RANDOM WALKS ON HYPERPLANE ARRANGEMENTS

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Introduction

In this paper, we will study a family of random walks presented by Bidigare, Hanlon, and Rockmore [4]. The walk is studied further by Brown, and Diaconis [2] and Brown, Billera, and Diaconis [3]. Brown [1] regards the walk as a walk on semigroup under some restriction. His work gives a method for calculating the stationary distribution and the rate of convergence of the walk. We will implement this method to the calculation for the hyperplane chamber walk.

1. Background

1.1. **Hyperplane Arrangement.** Let H_i be a hyperplane in $\mathbb{R}^d$. The complement of H_i in $\mathbb{R}^d$ is a pair of disjoint connected spaces. We pick one of these spaces and define H_i^+ to be its union with H_i. We define H_i^- to be the union of H_i with the other space.

Example 1.1.1. One of the two ways to define H_i^+, H_i^- for $H_i = \{0\}$ as a hyperplane in $\mathbb{R}$ is

$$H_i^+ = [0, \infty), \quad H_i^- = (-\infty, 0].$$

Regardless of prior intuition we may have, we call H_i^+ the positive side of H_i and H_i^- the negative side of H_i.

A set $\mathcal{A}$ of hyperplanes in $\mathbb{R}^d$ is called a *hyperplane arrangement* in $\mathbb{R}^d$. An arrangement of hyperplanes passing through the origin is called *central.* Although one might successfully develop a similar theory on arbitrary arrangements, we will consider only central arrangements in this paper.

1.2. **Faces, vertices, and chambers.** Intersections of collection of H_i^+'s and H_i^-'s are connected subspaces of $\mathbb{R}^d$. We called these subspaces *faces.* More precisely, we define

$$\mathcal{F} = \left\{ \bigcap_{i=1}^{n} H_i^{\sigma_i} : \sigma_i \in \{+, -, 0\} \right\}$$

to be the set of all faces in $\mathcal{A}$. Since $\mathcal{A}$ is central, the above intersection is never empty. Note that the dimensions of faces of an arrangement $\mathcal{A}$ in $\mathbb{R}^d$ range from 0 to d. We call 1 dimensional faces *vertices* and 2 dimensional faces *edges.* The maximal faces, those of dimension d, are

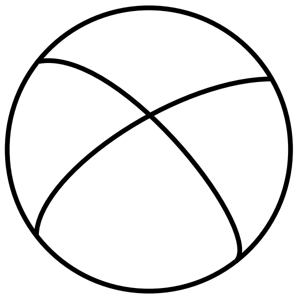

FIGURE 1. A visualization of an arrangement of 3 planes

called *chambers*. These notations come from the following geometric representation of the arrangement.

To help visualizing, we intersect the arrangement with a unit sphere in $\mathbb{R}^d$ centered at origin. Figure 1 shows an intersection of 3 planes with the sphere. This is a view of the northern hemisphere with the circle representing an intersection of the sphere with a plane while the arcs are the great circles which are the intersection of other planes with the sphere. The geometric property of great circles ensures that the view from southern hemisphere will look similar. In this picture, vertices are the intersections of 2 arcs, and edges are the arcs connecting two vertices. Chambers are the connected areas bounded by edges.

1.3. **Product of sign sequence.** From the definition of face, note that we have a unique sign sequence $\sigma_1\sigma_2\dots\sigma_n$ associates to each face $F = \bigcap_{i=1}^n H_i^{\sigma_i}$. The sign sequence helps to locate the face by telling which side of each H_i the face is in. We define a product of sign sequences by

$$\sigma_i(FG) = \begin{cases} \sigma_i(F) & \text{if } \sigma_i(F) \neq 0 \\ \sigma_i(G) & \text{otherwise,} \end{cases}$$

where $\sigma_i(K)$ is the sign sequence of face K at i-th position.

Example 1.3.1. Consider an arrangement of 4 planes as shown in Figure 2. Let $F = +0-0$ and $G = ++++$. Then $FG = ++-+$ which is a chamber adjacent to F closest to G in the sense that one can go from G to FG by crossing the least number of planes.

This product gives a semigroup structure on $\mathcal{F}$. Since the origin $\{0\} = 00\dots0$ acts as an identity under the product of sign sequence, $\mathcal{F}$ always contains an identity. The associativity of this product follows immediately from the definition. To see that $\mathcal{F}$ is closed under the product consider the following. If a face F contains no zero in its sign sequence, then $FG = F \in \mathcal{F}$ for any face $G \in \mathcal{F}$. So, assume that $\sigma_i(F) = 0$ for some i. Then F lies on the hyperplane H_i. Hence, there are two faces, F^+ and F^-, adjacent to F with all the same sign sequence as F except at the i-th position. These faces are the two

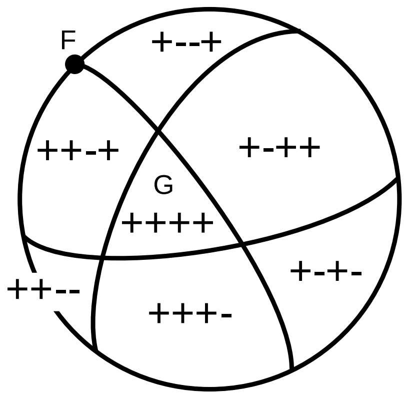

FIGURE 2. An arrangement of 4 planes with cells encoding

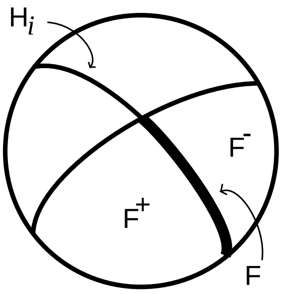

FIGURE 3. Changing zero at i-th position gives either F^+ or F^-.

faces separated by H_i at F and are indeed faces in $\mathcal{F}$ (see Figure 3). By changing the 0 in i-th position in the sign sequence of F, we have either F^+ or F^- as a resulting face. Any change of other zeroes in the sign sequence of F would have a similar result. Since a product FG is a change of a set of zeroes in the sign sequences of F, the product FG is a face for all $F, G \in \mathcal{F}$. Thus, the product of sign sequence is closed in $\mathcal{F}$.

1.4. **The random walk.** The product of sign sequence gives rise to a random walk on the set $\mathcal{F}$ of all faces of an arrangement. We start by giving weight to each face. Then we pick the starting face $F_0 \in \mathcal{F}$. Let F_j be the position of the walk at j-th step. We inductively define

$$F_j = FF_{j-1}$$

where $F \in \mathcal{F}$ is chosen according to the weight $\{w_F\}_{F\in\mathcal{F}}$.

From the definition of the product, we know that for any face F and G the number of zeroes in the sign sequence of FG cannot exceed that of G. It follows that, for large k, F_k should have no zero in its sign sequence as long as the weight does not concentrate on a hyperplane. This is amount to say that if there is no hyperplane containing all faces of nonzero weight, the stationary distribution of the walk is a linear combination of chambers.

Example 1.4.1. On an arrangement of 3 planes, let $w_{++0} = \frac{1}{2} = w_{--0}$. Observe that the weights concentrate on the plane corresponding to the last coordinate of the sign sequence. Suppose that the walk starts at $F_0 = \sigma_1\sigma_2\sigma_3$. Since

$$\begin{aligned} (++0)\cdot(***) &= ++* \\ \text{and } (--0)\cdot(***) &= --*, \quad * \in \{+,-,0\}\,, \end{aligned}$$

it is clear that after a large number of steps we can expect that the walk will be at $++\sigma_3$ or $--\sigma_3$ with the same probability. Thus,

$$\text{stationary distribution}(\pi) = \frac{1}{2}(++\sigma_3) + \frac{1}{2}(--\sigma_3).$$

Since σ_3 can be zero, π may involve terms of non-chamber faces as implied by the concentration of weights.

In order to analyze the walk further, we introduce the matrix of transition. Each row and column of the matrix represents each face. For simplicity, we use the order in such a way that the i-th row and the i-th column represent the same face. We define $K(F, F')$ to be the probability of the walk moving from F to F' in one step. This is exactly the sum of the weight of all faces G such that $GF = F'$. Hence, we have

$$K(F, F') = \sum_{GF=F'} w_G.$$

The transition matrix K is define by $K_{ij} = K(F, F')$ where F is represented by i-th column and F' is represented by j-th row. Since there is no possible confusion, we will refer to the entries of matrix K by the representation $K(F, F')$.

Example 1.4.2. Let $\mathcal{A}$ be an arrangement of two lines with uniform weight on the set of all vertices. That is each vertex has weight $\frac{1}{\#of vertices} = \frac{1}{4}$. We have the following matrix of transition.

$$\begin{array}{cccccccccc}
 & ++ & +- & -+ & -- & +0 & -0 & 0+ & 0- & 00 \\
++ & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\
+- & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
-+ & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
-- & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\
+0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
-0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
0+ & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0- & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Let $\vec{F}$ be the column vector representing the face F. Then the probability distribution of the next step of the walk is the column vector $K\vec{F}$. More generally, the walk starting at F has $K^\ell \vec{F}$ as the probability distribution of the walk after ℓ steps. It follows that the stationary distribution of the walk starting at F is

$$\pi_F = \lim_{\ell \to \infty} K^\ell \vec{F}.$$

A fundamental theorem of Markov chain theory implies that the stationary distribution is independent of the starting face. Thus,

$$\pi = \lim_{\ell \to \infty} K^\ell \vec{F}, \quad F \in \mathcal{F}.$$

Example 1.4.3. A computer computation shows that

$$\lim_{\ell \to \infty} K^\ell = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0)$. This means that after a large number of steps the walk is equally likely to be in one of the four chambers.

A natural question that one may ask is how large should ℓ be in order to be sure that the walk starting at F_0 is "close" to π after ℓ steps. To be more precise, we introduce a distance function. Let $\vec{u} = (u_0, \ldots, u_n)$ and $\vec{v} = (v_0, \ldots, v_n)$ be two vectors. Define the distance between $\vec{u}$ and $\vec{v}$ by

$$\|\vec{u} - \vec{v}\| = \max_{0 \le i \le n} |u_i - v_i|.$$

We call the distance $\|K^\ell \vec{F} - \pi\|$ the *convergence rate* of the walk starting at F. So, the question is equivalent to how large should ℓ be in order for the convergence rate to be smaller than some constant.

With enough effort, one can compute the convergence rate and the stationary distribution directly from the definition. However, most of the arrangements that we will encounter have large face sets which make the matrix multiplication too tedious. In this paper, we will develop a technique for computing the convergence rate and the stationary distribution of the walk.

2. Random walks on left regular bands

In studying the hyperplane chamber walk, it is useful to consider a structure called "left regular band". We will show that the hyperplane chamber walk is a walk on left regular bands. The walk on left regular bands admits properties that will be helpful in determining the stationary distribution and the convergence rate of the hyperplane chamber walk.

2.1. **Left regular band.**

A semigroup $\mathcal{F}$ is called a *left regular band* (LRB) if it satisfies the following conditions

$$F^2 = F \quad \text{and} \quad FGF = FG, \quad \forall F, G \in \mathcal{F}.$$

We call these conditions the "deletion properties". Intuitively, the reappearance of an element of $\mathcal{F}$ in the product is irrelevant to the overall outcome of the product. Since the set of all faces $\mathcal{F}$ of a hyperplane arrangement is a semigroup, we only need to check that $\mathcal{F}$ satisfies the deletion properties. To see this, recall that the product FG is just a change of a set of zeroes in the sign sequence of F to the corresponding sign of G. It is obvious that $F^2 = F$. Notice also that $\sigma_i(FG) = 0$ implies $\sigma_i(F) = 0$. So, right multiplication by F does not effect the overall product FGF. Therefore, $\mathcal{F}$ is a LRB. As we have seen earlier that $\mathcal{F}$ is always finite with identity, from now on we will only take our LRB to be finite and contains an identity.

2.2. **An action on $\mathbb{R}\mathcal{F}$.**

Consider vector space $\mathbb{R}\mathcal{F}$ of all linear combination $\sum_{F\in\mathcal{F}} a_F F$ where $a_F \in \mathbb{R}$. The product of sign sequence on $\mathcal{F}$ extends to $\mathbb{R}\mathcal{F}$. This gives a ring structure on $\mathbb{R}\mathcal{F}$.

Example 2.2.1. Let $\vec{a} = \sum_{F\in\mathcal{F}} a_F F$ and $\vec{b} = \sum_{F\in\mathcal{F}} b_F F$ be elements of $\mathbb{R}\mathcal{F}$. The product on $\mathcal{F}$ extends to the following product

$$\vec{a}\vec{b} = \sum_{F\in\mathcal{F}} a_F F \sum_{F\in\mathcal{F}} b_F F = \sum_{F\in\mathcal{F}} (\sum_{GG'=F} a_G b_{G'}) F$$

This is just a vector multiplication with the usual multiplication replaced by the product of sign sequence.

The weight distribution $w = \sum_{F\in\mathcal{F}} w_F F$ is also an element of this ring. It should be clear that each power w^ℓ is contained in this ring. In fact, $\mathbb{R}[w]$ is a subalgebra of $\mathbb{R}\mathcal{F}$ generated by the weight w.

We take the element $\vec{F} = 1 \cdot F \in \mathbb{R}\mathcal{F}$ to be the representation of the face $F \in \mathcal{F}$. Consider the product

$$w\vec{F} = \sum_{F'=GF} w_G F'$$

which is exactly the column vector of the transition matrix K represented by the face F. Since this is true for each $F \in \mathcal{F}$, we have $w\vec{F} = K\vec{F}$. That is, for each $F \in \mathcal{F}$ the right multiplication by w

acting on $\vec{F} \in \mathbb{R}\mathcal{F}$ is equivalent to the right multiplication by K acting on the column vector $\vec{F}$. More generally, we have

$$w^\ell \vec{F} = K^\ell \vec{F}.$$

By taking $\ell \to \infty$, we obtain a formula for the stationary distribution in term of w.

2.3. **Split semisimplicity.** An $\mathbb{R}$-algebra R is said to be *split semisimple* if it is isomorphic to $\mathbb{R}^I$ where I is a finite index set. This is equivalent to saying that R has an orthogonal basis $\{e_i\}_{i\in I}$ consisting of idempotents of R. We call e_i a *primitive idempotent* of R.

Let w be the generator of R. Consider the generating function

$$f(t) = \sum_{\ell=0}^{\infty} w^\ell t^\ell = \frac{1}{1-wt} \tag{2.3.1}$$

where the last equality follows from the formula for the sum of infinite power series. Let

$$g(z) = \frac{1}{z} f(1/z) = \frac{1}{z-w}. \tag{2.3.2}$$

Proposition 2.3.1. *An $\mathbb{R}$-algebra R is split semisimple if and only if the function $g(z)$ has the form*

$$g(z) = \sum_{i\in I} \frac{e_i}{z-\lambda_i} \tag{2.3.3}$$

where e_i is a primitive idempotent and $\lambda_i \in \mathbb{R}$. In this case, we have $w = \sum_{i\in I} \lambda_i e_i$.

Proof. Suppose R is split semisimple with basis $\{e_i\}_{i\in I}$ consisting of primitive idempotents. Since $w \in R$, we have a representation $w = \sum_{i\in I} \lambda_i e_i$ where $\lambda \in \mathbb{R}$. From the properties of primitive idempotents, we have that $e_i e_j = 0$ when $i \neq j$ and $e_i e_i = e_i$. It follows that $w^\ell = \sum_{i\in I} \lambda_i^\ell e_i$. Substitute this to the equation 2.3.1, we have

$$f(t) = \sum_{i\in I} (\sum_{\ell=0}^{\infty} \lambda_i^\ell t^\ell) e_i = \sum_{i\in I} \frac{e_i}{1-\lambda_i t}$$

where the last equality follows from the power series formula. Substitute this in equation 2.3.2 to obtain equation 2.3.3.

Conversely, suppose R is not split semisimple. Assume that the minimal polynomial p for w split into linear factor in $\mathbb{R}[x]$. Let $p(x) = \prod_{i\in I} (x-\lambda_i)^{r_i}$ where λ_i are distinct. By the Chinese remainder theorem, we have

$$R \cong \prod_{i\in I} \mathbb{R}[x]/(x-\lambda_i)^{r_i}.$$

Then some $r_i > 1$ because R is not split semisimple. Assume that $R = \mathbb{R}[x]/(x-\lambda)^r$ for some λ, where $r > 1$. Then $w = \lambda + b$ where $b^r = 0$ with $b^{r-1} \neq 0$. Hence,

$$\begin{aligned} g(z) &= \frac{1}{z-w} \\ &= \frac{1}{(z-\lambda)-b} \\ &= \frac{1}{z-\lambda} \cdot \frac{1}{1-(z-\lambda)^{-1}b} \\ &= \sum_{j=0}^{r-1} \frac{b^j}{(z-\lambda)^{j+1}} \end{aligned}$$

Thus, $g(z)$ has a pole of order $r > 1$ at $z = \lambda$. So, $g(z)$ does not have the form 2.3.3.

If p does not split into linear factors, extend scalars to a splitting field $\mathbb{R}'$ of p and apply the results above to $R' = \mathbb{R}' \otimes_{\mathbb{R}} R \cong \mathbb{R}'[x]/(p)$. Then $g(z)$ has poles at the roots of p, at least one of which is not in $\mathbb{R}$. Thus, $g(z)$ does not have the form 2.3.3. □

3. The primitive idempotents

In this section, we will calculate the primitive idempotents using proposition 2.3.1. We will do this by showing that $g(z)$ has the form 2.3.3. The proposition then gives us the primitive idempotents of the walk.

3.1. **Reduced words.** For simplicity in calculation, we admit some lost of generality and will consider only hyperplane arrangements in general position. An arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^d$ is said to be in *general position* if for each set of d distinct hyperplanes $\{H_i\}_{i=1}^{i=d}$, $\bigcap_{i=1}^d H_i = \{0\}$. Intuitively, if we form a central arrangement by choosing n hyperplanes at random from the set of all hyperplanes passing through the origin, the arrangement is more likely to be in general position. It follows from the definition that each vertex is an intersection of $d-1$ hyperplanes. In other words, each vertex of an arrangement in $\mathbb{R}^d$ lies on exactly $d-1$ hyperplanes, thus has $d-1$ zeroes in its sign sequence.

Product $F_1F_2\cdots F_\ell$ of ℓ faces with nonzero weight gives a *word* of length ℓ. We will use the representation $\vec{x} = (x_1, x_2, \ldots, x_\ell)$ for the word $x_1x_2\cdots x_\ell$. The deletion property on $\mathcal{F}$ suggest that some of the faces F_i may be removed without changing the outcome of the product. A word can be *reduced* by repeatedly perform the above process until such a removal is impossible. From the point of view of sign sequences, each face in a reduce word is nonzero and contains at least one nonzero sign at the position in which all the sign of the faces to its left are zeroes.

Example 3.1.1. Consider a reduced word of length 3

$$\vec{x} = (0+0)(-+0)(00+).$$

Observe that the second face has a nonzero sign at the first position while the first face has a zero sign. And the third face has a nonzero sign at the third position while the first and the second faces have zero signs. This makes $\vec{x}$ a reduced word.

Consider an arrangement $\mathcal{A}$ in $\mathbb{R}^d$ of $n \geq d$ hyperplanes. Recall that the vertices are nonzero faces with lowest dimension. So, a reduced word with maximal length are those consisting of vertices. Since $\mathcal{A}$ is in general position, each vertex has exactly $d-1$ zeroes in its sign sequence. It follows that the maximal length for a reduced word in $\mathcal{A}$ is d. Clearly, each reduced word of length d is a chamber, which has no zero in its sign sequence. Note that the terms in the stationary distribution of the walk should be the words of infinite length. This should leave no zero in the sign sequences of such words. We should then expect that the stationary distribution will involve only chambers. One might expect that the stationary distribution involves only the reduced word of length d. However, this is not the case since some chambers can be represented by reduced words of length less than d.

3.2. **Computation in $\mathbb{R}^2$.** Let $u_F = \sum_{FG=F} w_G$. This is to say u_F is the sum of the weights of faces G with $FG = F$. Consider an arrangement $\mathcal{A}$ of $n \geq 2$ lines in general position. Let $w = \sum_{F\in\mathcal{F}} w_F F$ be the weight distribution of the walk on the face semigroup $\mathcal{F}$ of $\mathcal{A}$. We will compute w^ℓ in terms of the coefficients of each face. In what follows, we will abbreviate the notation by letting $w_i = w_{x_i}$ and $u_i = u_{x_i}$ and $u_{i,j} = u_{x_i x_j}$.

Coefficients of w^ℓ:

- x_i = reduced word of length 1
$$coeff_{w^\ell}(x_i) = w_i(u_i)^{\ell-1}$$
Note that $w^\ell = ww\cdots w$. To obtain x_i from this product, we must first pick $w_i x_i$ from the first w. This gives us the w_i in the above formula. And for the rest of the w's, we can pick any face F with $x_i F = x_i$. Since the sum of the weight on these faces is u_i and we have to pick exactly one face from each w, we have the above formula.

- $x_i x_j$ = reduced word of length 2
$$coeff_{w^\ell}(x_i x_j) = w_i w_j \sum_{r,s\geq 0,\ r+s+2=\ell} u_i^r u_{i,j}^s$$

As in the previous case, we have to pick $w_i x_i$ from the first w. Then we can pick any face F with $x_i F = x_i$ for r of the following w before picking $w_i x_j$. This accounts for u_i^r. And for the rest of the w, where there are $s = \ell - r - 2$ of them left, we can pick any face F with $x_i x_j F = x_i x_j$. This gives us the above formula.

Since $\mathcal{A}$ is in general position, the maximal length of reduced word is 2. We do not have to compute further. The computation for w^ℓ when $\mathcal{A}$ is not in general position is the same as above but we have to compute up to the reduced word of length ℓ because w^ℓ may contain a factor of such reduced word. In further computation, it is useful to note that w^ℓ cannot produce a reduced word of length greater than ℓ. Using the result above, we proceed to the formula for $g(z)$.

Coefficients of $g(z) = \sum_{\ell \geq 0} w^\ell z^{-\ell-1}$:

- x_i = reduced words of length 1

$$\begin{aligned} coeff_g(x_i) &= \sum_{\ell \geq 1} w_i u_i^{\ell-1} z^{-\ell-1} \\ &= \frac{w_i}{z(z-u_i)} \\ &= \frac{-w_i/u_i}{z} + \frac{w_i/u_i}{z} \end{aligned}$$

- $x_i x_j$ = reduced word of length 2

$$\begin{aligned} coeff_g(x_i x_j) &= \sum_{\ell \geq 2} \{ w_i w_j \sum_{r,s \geq 0,\ r+s+2=\ell} u_i^r u_{i,j}^s \} \\ &= w_i w_j \sum_{r \geq 0} u_i^r \sum_{s \geq 0} u_{i,j}^s z^{-r-1} z^{-s-1} z^{-1} \\ &= w_i w_j \frac{1}{z-u_i} \frac{1}{z-u_{i,j}} \frac{1}{z} \\ &= \frac{1}{z}\left(\frac{w_i w_j}{u_i u_{i,j}}\right) + \frac{1}{z-u_i}\left(\frac{-w_i w_j}{u_i(u_{i,j}-u_i)}\right) \\ &\quad + \frac{1}{z-u_{i,j}}\left(\frac{w_i w_j}{u_{i,j}(u_{i,j}-u_i)}\right) \end{aligned}$$

Observe that from the coefficients of $g(z)$ each pole of $g(z)$ has order 1. Hence, $g(z)$ has the form 2.3.3. By proposition 2.3.1, the algebra $\mathbb{R}[w]$ is split semisimple with primitive idempotents $\{e_i\}$. Using the equation 2.3.3, we obtain formula for primitive idempotent e_i by lumping up all the coefficients of $1/(z-\lambda_i)$ in $g(z)$. Let $\mathcal{C}_i$ be the set of all reduced words of length i. We have the following:

- $\lambda = 0$,

$$e_0 = 1 - \sum_{x_i \in \mathcal{C}_1} \frac{w_i x_i}{u_i} + \sum_{x_i x_j \in \mathcal{C}_2} \frac{w_i w_j x_i x_j}{u_i u_{i,j}}$$

- $\lambda \neq 0$,

$$e_\lambda = \sum_{u_i = \lambda} \left(\frac{w_i x_i}{u_i} - \frac{w_i w_j x_i x_j}{u_i(u_{i,j} - u_i)}\right) + \sum_{u_{i,j} = \lambda} \frac{w_i w_j x_i x_j}{u_{i,j}(u_{i,j} - u_i)}$$

3.3. **Generalization to $\mathbb{R}^d$.** For an arrangement in general position in $\mathbb{R}^d$, the maximal length for reduced word becomes d. So, to obtain the formula for the primitive idempotents in $\mathbb{R}^d$, we have to compute the coefficients of reduced words of lengths up to d. Fortunately, the technique used to derived the primitive idempotent in $\mathbb{R}^2$ in previous section works equally well in higher dimension. In fact, if the formula for the primitive idempotents of the walk in $\mathbb{R}^{d-1}$ is known, we only need to compute the coefficients of reduced word of length d to obtain the formula for primitive idempotents of the walk in $\mathbb{R}^d$. On the other hand, if the formula for the primitive idempotents of the walk in $\mathbb{R}^d$ is known, we can easily delete the terms involving reduced words of length d to obtain the formula for the primitive idempotents of the walk in $\mathbb{R}^{d-1}$.

4. Stationary distribution and the rate of convergence

Recall that the properties of primitive idempotents gives the identity

$$w^\ell = \sum_{i \in I} \lambda_i^\ell e_i.$$

Taking the limit as $\ell \to \infty$ all the terms in the right vanish except for $\lambda = 1$. Thus, the formula for the stationary distribution is simply the primitive idempotent corresponding to $\lambda = 1$, i.e., $\pi = e_1$. Hence,

$$w^\ell = \pi + \sum_{\lambda \neq 1} \lambda^\ell e_\lambda.$$

By definition of π, we have $\pi \vec{F} = \vec{F}$. Then

$$\begin{aligned}
\left\| K^\ell \vec{F} - \pi \right\| &= \left\| w^\ell \vec{F} - \pi \right\| \\
&= \left\| \pi \vec{F} + \sum_{\lambda \neq 1} \lambda^\ell e_\lambda \vec{F} - \pi \right\| \\
&= \left\| \sum_{\lambda \neq 1} \lambda^\ell e_\lambda \vec{F} \right\|,
\end{aligned}$$

which is the formula of the rate of convergence.

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