, and* $b \in G$*, then* $\langle a, b\rangle \in \mathsf{V}$.

Proof. Let $H = \langle a, b\rangle$. Then H is a cyclic extension of its normal subgroup $N = H \cap G_{\mathsf{V}}$. Since $N \in \mathsf{V}$ and V contains Ab, it follows that $H \in \mathsf{V}$. □

The following notation will be convenient for a pseudovariety V:

$$(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}} = \{u \in \overline{\Omega}_2\mathsf{S} : \mathsf{V} \models u = 1\}.$$

Note that, if V is a Fitting pseudovariety of groups and W is a binary characterization of the V-radical G_{V}, then $W \subseteq (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$.

Theorem 4.1 may be formulated in the language of characterizations of radicals as stating that the solvable radical admits a binary characterization. More generally, we have the following result.

PROPOSITION 4.4. *Let* V *be an extension-closed pseudovariety of groups containing* Ab. *Then the* V*-radical admits a binary characterization if and only if, for every finite group* G,

$$G_{\mathsf{V}} = \{a \in G : \forall b \in G, \langle a, b\rangle \in \mathsf{V}\}. \tag{4.5}$$

Proof. Suppose first that W is a binary characterization of the V-radical and let $G \in \mathsf{G}$ and $a, b \in G$. Consider the subgroup $H_b = \langle a, b\rangle$. If $a \in G_{\mathsf{V}}$, then $H_b \in \mathsf{V}$ by Lemma 4.3. On the other hand, if $H_b \in \mathsf{V}$ for every $b \in G$, then $w(a, b) = 1$ for every $w \in W$ since $W \subseteq (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$. Since W is a characterization of the V-radical, it follows that $a \in G_{\mathsf{V}}$. Hence, the equality (4.5) holds.

Conversely, suppose that the V-radical of every finite group G is given by (4.5). By Theorem 3.2, there exists $u \in \overline{\Omega}_2\mathsf{S}$ such that $\mathsf{V} = [\![u = 1]\!]$. Let

$$W = \big\{u(x, y) : x, y \in \{x_1, x_2\}^+\big\},$$

where x_1, x_2 are the free generators of $\overline{\Omega}_2\mathsf{S}$. We claim that W characterizes the V-radical. Let G be a finite group and let $a \in G$. By (4.5), $a \in G_{\mathsf{V}}$ if and only if, for every $b \in G$, the subgroup $\langle a, b\rangle$ belongs to V, that is if it satisfies the pseudoidentity $u = 1$. Since the elements of $\langle a, b\rangle$ are described by arbitrary positive words in a and b, the latter condition is equivalent to $w(a, b) = 1$ for all $w \in W$, which shows that W is a binary characterization of the V-radical. □

Further evidence towards the conjecture of Bandman et al. is given by the following recent result [39], which also depends on Theorem 4.1 and whose finite version translates in our language by saying that there is a singleton binary characterization of the solvable radical over the class of all finite linear groups.

THEOREM 4.5. *There is a sequence $(w_n)_n$ of group words in the free group on x_1, x_2 which converges in $\overline{\Omega}_2\mathsf{G}$ such that, for every linear group G and element $g \in G$, g lies in the solvable radical of G if and only if, for all $h \in G$, we have $w_n(g,h) = 1$ for all sufficiently large n.*

For the remainder of this section, V denotes a Fitting pseudovariety of groups.

We observe that there is a formulation of the existence of characterizations by sets of implicit operations similar to the property in Theorem 4.5. For simplicity, we illustrate it in the case of binary characterizations.

PROPOSITION 4.6. *The V-radical admits a binary characterization if and only if there is a sequence $(w_n)_n$ of $\{x_1, x_2\}^+$ such that, for every finite group G,*

$$G_{\mathsf{V}} = \{g \in G : \forall h \in G \; \exists n_0 \; \forall n \geq n_0, w_n(g,h) = 1\}. \tag{4.6}$$

Proof. Suppose first that W is a binary characterization of the V-radical. As has been observed, we may assume that it is countable. Let $v_1, v_2, \ldots$ be an enumeration of its elements. For each pair of positive integers n, k, let $v_{n,k} \in \{x_1, x_2\}^+$ be such that $d(v_{n,k}, v_n) \leq 2^{-k}$. Let $w_1, w_2, \ldots$ be an enumeration of the list of words $v_{n,k}$ with $k \geq n$. Then we claim that equation (4.6) holds for every finite group G. Indeed, given $g \in G_{\mathsf{V}}$ and $h \in G$, $v_n(g,h) = 1$ for all n and so $v_{n,k} = 1$ for all $k \geq |G|$, which implies that $w_n(g,h) = 1$ for every sufficiently large n. On the other hand, if $g \in G$ is such that, for all $h \in G$, $w_n(g,h) = 1$ for every sufficiently large n, then certainly, for every n and sufficiently large k, $v_{n,k}(g,h) = 1$, which implies that $v_n(g,h) = 1$ for every n, whence $g \in G_{\mathsf{V}}$.

Conversely, suppose that the sequence of words $(w_n)_n$ satisfies (4.6) for every finite group G. Let W denote the set of all accumulation points of the sequence $(w_n)_n$ in $\overline{\Omega}_2\mathsf{S}$. Then, given a finite group G and $g, h \in G$, we have $w(g,h) = 1$ for every $w \in W$ if and only if $w_n(g,h) = 1$ for every sufficiently large n. Hence, W is a binary characterization of the V-radical. □

For each finite group G, we let $U_{\mathsf{V}}(G)$ denote the set of all $u \in \overline{\Omega}_2\mathsf{S}$ such that the following two conditions hold:

(1) $\mathsf{V} \models u = 1$;
(2) for every $a \in G \setminus G_{\mathsf{V}}$ there exists $b \in G$ such that $u(a,b) \neq 1$.

If, additionally, a, b are specific elements of G, then we let

$$U^b_{\mathsf{V},a}(G) = \{u \in \overline{\Omega}_2\mathsf{S} : u(a,b) \neq 1, \mathsf{V} \models u = 1\}$$

and

$$U_{\mathsf{V},a}(G) = \bigcup_{g\in G} U^g_{\mathsf{V},a}(G),$$

so that

$$U_{\mathsf{V}}(G) = \bigcap_{a\in G\setminus G_{\mathsf{V}}} U_{\mathsf{V},a}(G), \tag{4.7}$$

where the intersection is viewed as specifying a subset of $(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ and so it is taken to be $(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ in case the intersected family is empty, that is $G\in \mathsf{V}$. Note that $U^b_{\mathsf{V},a}(G)$ is a closed subset of $\overline{\Omega}_2\mathsf{S}$ as it is the intersection of $(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ with the clopen set $\varphi^{-1}(G\setminus\{1\})$, where $\varphi:\overline{\Omega}_2\mathsf{S}\to G$ is the continuous homomorphism which maps x_1 to a and x_2 to b. Hence, each of the sets $U_{\mathsf{V},a}(G)$ and $U_{\mathsf{V}}(G)$ is closed in $\overline{\Omega}_2\mathsf{S}$.

LEMMA 4.7. *The following formula holds for every pseudovariety* V *containing* Ab *and all finite groups* $G_1,\dots,G_n$*:*

$$U_{\mathsf{V}}(G_1\times\cdots\times G_n) = \bigcap_{i=1}^{n} U_{\mathsf{V}}(G_i). \tag{4.8}$$

Proof. We start by observing that the hypothesis that V contains Ab implies that G satisfies the pseudoidentity $u(1,x_2)=1$ whenever $u\in(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$. Indeed, the assumption on u implies that it holds in V and therefore also in every finite cyclic group. Since the pseudoidentity $u(1,x_2)=1$ involves only one variable, it holds in G.

Let $G=G_1\times\cdots\times G_n$. It can be easily verified that

$$G_{\mathsf{V}} = (G_1)_{\mathsf{V}}\times\cdots\times(G_n)_{\mathsf{V}}. \tag{4.9}$$

To prove the inclusion from left to right in (4.8), take $u\in U_{\mathsf{V}}(G)$ and let $a_i\in G\setminus(G_i)_{\mathsf{V}}$. By (4.9), the n-tuple $(1,\dots,1,a_i,1,\dots,1)$, with a_i in the ith position, belongs to $G\setminus G_{\mathsf{V}}$. Hence, there exists an n-tuple $(b_1,\dots,b_n)\in G$ such that

$$u((1,\dots,1,a_i,1,\dots,1),(b_1,\dots,b_n))\neq 1. \tag{4.10}$$

Now, the left side of (4.10) has ith component $u(a_i,b_i)$, in G_i, and remaining components of the form $u(1,b_j)$, in G_j. Since $u\in U_{\mathsf{V}}(G_1\times\cdots\times G_n)\subseteq(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ and $\mathsf{G}\models u(1,x_2)=1$, it follows from (4.10) that $u(a_i,b_i)\neq 1$. Hence $u\in U_{\mathsf{V}}(G_i)$.

For the reverse inclusion, let $u\in\bigcap_{i=1}^n U_{\mathsf{V}}(G_i)$ and suppose that $a=(a_1,\dots,a_n)$ is an element of $G\setminus G_{\mathsf{V}}$. By (4.11), there is some index i such that $a_i\notin(G_i)_{\mathsf{V}}$. Since $u\in U_{\mathsf{V}}(G_i)$, there exists $b_i\in G_i$ such that $u(a_i,b_i)\neq 1$. Hence, for $b=(1,\dots,1,b_i,1,\dots,1)$, with ith component b_i, we have $u(a,b)\neq 1$, which shows that $u\in U_{\mathsf{V}}(G)$. $\square$

The relevance of the sets $U_{\mathsf{V}}(G)$ comes from the following result.

Proposition 4.8. *Let* V *be an extension-closed pseudovariety of groups containing* Ab. *Then the set* $\bigcap_{G\in\mathsf{G}} U_{\mathsf{V}}(G)$ *consists precisely of the binary implicit operations* u *that characterize the* V*-radical of finite groups.*

Proof. Suppose that $u \in U_{\mathsf{V}}(G)$ for every finite group G. We show that u characterizes the V-radical of finite groups, that is, for every finite group G, its V-radical is given by the formula

$$G_{\mathsf{V}} = \{a \in G : \forall b \in G, u(a,b) = 1\}. \tag{4.11}$$

Indeed, if $a \in G \setminus G_{\mathsf{V}}$ then $u \in U_{\mathsf{V}}(G) \subseteq U_{\mathsf{V},a}(G)$ and, therefore, there exists $b \in G$ such that $u(a,b) \neq 1$. Suppose next that $a \in G_{\mathsf{V}}$. Given $b \in G$, the equality $u(a,b) = 1$ holds by Lemma 4.3 since $u \in (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$, which completes the proof of equation (4.11).

Conversely, suppose that u is a binary implicit operation which characterizes the V-radical of finite groups. If G is a group in V then $G_{\mathsf{V}} = G$ and so, in view of (4.11), we obtain $u(a,b) = 1$ for all $a, b \in G$. Hence the pseudoidentity $u = 1$ holds in V, which shows that $u \in (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$. On the other hand, for an arbitrary finite group G, from (4.11) it also follows that, if $a \in G \setminus G_{\mathsf{V}}$, then there exists $b \in G$ such that $u(a,b) \neq 1$, whence $u \in U_{\mathsf{V}}(G)$. □

The following result is a simple compactness theorem which reformulates the existence of binary singleton characterizations of the V-radical which work for all finite groups in terms of binary singleton characterizations of the V-radical for each specific finite group.

Theorem 4.9. *Let* V *be an extension-closed pseudovariety of groups containing* Ab. *Then the set* $U_{\mathsf{V}}(G)$ *is nonempty for every finite group* G *if and only if the* V*-radical admits a binary singleton characterization.*

Proof. By Proposition 4.8, it suffices to show that, if each of the sets $U_{\mathsf{V}}(G)$ $(G \in \mathsf{G})$ is nonempty, then so is their intersection. Now, from (4.8) we conclude that the family of closed subsets $(U_{\mathsf{V}}(G))_{G\in\mathsf{G}}$ of $(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ has the nonempty finite intersection property. By compactness, the intersection of the family is nonempty. □

We proceed to formulate the existence of binary characterizations of the V-radical in terms of properties of the sets $U_{\mathsf{V},a}(G)$.

Proposition 4.10. *For an extension-closed pseudovariety of groups* V *containing* Ab, *the* V*-radical admits a binary characterization if and only if, for every finite group* G *and every* $a \in G \setminus G_{\mathsf{V}}$, *the set* $U_{\mathsf{V},a}(G)$ *is nonempty.*

Proof. Suppose that W is a binary characterization of the V-radical and let $G \in \mathsf{G}$ and $a \in G \setminus G_{\mathsf{V}}$. Then there exist $b \in G$ and $w \in W$ such that $w(a,b) \neq 1$. Since $w \in W \subseteq (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$, it follows that $w \in U^b_{\mathsf{V},a}(G) \subseteq U_{\mathsf{V},a}(G)$, which shows that $U_{\mathsf{V},a}(G) \neq \emptyset$.

For the converse, let W be the union of all $U_{\mathsf{V},a}(G)$ with $G \in \mathsf{G}$ and $a \in G \setminus G_{\mathsf{V}}$. We claim that W characterizes the V-radical. Indeed, given $a \in G_{\mathsf{V}}$ and $b \in G$, $w(a,b)=1$ for all $w \in W$ by Lemma 4.3 since $W \subseteq (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$. On the other hand, if $a \in G \setminus G_{\mathsf{V}}$, then by hypothesis there exists $w \in W$ such that $w(a,b) \neq 1$ for some $b \in G$. Hence, W characterizes the V-radical. □

Combining Theorem 4.1 with Propositions 4.4 and 4.10, we deduce that, for every finite group G and every $a \in G$, the set $U_{\mathsf{G}_{\mathrm{sol}},a}(G)$ is nonempty. On the other hand, in view of Theorem 4.9, the conjecture of Bandman et al. about the solvable radical amounts to the set

$$U_{\mathsf{G}_{\mathrm{sol}}}(G) = \bigcap_{a \in G \setminus G_{\mathsf{G}_{\mathrm{sol}}}} U_{\mathsf{G}_{\mathrm{sol}},a}(G)$$

being nonempty for every finite group G.

Lemma 4.11. *Let* V *be an extension-closed pseudovariety containing* Ab. *If the sets* $U_{\mathsf{V},a_1}(G)$ *and* $U_{\mathsf{V},a_2}(G)$ *are nonempty for a given finite group* G *and elements* $a_1, a_2 \in G$ *then the intersection* $U_{\mathsf{V},a_1}(G) \cap U_{\mathsf{V},a_2}(G)$ *is also nonempty.*

Proof. If at least one of the a_i belongs to G_{V}, then $U_{\mathsf{V},a_i}(G) = (\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ by Lemma 4.3. Hence, the intersection $U_{\mathsf{V},a_1}(G) \cap U_{\mathsf{V},a_2}(G)$ is the other $U_{\mathsf{V},a_j}(G)$, which is nonempty by hypothesis. Hence, we may assume that neither a_1 nor a_2 belong to G_{V}. Let $u_i \in U_{\mathsf{V},a_i}(G)$ $(i=1,2)$. Then there exist $b_i \in G$ such that $u_i(a_i,b_i) \neq 1$ $(i=1,2)$.

If, for some $i \in \{1,2\}$, there is $g \in G$ such that $u_i(a_j,g) \neq 1$, where $\{i,j\} = \{1,2\}$, then $u_i \in U_{\mathsf{V},a_1}(G) \cap U_{\mathsf{V},a_2}(G)$, and we are done. Hence, we may assume that $u_i(a_j,g)=1$ whenever $i \neq j$ and $g \in G$. Let $u = u_1u_2$. Then u is an element of $(\overline{\Omega}_2\mathsf{S})^{\mathsf{V}}$ such that $u(a_i,b_i) = u_i(a_i,b_i) \neq 1$ $(i=1,2)$, and so $u \in U_{\mathsf{V},a_1}(G) \cap U_{\mathsf{V},a_2}(G)$. □

We did not manage to show that $U_{\mathsf{V}}(G)$ is always nonempty for every finite group under the hypothesis that $U_{\mathsf{V},a}(G) \neq \emptyset$ for every finite group G and $a \in G$. To illustrate the difficulty, we consider the case of three elements a_1, a_2, a_3 of a finite group for which we assume that each $U_{\mathsf{V},a_i}(G)$ is nonempty. The aim is to show that $\bigcap_{i=1,2,3} U_{\mathsf{V},a_i}(G) \neq \emptyset$. Assuming that V is extension closed and contains Ab, as in the proof of Lemma 4.11 it suffices to consider the case in which none of the a_i belongs to G_{V}. By Lemma 4.11, for each $i \in \{1,2,3\}$, there exists $v_i \in \bigcap_{j \neq i} U_{\mathsf{V},a_j}(G)$. We may further assume that $v_i(a_i,g)=1$ for every $g \in G$ for, otherwise, $v_i \in \bigcap_{i=1,2,3} U_{\mathsf{V},a_i}(G)$ and we are done. Moreover, if $v_1(a_3,c)^2 \neq 1$ for some $c \in G$, then either $w = v_1v_2$ or $w = v_1^{\omega-1}v_2$ belongs to $\bigcap_{i=1,2,3} U_{\mathsf{V},a_i}(G)$: indeed, for $\{i,j\} = \{1,2\}$ $w(a_i,g) = v_j(a_i,g)^{\pm 1}$ is not the identity element for some $g \in G$; on the other hand, if $v_1(a_3,c)^{-1}v_2(a_3,c) = 1$ then $v_1(a_3,c)v_2(a_3,c) \neq 1$ by hypothesis. It remains to consider the case where $v_i(a_j,g)^2 = 1$ whenever $g \in G$ and $i \neq j$, which we do not know how to handle.

PROBLEM 4.12. Let V be an extension-closed pseudovariety. Is it true that, for every finite group G, the set $U_{\mathsf{V}}(G)$ is nonempty?

In view of Theorem 4.9, for an extension-closed pseudovariety V containing Ab, an affirmative answer is equivalent to the existence of a binary implicit characterization of the V-radical. Equivalently, it means that there exists a sequence $w_n(x_1, x_2)$ of words in the letters x_1, x_2 which converges in $\overline{\Omega}_2\mathsf{S}$ such that, for every finite group G and every $a \in G$, $a \in G_{\mathsf{V}}$ if and only if, for every $b \in G$, $w_n(a, b) = 1$ for all sufficiently large n. In particular, Problem 4.12 generalizes to arbitrary extension-closed pseudovarieties of groups the Bandman et al. conjecture for the case of solvable groups.

An alternative characterization of radicals has been receiving a lot of attention from group theorists. It is based on the observation that, for a finite group G, an element g lies in the V-radical if and only if its conjugacy class g^G generates a subgroup from V. Thus, one may ask, if one needs to consider the subgroup generated by the whole conjugacy class g^G or whether a much smaller subset, of size bounded by some number independent of G suffices. The Baer–Suzuki theorem shows that two elements suffice for $\mathsf{V} = \mathsf{G}_{\mathrm{nil}}$. For $\mathsf{V} = \mathsf{G}_{\mathrm{sol}}$, it has been recently shown that four elements suffice, while two suffice if they have prime order $p > 3$ [17, 19–22]. There seems to be no obvious relationship between this type of characterization of radicals and the implicit characterizations considered in this section.

5. Semigroup radicals

Let V be a pseudovariety of semigroups. We denote by LV the class of all finite semigroups S such that, for every idempotent $e \in S$, the monoid eSe belongs to V. We say that a congruence on a finite semigroup is a *congruence over* V if its idempotent classes belong to V.

The purpose of this section is to give a description of the largest congruence over LH on a finite semigroup S when H is a Fitting pseudovariety. There is already such a description available [25]. It is formulated in terms of the Rees matrix structure of regular $\mathcal{J}$-classes. Ours, which appears to be more suitable for the applications in Section 6, is essentially an extension of the description given in [26] for the case of $\mathsf{H} = \mathsf{G}$ (see also [3] for the case of $\mathsf{H} = \mathsf{G}_p$ and the connections of both with representation theory).

Let J be a regular $\mathcal{J}$-class of a finite semigroup S and let G_J be a maximal subgroup contained in J. Let N be a normal subgroup of G_J. We denote by R_i $(i \in I)$ the $\mathcal{R}$-classes of J and by L_λ $(\lambda \in \Lambda)$ the $\mathcal{L}$-classes of J. Suppose that $G_J = R_1 \cap L_1$. For each $i \in I$ and $\lambda \in \Lambda$, choose *coordinates* $r_i \in J$ such that $s \mapsto r_i s$ is a bijection $R_i \to R_1$ and $l_\lambda \in J$ such that $s \mapsto s l_\lambda$ is a bijection $L_\lambda \to L_1$. With this notation, if $H_{i\lambda} = R_i \cap L_\lambda$, then $s \mapsto r_i s l_\lambda$ is a bijection $H_{i\lambda} \to G_J$.

We define a congruence by $s \equiv_{(J,G_J,N)} t$ if and only if, for all $x, y \in J$,

$$xsy \in J \iff xty \in J \tag{5.1}$$

and in this case if $x \in R_i$ and $y \in L_\lambda$, then

$$r_i xsyl_\lambda N = r_i xtyl_\lambda N. \tag{5.2}$$

The quotient $S/{\equiv_{(J,G_J,N)}}$ is denoted $\mathsf{GGM}(J,G_J,N)$ [26]. In case S is a group, $S = J = G_J$ and $\equiv_{(J,G_J,N)}$ is the congruence determined by the normal subgroup N. Note also that, if K is another normal subgroup of G_J then

$$N \subseteq K \implies {\equiv_{(J,G_J,N)}} \subseteq {\equiv_{(J,G_J,K)}}. \tag{5.3}$$

From hereon, H always denotes a Fitting pseudovariety of groups. For a finite semigroup S, we define $\mathrm{Rad}_{\mathsf{H}}(S)$ to be the congruence on S which is obtained by taking the intersection of all congruences of the form $\equiv_{(J,G_J,(G_J)_{\mathsf{H}})}$. It is a standard exercise in semigroup theory to show that the congruence $\equiv_{(J,G_J,(G_J)_{\mathsf{H}})}$ depends only on J and not on the choice of the maximal subgroup G_J and of the coordinates.

Theorem 5.1. *The congruence* $\mathrm{Rad}_{\mathsf{H}}(S)$ *on a finite semigroup* S *is the largest congruence over* LH *on* S.

Proof. Suppose that θ is a congruence over LH on S and let $(s,t) \in \theta$. We show that $(s,t) \in \mathrm{Rad}_{\mathsf{H}}(S)$. Let J be a regular $\mathcal{J}$-class of S and suppose that $x, y, xsy \in J$. Let $z \in S$ be such that $xsyz$ is an idempotent in J. Then $xsyz$ and $xtyz$ lie in the same idempotent θ-class T. Since θ is a congruence over LH by hypothesis, the subsemigroup T belongs to LH. As the elements $xsyz$ and $xtyz$ both lie in T and $xsyz$ is regular, it follows that we have the following chain of relations in $S: x \geq_{\mathcal{J}} xty \geq_{\mathcal{J}} xtyz \geq_{\mathcal{J}} xsyz \geq_{\mathcal{J}} x$. Hence, $xty \in J$ which, together with the dual argument, establishes condition (5.1). Suppose next that $x, y, xsy, xty \in J$, say $x \in R_i$ and $y \in L_\lambda$. Choose G_J to be the maximal subgroup containing the idempotent $xsyz$ and let $N = G_J \cap T$. Then N is a normal subgroup of G_J which is contained in the semigroup T from LH, and so $N \in \mathsf{H}$. In particular, $N \subseteq (G_J)_{\mathsf{H}}$ and the congruence $\equiv_{(J,G_J,N)}$ is contained in $\equiv_{(J,G_J,(G_J)_{\mathsf{H}})}$ by (5.3). Since the elements $r_i xsyl_\lambda$ and $r_i xtyl_\lambda$ lie in G_J and they are θ-equivalent, they define the same N-coset. Hence, s and t are $\equiv_{(J,G_J,N)}$-equivalent and therefore they are also $\equiv_{(J,G_J,(G_J)_{\mathsf{H}})}$-equivalent. Since the regular $\mathcal{J}$-class J of S is arbitrary, we conclude that $(s,t) \in \mathrm{Rad}_{\mathsf{H}}(S)$. This establishes that $\theta \subseteq \mathrm{Rad}_{\mathsf{H}}(S)$.

It remains to show that $\mathrm{Rad}_{\mathsf{H}}(S)$ is itself a congruence over LH. Let T be an idempotent class of $\mathrm{Rad}_{\mathsf{H}}(S)$. We must verify that, for every idempotent e of T, eTe is a group from H. Let J be the $\mathcal{J}$-class of S which contains e and let G_J be the maximal subgroup containing e. Since T is a $\mathrm{Rad}_{\mathsf{H}}(S)$-class, in particular every element x of eTe is such that $x \equiv_{(J,G_J,(G_J)_{\mathsf{H}})} e$, hence x lies in J. Since $x \in eTe$, it follows that $x \in G_J$ and so $x \in (G_J)_{\mathsf{H}}$ by (5.2). Hence, eTe is a subgroup of $(G_J)_{\mathsf{H}}$, which shows that $eTe \in \mathsf{H}$ and completes the proof of the theorem. $\square$

For two pseudovarieties V and W, denote by $\mathsf{V} \textcircled{m} \mathsf{W}$ the pseudovariety generated by the class of all finite semigroups S which admit a congruence ρ over V such that $S/\rho \in \mathsf{W}$. The following result can be easily deduced from Theorem 5.1 (cf. [25]).

Theorem 5.2. *Let S be a finite semigroup. Then $S \in \mathsf{LH} \textcircled{m} \mathsf{V}$ if and only if the quotient $S/\operatorname{Rad}_{\mathsf{H}}(S)$ belongs to V.*

An immediate application is the following decidability result, where a pseudovariety is said to be *decidable* if there is an algorithm for testing membership of finite semigroups in it. It is a particular case of a more general result from [25, Corollary 2.12].

Corollary 5.3. *If H is a decidable Fitting pseudovariety of groups and V is a decidable pseudovariety of semigroups then the Mal'cev product $\mathsf{LH} \textcircled{m} \mathsf{V}$ is decidable.*

Note that in general the Mal'cev product of decidable pseudovarieties may not be decidable [6, 34].

6. Bases of pseudoidentities

We say that an n-tuple $(\alpha_1, \dots, \alpha_n)$ of members of $\overline{\Omega}_n\mathsf{S}$ is *group-generic* if the following conditions hold:

- given a finite semigroup S and n elements $s_1, \dots, s_n \in S$, the elements $\alpha_i(s_1, \dots, s_n)$ $(i = 1, \dots, n)$ lie all in the same subgroup of S;
- if G is a finite group and $g_1, \dots, g_n \in G$ then $\alpha_i(g_1, \dots, g_n) = g_i$ $(i = 1, \dots, n)$.

The existence and characterizations of such tuples have been extensively investigated in [4]. A simple example is obtained by considering the continuous endomorphism φ of the free profinite semigroup $\overline{\Omega}_n\mathsf{S}$ which maps x_i to $x_1 \cdots x_i^{\varepsilon_i} \cdots x_n$ $(i = 1, \dots, n)$, where $\varepsilon_i = 2$ for $i < n$ and $\varepsilon_n = 1$. Since the monoid of continuous endomorphisms of a finitely generated profinite semigroup is itself profinite [2], there is a unique idempotent limit φ^ω of sequences of finite powers of φ, namely $\varphi^\omega = \lim_{n\to\infty} \varphi^{n!}$. We can take $\alpha_i = \varphi^\omega(x_i)$ $(i = 1, \dots, n)$ [4].

Throughout this section, we suppose again that H is a Fitting pseudovariety. We now show how characterizations of the radical may be used to obtain bases of pseudoidentities for pseudovarieties of the form $\mathsf{LH} \textcircled{m} \mathsf{V}$.

THEOREM 6.1. *Let $\Sigma = \{u_i = v_i : i \in I\}$ be a set of pseudoidentities and let $\mathsf{V} = [\![\Sigma]\!]$. Suppose that W is an $(m+1)$-ary characterization of the H-radical. Then the Mal'cev product $\mathsf{LH} \,ⓜ\, \mathsf{V}$ is defined by the following pseudoidentities*:

$$((xu_iy)^\omega xv_iy(xu_iy)^\omega)^\omega = (xu_iy)^\omega, \tag{6.1}$$

$$\begin{aligned} &w(\alpha_1(xv_iy, z_1, \dots, z_m)^{\omega-1}\alpha_1(xu_iy, z_1, \dots, z_m)\alpha_1(xv_iy, z_1, \dots, z_m)^\omega,\\ &\quad\alpha_2(xv_iy, z_1, \dots, z_m), \dots, \alpha_{m+1}(xv_iy, z_1, \dots, z_m))\\ &\quad= \alpha_1(xv_iy, z_1, \dots, z_m)^\omega, \end{aligned} \tag{6.2}$$

with $i \in I$ and $w \in W$, where $x, y, z_1, \dots, z_m$ are new variables and the α_j are such that $(\alpha_1, \dots, \alpha_{m+1})$ is a group-generic $(m+1)$-tuple of implicit operations. In particular, *if V is finitely based and W is finite*, *then $\mathsf{LH} \,ⓜ\, \mathsf{V}$ is also finitely based.*

Proof. We first show that $\mathsf{LH} \,ⓜ\, \mathsf{V}$ satisfies the pseudoidentities (6.1) and (6.2). Let S be a semigroup in $\mathsf{LH} \,ⓜ\, \mathsf{V}$. By Theorem 5.2, the quotient $S/\operatorname{Rad}_{\mathsf{H}}(S)$ belongs to V. Consider the values s and t resulting from an evaluation of the implicit operations xu_iy and xv_iy in S. Let $\sigma(z_0, z_1, \dots, z_n) \in \overline{\Omega}_{m+1}\mathsf{S}$ be an implicit operation which is an element of a subgroup, whose idempotent we denote by e. Since $\operatorname{Rad}_{\mathsf{H}}(S)$ is a congruence, given any $r_1, \dots, r_n \in S$, the elements $\sigma(s, r_1, \dots, r_n)$ and $\sigma(t, r_1, \dots, r_n)$ are in the same $\operatorname{Rad}_{\mathsf{H}}(S)$-class. Consider the idempotent $\bar{e} = e(s, r_1, \dots, r_n)$. We claim that, as a consequence of Theorem 5.1, the element $\bar{e}\sigma(t, r_1, \dots, r_n)\bar{e}$ belongs to the maximal subgroup of S containing $\bar{e}$:

$$\bar{e} \,\mathcal{H}\, \sigma(s, r_1, \dots, r_n) \,\mathcal{H}\, \bar{e}\sigma(t, r_1, \dots, r_n)\bar{e}. \tag{6.3}$$

Indeed, since S is a finite semigroup, there is some finite word w such that $S \models \sigma = w$. Note that $\bar{e} = (w(s, r_1, \dots, r_n))^\omega$. We consider w as a word in the variables $z_0, z_0', z_1, \dots, z_n$ and we show that changing the first occurrence of z_0 to z_0' in w leads to a word w' such that

$$\bar{e}w'(s, t, r_1, \dots, r_n)\bar{e} \,\mathcal{H}\, \bar{e}. \tag{6.4}$$

Let $w = w_1 z_0 w_2$, where z_0 does not occur in w_1. Then the products $\bar{e}w_1(s, t, r_1, \dots, r_n)$ and $w_2(s, t, r_1, \dots, r_n)\bar{e}$ are both elements of the $\mathcal{J}$-class J of $\bar{e}$. Let G be the maximal subgroup of S containing $\bar{e}$. Since $s \equiv_{(J,G,G_{\mathsf{H}})} t$ by the definition of $\operatorname{Rad}_{\mathsf{H}}(S)$, we conclude that $\bar{e}w'(s, t, r_1, \dots, r_n)\bar{e}$ belongs to J and, therefore, it belongs to G, which proves (6.4). The claim (6.3) now follows by induction on the number of occurrences of z_0 in w.

We first apply (6.3) to the implicit operation

$$\sigma(z_0, z_1) = (z_1^\omega z_0 z_1^\omega)^\omega,$$

with $r_1 = t$. The claim yields the first of the following equalities

$$(t^\omega s t^\omega)^\omega = (t^\omega t t^\omega)^\omega = t^\omega,$$

which shows that S satisfies (6.1). On the other hand, if $\alpha \in \overline{\Omega}_{m+1}\mathsf{S}$ lies in a subgroup and we let

$$\sigma(z_0, z_1, \vec{z}) = (\alpha(z_1, \vec{z})^{\omega}\alpha(z_0, \vec{z})\alpha(z_1, \vec{z})^{\omega})^{\omega+1},$$

where $\vec{z}$ abbreviates $z_2, \dots, z_{m+1}$, then by (6.3), for any m-tuple $\vec{r}$ of elements of S the elements $\bar{s} = \sigma(s, t, \vec{r})$ and $\bar{t} = \sigma(t, t, \vec{r}) = \alpha(t, \vec{r})$ lie in a maximal subgroup G of S. Since $\bar{s}$ and $\bar{t}$ are $\operatorname{Rad}_{\mathsf{H}}(S)$-equivalent, the element

$$\begin{aligned}\bar{t}^{-1}\bar{s} &= \alpha(t, \vec{r})^{\omega-1}(\alpha(t, \vec{r})^{\omega}\alpha(s, \vec{r})\alpha(t, \vec{r})^{\omega})^{\omega+1}\\ &= \alpha(t, \vec{r})^{\omega-1}\alpha(s, \vec{r})\alpha(t, \vec{r})^{\omega}\end{aligned}$$

belongs to the unipotent radical G_{H}. In particular, if we let $\alpha = \alpha_1$, since the elements $(\alpha_k)(t, \vec{r})$ $(k = 1, \dots, m+1)$ all lie in G and W characterizes the H-radical, the following equality holds for every $w \in W$:

$$w(\bar{t}^{-1}\bar{s}, \alpha_2(t, \vec{r}), \dots, \alpha_{m+1}(t, \vec{r})) = \alpha_1(t, \vec{r})^{\omega},$$

which shows that S satisfies (6.2).

Conversely, let S be a finite semigroup that satisfies the pseudoidentities (6.1) and (6.2). By Theorem 5.2, it suffices to show that $S/\operatorname{Rad}_{\mathsf{H}}(S)$ satisfies each of the pseudoidentities $u_i = v_i$. Consider again the values s and t resulting from an evaluation of the implicit operations u_i and v_i in S, respectively. We claim that $(s, t) \in \operatorname{Rad}_{\mathsf{H}}(S)$. By the definition of $\operatorname{Rad}_{\mathsf{H}}(S)$, we should show that $s \equiv_{(J,G,G_{\mathsf{H}})} t$ for every regular $\mathcal{J}$-class J of S, any maximal subgroup G of S contained in J, and "coordinates" r_a, l_b. Recall that the subgroup and coordinates may be suitably chosen since the congruence $\equiv_{(J,G,G_{\mathsf{H}})}$ does not depend on them.

Let $\bar{x}, \bar{y} \in J$, for a regular $\mathcal{J}$-class J, and suppose that $\bar{x}s\bar{y} \in J$. Let $\bar{z} \in S$ be such that $\bar{x}s\bar{y}\bar{z}$ is an idempotent in J. Then, from the pseudoidentity (6.1) we deduce that $\bar{x}s\bar{y}\bar{z} = ((\bar{x}s\bar{y}\bar{z})^{\omega}\bar{x}t\bar{y}\bar{z}(\bar{x}s\bar{y}\bar{z})^{\omega})^{\omega}$ which shows that

$$\bar{x} \leq_{\mathcal{J}} \bar{x}s\bar{y}\bar{z} \leq_{\mathcal{J}} \bar{x}t\bar{y}\bar{z} \leq_{\mathcal{J}} \bar{x}t\bar{y} \leq_{\mathcal{J}} \bar{x} \tag{6.5}$$

and so $\bar{x}t\bar{y} \in J$. Conversely, assuming that $\bar{x}t\bar{y} \in J$, let $\bar{z} \in S$ be such that $\bar{x}t\bar{y}\bar{z}$ is an idempotent in J. Complete the evaluation of the variables in the pseudoidentity $u_i = v_i$ to an evaluation of those in any pseudoidentity from (6.2), by making the following assignment to the new variables: $x \mapsto \bar{x}$, $y \mapsto \bar{y}\bar{z}$, $z_i \mapsto \bar{x}t\bar{y}\bar{z}$ $(i = 1, \dots, m)$. Then (6.2) yields that $\bar{x}s\bar{y}\bar{z}$ is a factor of $\bar{x}t\bar{y}\bar{z}$ from which it follows, as in (6.5) with s and t interchanged, that $\bar{x}s\bar{y} \in J$.[2]

Suppose next that the six elements $a, b, r, l, rasbl, ratbl$ lie in J and that the $\mathcal{H}$-class of $rasbl$ is a group G. Then $\bar{s} = rasbl$ and $\bar{t} = ratbl$ are both elements of G. Let $c_1, \dots, c_m$ be arbitrary elements of G and, for brevity, denote $(c_1, \dots, c_m)$ by $\vec{c}$. Since the $(m+1)$-tuple of implicit operations $(\alpha_1, \dots, \alpha_{m+1})$

[2] This argument is adapted from [30] where, among other results, a basis of pseudoidentities for $\mathsf{LG} \textcircled{m} \mathsf{V}$ is given in terms of a basis of pseudoidentities for V. The basis in question consists precisely of the pseudoidentities $((xu_iy)^{\omega}xv_iy(xu_iy)^{\omega})^{\omega} = (xu_iy)^{\omega}$ and its dual, which is obtained by interchanging u_i and v_i.

is group-generic, $\bar{s}=\alpha_1(\bar{s},\vec{c})$, $\bar{t}=\alpha_1(\bar{t},\vec{c})$, and $c_i=\alpha_{i+1}(\bar{s},\vec{c})$ $(i=1,\dots,m)$. We apply the pseudoidentities (6.2) with the evaluation of the new variables defined by $x\mapsto ra$, $y\mapsto bl$, and $z_i\mapsto c_i$, to obtain $w(\bar{t}^{-1}\bar{s},c_1,\dots,c_m)=\bar{t}^{\omega}$ whenever $w\in W$. Since W is assumed to be a characterization of the H-radical, it follows that $\bar{t}^{-1}\bar{s}\in\mathrm{Rad}_{\mathsf{H}}(G)$ which shows that $s\equiv_{(J,G,G_{\mathsf{H}})}t$. $\square$

As particular cases of Theorem 6.1, we exhibit bases of pseudoidentities for pseudovarieties of the form $\mathsf{LH}\,ⓜ\,\mathsf{V}$ for Fitting pseudovarieties H of special interest.

COROLLARY 6.2. *Let $\Sigma=\{u_i=v_i : i\in I\}$ be a set of pseudoidentities and let $\mathsf{V}=[\![\Sigma]\!]$. Then the Mal'cev product $\mathsf{LG}_{\mathrm{nil}}\,ⓜ\,\mathsf{V}$ is defined by the pseudoidentities (6.1) together with:*

$$[\beta(xv_iy,z),{}_{\omega}\alpha(xv_iy,z)^{\omega-1}\alpha(xv_iy,z)\alpha(xu_iy,z)^{\omega}]=\beta(xv_iy,z)^{\omega}, \tag{6.6}$$

with $i\in I$, where x,y,z are new variables and (α,β) is a fixed group-generic pair of implicit operations. If V is finitely based, then so is $\mathsf{LG}_{\mathrm{nil}}\,ⓜ\,\mathsf{V}$.

COROLLARY 6.3. *Let $\Sigma=\{u_i=v_i : i\in I\}$ be a set of pseudoidentities and let $\mathsf{V}=[\![\Sigma]\!]$. Then the Mal'cev product $\mathsf{LG}_p\,ⓜ\,\mathsf{V}$ is defined by the pseudoidentities (6.6) together with:*

$$((xv_iy)^{\omega-1}(xu_iy(xv_iy)^{\omega})^{\omega+1})^{p^{\omega}}=(xv_iy)^{\omega} \tag{6.7}$$

with $i\in I$, where x,y,z are new variables and (α,β) is a fixed group-generic pair of implicit operations. If V is finitely based, then so is $\mathsf{LG}_p\,ⓜ\,\mathsf{V}$.

Proof. The proof is obtained by minor adaptations of the proof of Theorem 6.1 taking into account that, in a finite group G, an element a lies in G_{G_p} if and only if it lies in $G_{\mathsf{G}_{\mathrm{nil}}}$ and it has order a power of p. $\square$

Note that if the implicit operations u_i,v_i of the basis of pseudoidentities of V are computable then so are the implicit operations of the bases of the Mal'cev products given by Corollaries 6.2 and 6.3.

The following result depends on the Bandman et al. conjecture.

COROLLARY 6.4. *Let $\Sigma=\{u_i=v_i : i\in I\}$ be a set of pseudoidentities and let $\mathsf{V}=[\![\Sigma]\!]$. If the Bandman et al. conjecture holds and $\{u\}$ is a binary characterization of the solvable radical, then the Mal'cev product $\mathsf{LG}_{\mathrm{sol}}\,ⓜ\,\mathsf{V}$ is defined by the pseudoidentities (6.1) together with:*

$$u(\alpha(xv_iy,z)^{\omega-1}\alpha(xv_iy,z)\alpha(xu_iy,z)^{\omega},\beta(xv_iy,z))=\beta(xv_iy,z)^{\omega},$$

with $i\in I$, where x,y,z are new variables and (α,β) is a fixed group-generic pair of implicit operations. Hence, still under the hypothesis that the Bandman et al. conjecture holds, if V is finitely based then so is $\mathsf{LG}_{\mathrm{sol}}\,ⓜ\,\mathsf{V}$.

Another type of application is the following. Say that a pseudovariety V has *rank* n if it admits a basis of pseudoidentities in n variables. Equivalently, V has rank n if a finite semigroup S lies in V if and only if all its n-generated subsemigroups lie in V.

Corollary 6.5. *Suppose that the Fitting pseudovariety* H *admits an* $(m+1)$*-ary characterization and that the pseudovariety* V *has rank* n. *Then* $\mathsf{LH} \textcircled{m} \mathsf{V}$ *has rank at most* $n+m+2$.

In particular, in view of Theorem 4.1 and Proposition 4.4, if V has rank n then $\mathsf{LG}_{\mathrm{sol}} \textcircled{m} \mathsf{V}$ has rank at most $n+4$.

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