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Journal of Algebra 273 (2004) 227–251

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Representations of the q-rook monoid

Tom Halverson [1]

Department of Mathematics and Computer Science, Macalester College, Saint Paul, MN 55105, USA

Communicated by Peter Littelmann

Abstract

The q-rook monoid $\mathcal{I}_n(q)$ is a semisimple algebra over $\mathbb{C}(q)$ that specializes when $q \to 1$ to $\mathbb{C}[R_n]$, where R_n is the monoid of $n \times n$ matrices with entries from $\{0, 1\}$ and at most one nonzero entry in each row and column. When q is specialized to a prime power, $\mathcal{I}_n(q)$ is isomorphic to the Iwahori algebra $\mathcal{H}_{\mathbb{C}}(M, B)$, where $M = \mathbf{M}_n(\mathbb{F}_q)$ is the monoid of $n \times n$ matrices with entries from a finite field having q-elements and $B \subseteq M$ is the Borel subgroup of invertible upper triangular matrices. In this paper, we (i) give a new presentation for $\mathcal{I}_n(q)$ on generators and relations and determine a set of standard words which form a basis; (ii) explicitly construct a complete set of "seminormal" irreducible representations of $\mathcal{I}_n(q)$; and (iii) show that $\mathcal{I}_n(q)$ is the centralizer of the quantum general linear group $U_q\mathfrak{gl}(r)$ acting on the tensor product $(W \oplus V)^{\otimes n}$, where V is the fundamental $U_q\mathfrak{gl}(r)$ module and W is the trivial $U_q\mathfrak{gl}(r)$ module.
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Keywords: Quantum group; Iwahori Hecke algebra; Rook monoid; Representation

0. Introduction

N. Iwahori [8] discovered the marvelous structure in the "Hecke algebra" $\mathcal{H}_{\mathbb{C}}(G, B)$, where $G = \mathbf{GL}_n(\mathbb{F}_q)$ is the general linear group of invertible $n \times n$ matrices over the field $\mathbb{F}_q$ with q elements and B is the Borel subgroup of upper triangular matrices. He proved that $\mathcal{H}_{\mathbb{C}}(G, B) \cong \mathbb{C}[S_n]$, where $\mathbb{C}[S_n]$ is the group algebra of the symmetric group S_n, and he showed that $\mathcal{H}_{\mathbb{C}}(G, B)$ has a presentation given on generators $T_1, T_2, \ldots, T_{n-1}$ and relations

E-mail address: halverson@macalester.edu.

[1] Research supported in part by National Science Foundation grant DMS-9800851.

0021-8693/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2003.11.002

$$\begin{aligned} &\text{(I1)} \quad T_i^2 = q \cdot 1 + (q-1)T_i, \quad \text{for } 1 \leqslant i \leqslant n-1, \\ &\text{(I2)} \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leqslant i \leqslant n-2, \\ &\text{(I3)} \quad T_i T_j = T_j T_i, \quad \text{when } |i-j| \geqslant 2. \end{aligned} \tag{0.1}$$

At $q = 1$ this becomes the well-known presentation of S_n due to E.H. Moore [12] in 1897. The generators T_i specialize to the simple transpositions $s_i = (i, i+1)$ in S_n.

Now let q be an indeterminate, and let $\mathcal{H}_n(q)$ be the associative $\mathbb{C}(q)$-algebra generated by $1, T_1, T_2, \ldots, T_{n-1}$ subject to (I1)–(I3). We refer to $\mathcal{H}_n(q)$ and $\mathcal{H}_{\mathbb{C}}(G, B)$ as Iwahori algebras (see the historical remarks in [19]).

L. Solomon [19] studied the Iwahori algebra $\mathcal{H}_{\mathbb{C}}(M, B)$, where now $M = \mathbf{M}_n(\mathbb{F}_q)$ is the monoid of $n \times n$ matrices over $\mathbb{F}_q$ and B is again the group of invertible upper triangular matrices. He showed that $\mathcal{H}_{\mathbb{C}}(G, B) \cong \mathbb{C}[R_n]$, where R_n is the rook monoid consisting of $n \times n$ matrices with entries from $\{0, 1\}$ and *at most* one nonzero entry in each row and column. The symmetric group S_n lives inside the rook monoid R_n as the rank n matrices. In [21], Solomon defines a $\mathbb{C}(q)$-algebra presented on generators $1, T_1, T_2, \ldots, T_{n-1}, N$ and relations (I1)–(I3), and

$$\begin{aligned} &\text{(I4)} \quad T_i N = N T_{i+1}, \quad \text{for } 1 \leqslant i \leqslant n-2, \\ &\text{(I5)} \quad T_i N^k = q N^k, \quad \text{when } i > n-k, \\ &\text{(I6)} \quad N^k T_i = q N^k, \quad \text{when } i < k, \\ &\text{(I7)} \quad N(T_1 T_2 \cdots T_{n-1}) N = q^{n-1} N. \end{aligned} \tag{0.2}$$

When q is a prime power, $\mathcal{I}_n(q)$ specializes to $\mathcal{H}_{\mathbb{C}}(M, B)$. At $q = 1$, (0.2) is the presentation of R_n found by Solomon in [20]. The T_i specialize to s_i and the new generator N specializes to $\nu = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}$, where $E_{i,j}$ is a matrix unit with a 1 in row i and column j.

In this paper we study the representation theory of $\mathcal{I}_n(q)$. The main results are as follows:

(1) We find a new presentation of $\mathcal{I}_n(q)$ on generators $T_1, \ldots, T_{n-1}, P_1, \ldots, P_n$ and relations given in (2.1). When $q \to 1$, the idempotent P_i specializes to $\varepsilon_i = E_{i+1,i+1} + E_{i+2,i+2} + \cdots + E_{n,n} \in R_n$ for $1 \leqslant i \leqslant n-1$ (and P_n specializes to the zero matrix). This presentation has several advantages:
   - (a) The action of P_i is simple and natural in the representations that we define in Sections 3 and 4.
   - (b) It is a close generalization of the presentation of the rook monoid given by Lipscomb [10], who uses generators $s_1, s_2, \ldots, s_{n-1}$, and ε_1.
   - (c) The idempotents P_i allow us to define a "basic construction" for $\mathcal{I}_n(q)$ in [4] that is analogous to a Jones basic construction. We use this construction in [4] to define a set of elements in $\mathcal{I}_n(q)$ on which it is sufficient to determine irreducible characters (i.e., analogs of conjugacy class representatives).

(d) The idempotents P_i appear in the general theory of reductive monoids. The set $\Lambda = \{1, P_1, \ldots, P_n\}$ is (up to scalar multiples) the set of cross-sectional idempotents used by Putcha [16] to naturally represent G-orbits in $G\backslash M/G$. However, Solomon's generators $\mathcal{N} = \{1, N, N^2, \ldots, N^n\}$ also index the these orbits. Furthermore, $\mathcal{N}$, and not Λ, behaves well with respect to the length function on R_n (see [18]), and N arises naturally in Solomon's definition of $\mathcal{H}_{\mathbb{C}}(M, B)$ (see (1.7)).

Note that a presentation using elements that specialize at $q \to 1$ to $\pi_i = I_n - E_{i,i}$ appears difficult. See Remark 4.4 and the comments in [20].

(2) For each partition λ with $0 \leqslant |\lambda| \leqslant n$ we define, in Section 3, a vector space V^λ. The dimension of V^λ is $\binom{n}{|\lambda|} f_\lambda$, where f_λ is the dimension of the irreducible $S_{|\lambda|}$ module indexed by λ. We define a basis of V^λ indexed by standard tableaux of shape λ and give explicit actions of the generators T_i, P_j on the basis. We show that these V^λ form a complete set of irreducible, pairwise non-isomorphic $\mathcal{I}_n(q)$-modules. These are generalizations of Young's [22] seminormal representations of S_n and Hoefsmit's [7] seminormal representations of $\mathcal{H}_n(q)$, and we explicitly realize the decomposition of V^λ into irreducibles for the subalgebra $\mathcal{I}_{n-1}(q) \subseteq \mathcal{I}_n(q)$. We also produce elements X_i, $1 \leqslant i \leqslant n$, which are analogs of Jucys–Murphy elements and which act diagonally on these representations.

When $q = 1$ we obtain seminormal representations of R_n. The representation theory of R_n was originally determined by Munn [13,14] and furthered by Solomon [20]. An analog Young's natural representation for R_n, using rook-monoid analogues of Young symmetrizers, is computed by Grood [5].

(3) Solomon [21] defined an action of $\mathcal{I}_n(q)$ on tensor space. In Section 4, we use this action to determine a Schur–Weyl duality between $\mathcal{I}_n(q)$ and the quantum general linear group $U_q\mathfrak{gl}(r)$. Let W and V be the trivial and fundamental representation of $U_q\mathfrak{gl}(r)$, respectively, and let $C_n = \mathrm{End}_{U_q\mathfrak{gl}(r)}((W \oplus V)^{\otimes n})$ be the centralizer of tensor powers of these representations. We compute R-matrices $\check{R}_i$ and $\check{E}_j$ in C_n and show that these are images of T_i and P_j, respectively. We show that when $r \geqslant n$, this is an isomorphism and $\mathcal{I}_n(q) \cong C_n$.

This duality is a generalization of the original Schur–Weyl duality between S_n and the general linear group $GL(r, \mathbb{C})$ on tensor space and of Jimbo's duality between $\mathcal{H}_n(q)$ and $U_q\mathfrak{gl}(r)$ on $V^{\otimes n}$. When $q \to 1$, this specializes to Solomon's [20] duality between $GL(r, \mathbb{C})$ and R_n on tensor space. In [4] we use the duality between $\mathcal{I}_n(q)$ and $U_q\mathfrak{gl}(r)$ to compute a Frobenius formula and a Murnaghan–Nakayama rule for the irreducible characters of $\mathcal{I}_n(q)$.

(4) We can define $\mathcal{I}_n(q)$ with parameter $q \in \mathbb{C}^*$. In [6], Halverson and Ram prove that $\mathcal{I}_n(q)$ is semisimple whenever $[n]! \neq 0$, where $[n]! = [n][n-1]\cdots[1]$ and $[k] = q^{k-1} + q^{k-2} + \cdots + 1$. The results in this paper work equally well for $\mathcal{I}_n(q)$ with $q \in \mathbb{C}^*$ and $[n]! \neq 0$.

Remark. The results of this paper inspired the work of Halverson and Ram [6], where we show that $R_n(q)$ is a quotient of the Iwahori Hecke algebra $H_n(u_1, u_2; q)$ of type B_n and that many of the results in this paper come from $H_n(u_1, u_2; q)$.

1. The Iwahori algebra $\mathcal{H}_{\mathbb{C}}(M, B)$ and the q-rook monoid $\mathcal{I}_n(q)$

1.1. The rook monoid

The symmetric group S_n of permutations of $\{1, 2, \dots, n\}$ can be identified with the group of $n \times n$ matrices with entries from $\{0, 1\}$ and *precisely* one nonzero entry in each row and in each column. The rook monoid R_n is the monoid (semigroup with identity) of $n \times n$ matrices with entries from $\{0, 1\}$ and *at most* one nonzero entry in each row and in each column. There are $\binom{n}{k}^2 k!$ matrices in R_n having rank k, and thus

$$|R_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!. \tag{1.1}$$

The rook monoid gets its name from the fact that the elements in R_n are in one-to-one correspondence with placements of non-attacking rooks on an $n \times n$ chessboard. The rook monoid is isomorphic to the monoid consisting of all one-to-one functions σ whose domain and range are subsets of $\{1, 2, \dots, n\}$. The bijection is given by assigning $\sigma(i) = j$ if the corresponding matrix has a 1 in the (i, j)-position. This monoid is commonly called the *symmetric inverse semigroup*.

Let $s_i \in S_n$ denote the transposition that exchanges i and $i + 1$. In R_n, the identity 1 is the $n \times n$ identity matrix and $E_{i,j}$ is the matrix unit with a 1 in the (i, j) position and 0s elsewhere. Let

$$\nu = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}. \tag{1.2}$$

If $0 \leqslant r \leqslant n$, then

$$\nu_r = \nu^{n-r} = E_{1,n-r+1} + E_{2,n-r+2} + \cdots + E_{r,n} \tag{1.3}$$

has rank r. Let

$$\begin{aligned} \varepsilon_i &= E_{i+1,i+1} + E_{i+2,i+2} + \cdots + E_{n,n}, \quad \text{for } 0 \leqslant i \leqslant n-1, \\ \pi_i &= I_n - E_{i,i}, \quad \text{for } 1 \leqslant i \leqslant n, \end{aligned} \tag{1.4}$$

then ε_i has rank $n - i$ and π_i has rank $n - 1$. We agree that ε_n is the zero matrix, and we have $\pi_1 = \varepsilon_1$.

A reduced word for $w \in S_n$ is an expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ with k minimal. The length of w is $\ell(w) = k$ and is independent of the choice of reduced word. Solomon [19] defined a length function for the rook monoid: if $\sigma \in R_n$ with $\operatorname{rank}(\sigma) = r$, then

$$\ell(\sigma) = \min\bigl\{\ell(w) + \ell(w') \bigm| w, w' \in S_n \text{ and } \sigma = w \nu_r w'\bigr\}. \tag{1.5}$$

1.2. The Iwahori algebra $\mathcal{H}_{\mathbb{C}}(M, B)$

Let q be a prime power and let $M = \mathbf{M}_n(\mathbb{F}_q)$ be the monoid of all $n \times n$ matrices over $\mathbb{F}_q$. Let $G = \mathbf{GL}_n(\mathbb{F}_q) \subseteq M$ be the general linear group of invertible matrices, and let $B \subseteq G$ be the Borel subgroup of upper triangular matrices. Renner [18] proves that there is a disjoint union

$$M = \bigsqcup_{\sigma \in R_n} B\sigma B,$$

and that $B\sigma B = B\sigma' B$ implies that $\sigma = \sigma'$.

Define the idempotent

$$\varepsilon = \frac{1}{|B|} \sum_{b \in B} b \in \mathbb{C}[M].$$

Following [19], define the Iwahori algebra

$$\mathcal{H} = \mathcal{H}_{\mathbb{C}}(M, B) = \varepsilon \mathbb{C}[M] \varepsilon.$$

If we consider $\mathbb{C}[M]$ acting on the left ideal $\mathbb{C}[M]\varepsilon$ by left multiplication, then $\mathcal{H}$ is the centralizer of this action; it acts by right multiplication on $\mathbb{C}[M]\varepsilon$. Okniński and Putcha [15] proved that $\mathbb{C}[M]$ is semisimple, and so it follows from general double-centralizer results that $\mathcal{H}$ is semisimple.

The elements

$$T_\sigma = q^{\ell(\sigma)} \varepsilon \sigma \varepsilon, \quad \sigma \in R_n,$$

form a basis for $\mathcal{H}$. Solomon [19] proved that the elements $T_{s_1}, \ldots, T_{s_{n-1}}, T_\nu$ generate $\mathcal{H}$ and

$$T_{s_i} T_\sigma = \begin{cases} qT_\sigma, & \text{if } \ell(s_i\sigma) = \ell(\sigma), \\ T_{s_i\sigma}, & \text{if } \ell(s_i\sigma) = \ell(\sigma) + 1, \\ qT_{s_i\sigma} + (q-1)T_\sigma, & \text{if } \ell(s_i\sigma) = \ell(\sigma) - 1, \end{cases}$$
$$T_\sigma T_{s_i} = \begin{cases} qT_\sigma, & \text{if } \ell(\sigma s_i) = \ell(\sigma), \\ T_{\sigma s_i}, & \text{if } \ell(\sigma s_i) = \ell(\sigma) + 1, \\ qT_{\sigma s_i} + (q-1)T_\sigma, & \text{if } \ell(\sigma s_i) = \ell(\sigma) - 1, \end{cases} \tag{1.6}$$

and

$$T_\nu T_\sigma = q^{\ell(\sigma) - \ell(\nu\sigma)} T_{\nu\sigma}, \qquad T_\sigma T_\nu = q^{\ell(\sigma) - \ell(\sigma\nu)} T_{\sigma\nu} \tag{1.7}$$

for all $\sigma \in R_n$.

Using (1.6), it is easy to verify the following lemma.

Lemma 1.1 (Iwahori [8]).

(1) $T_{s_i}^2 = (q-1)T_{s_i} + q \cdot 1,\ 1 \leqslant i \leqslant n-1,$
(2) $T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}},\ 1 \leqslant i \leqslant n-2,$
(3) $T_{s_i} T_{s_j} = T_{s_i} T_{s_j},\ |i-j| > 1.$

In [21], Solomon proves that $T_{s_1}, T_{s_2}, \ldots, T_{s_{n-1}}, T_\nu$ generate $\mathcal{H}_{\mathbb{C}}(M, B)$ and in [19] he extended Iwahori's relations to describe the interaction between T_{s_i} and T_ν:

Lemma 1.2 (Solomon [19]).

(1) $T_{s_i} T_\nu = T_\nu T_{s_{i+1}},\ 1 \leqslant i \leqslant n-2,$
(2) $T_{s_i} T_\nu^k = q T_\nu^k,\ i > n-k,$
(3) $T_\nu^k T_{s_i} = q T_\nu^k,\ i < k,$
(4) $T_\nu (T_{s_1} T_{s_2} \cdots T_{s_{n-1}}) T_\nu = q^{n-1} T_\nu,\ |i-j| > 1.$

1.3. The q-rook monoid

Let q be an indeterminate. For integers $n \geqslant 2$, define $\mathcal{I}_n(q)$ to be the associative $\mathbb{C}(q)$-algebra with 1 generated by $T_1, \ldots, T_{n-1}$ and N subject to the relations

$$\begin{aligned}
&(\text{I1}) \quad T_i^2 = q \cdot 1 + (q-1)T_i, \quad \text{for } 1 \leqslant i \leqslant n-1,\\
&(\text{I2}) \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leqslant i \leqslant n-2,\\
&(\text{I3}) \quad T_i T_j = T_j T_i, \quad \text{when } |i-j| \geqslant 2.\\
&(\text{I4}) \quad T_i N = N T_{i+1}, \quad \text{for } 1 \leqslant i \leqslant n-2,\\
&(\text{I5}) \quad T_i N^k = q N^k, \quad \text{for } i > n-k,\\
&(\text{I6}) \quad N^k T_i = q N^k, \quad \text{when } i < k,\\
&(\text{I7}) \quad N (T_1 T_2 \cdots T_{n-1}) N = q^{n-1} N.
\end{aligned} \tag{1.8}$$

Let $\mathcal{I}_0(q) = \mathbb{C}(q)$, and let $\mathcal{I}_1(q)$ be the $\mathbb{C}(q)$-span of 1 and N subject to $N^2 = N$. We see from Lemmas 1.1 and 1.2 that, when q is specialized to a prime power, we have a surjection, $\mathcal{I}_n(q) \to \mathcal{H}_{\mathbb{C}}(M, B)$ given by $T_i \to T_{s_i}$ and $N \to T_\nu$. In [21], Solomon finds a set of $|R_n|$ words in the generators of $\mathcal{I}_n(q)$ which span $\mathcal{I}_n(q)$. Thus,

Theorem 1.3 (Solomon [21]). *The $\mathbb{C}(q)$-algebra $\mathcal{I}_n(q)$ is semisimple of dimension $|R_n|$, and when q is specialized to a prime power, we have $\mathcal{I}_n(q) \cong \mathcal{H}_{\mathbb{C}}(M, B)$.*

Now, working in $\mathcal{I}_n(q)$, we define

$$T_{\gamma_n} = T_1 T_2 \cdots T_{n-1}, \qquad P_j = \left(q^{1-n}\right)^j T_{\gamma_n}^j N^j, \quad \text{for } 1 \leqslant j \leqslant n. \tag{1.9}$$

Using (I2) one can easily verify the well-known fact that

$$T_{\gamma_n} T_i = T_{i+1} T_{\gamma_n}, \quad 1 \leqslant i \leqslant n-2. \tag{1.10}$$

Furthermore, $N = q^{n-1} T_{\gamma_n}^{-1} P_1$, so $T_1, \ldots, T_{n-1}$ and P_1 generate $\mathcal{I}_n(q)$, and we have the following lemma.

Lemma 1.4.

(1) $T_i P_j = P_j T_i = q P_j,\ 1 \leqslant i < j \leqslant n,$
(2) $T_i P_j = P_j T_i,\ 1 \leqslant j < i \leqslant n,$
(3) $P_j^2 = P_j,\ 1 \leqslant i \leqslant n,$
(4) $P_{j+1} = q P_j T_i^{-1} P_j,\ 2 \leqslant i \leqslant n.$

Proof. Let $x = q^{1-n}$. For part (1), assume that $1 \leqslant i < j \leqslant n$. We use Lemma 1.1(1) to expand T_1^2 in the following calculation:

$$\begin{aligned}
T_i P_j &= x^j T_i T_{\gamma_n}^j N^j \\
&= x^j T_{\gamma_n}^{i-1} T_1 T_{\gamma_n}^{j-(i-1)} N^j \quad \text{by (1.8)} \\
&= x^j T_{\gamma_n}^{i-1} \big(T_1^2 T_2 \cdots T_{n-1}\big) T_{\gamma_n}^{j-i} N^j \\
&= (q-1) x^j T_{\gamma_n}^{i-1} (T_1 \cdots T_{n-1}) T_{\gamma_n}^{j-i} N^j + q x^j T_{\gamma_n}^{i-1} (T_2 \cdots T_{n-1}) T_{\gamma_n}^{j-i} N^j \\
&= (q-1) P_j + q x^j T_{\gamma_n}^{i} (T_1 \cdots T_{n-2}) T_{\gamma_n}^{j-i-1} N^j \quad \text{by (1.8)} \\
&= (q-1) P_j + x^j T_{\gamma_n}^{i} (T_1 \cdots T_{n-2}) T_{\gamma_n}^{j-i-1} T_{n-j+i} N^j \quad \text{by Lemma 1.2(2)} \\
&= (q-1) P_j + x^j T_{\gamma_n}^{i} (T_1 \cdots T_{n-2} T_{n-1}) T_{\gamma_n}^{j-i-1} N^j \quad \text{by (1.8)} \\
&= (q-1) P_j + P_j \\
&= q P_j.
\end{aligned}$$

On the other hand, by Lemma 1.2(1) and 1.2(2), we have

$$P_j T_i = x^j T_{\gamma_n}^j N^j T_i = x^j T_{\gamma_n}^j N^{j-(i-1)} T_1 N^{i-1} = q x^j T_{\gamma_n}^j N^{j-(i-1)} N^{i-1} = q P_j.$$

For part (2), if $j < i$, then using Lemma 1.2(1) and (1.8), we have

$$P_j T_i = x^j T_{\gamma_n}^j N^j T_i = x^j T_{\gamma_n}^j T_{i-j} N^j = x^j T_i T_{\gamma_n}^j N^j = T_i P_j.$$

Part (3) follows from Lemma 1.2(4):

$$P_i^2 = x^{2i}\, T_{\gamma_n}^i \big(N^i T_{\gamma_n}^i N^i\big) = x^i\, T_{\gamma_n}^i N^i = P_i.$$

For (4), we have

$$\begin{aligned} qP_iT_i^{-1}P_i &= q^iP_i\big(T_i^{-1}T_{i-1}^{-1}\cdots T_1^{-1}\big)P_i \quad \text{by part (1)} \\ &= q^ix^{2i}T_{\gamma_n}^iN^i\big(T_i^{-1}T_{i-1}^{-1}\cdots T_1^{-1}\big)T_{\gamma_n}^iN^i \\ &= q^ix^{2i}T_{\gamma_n}^iN^i(T_{i+1}T_{i+2}\cdots T_{n-1})T_{\gamma_n}^{i-1}N^i \\ &= q^ix^{2i}T_{\gamma_n}^i(T_1T_2\cdots T_{n-1-i})N^iT_{\gamma_n}^{i-1}N^i \quad \text{by Lemma 1.2(1)} \\ &= q^ix^{i+1}T_{\gamma_n}^i(T_1T_2\cdots T_{n-1-i})N^{i+1} \quad \text{by Lemma 1.2(4)} \\ &= x^{i+1}T_{\gamma_n}^{i+1}N^{i+1} = P_{i+1} \quad \text{by part (1).} \qquad \square \end{aligned}$$

Lemma 1.5. *Let q be a prime power. Under the isomorphism $\mathcal{I}_n(q) \to \mathcal{H}_{\mathbb{C}}(M, B)$ given by $T_i \to T_{s_i}$ and $N \to T_\nu$, we have $P_i \to q^{j(j-n)}T_{\varepsilon_i}$.*

Proof. We use induction to prove the following equivalent condition (see (1.9)):

$$T_{\gamma_n}^jT_\nu^j = q^{j(j-1)}T_{\varepsilon_j}.$$

Note that $\gamma_n\nu = 1$, $\ell(\gamma_n) = n-1$, and $\ell(\varepsilon_j) = j(n-j)$. Then the case $j = 1$ follows immediately from (1.7): $T_{\gamma_n}T_\nu = q^{\ell(\gamma_n)-\ell(\varepsilon_1)}T_{\varepsilon_1} = T_{\varepsilon_1}$.

Now let $j > 1$, and define

$$\sigma = (s_js_{j+1}\cdots s_{n-1})\varepsilon_j = \varepsilon_j(s_js_{j+1}\cdots s_{n-1}),$$

so that $\sigma\nu = \varepsilon_j$ and $\ell(\sigma) = \ell(\varepsilon_{j-1}) + n - j = j(n-j) + j - 1$. Thus, by induction,

$$\begin{aligned} T_{\gamma_n}^jT_\nu^j &= q^{(j-1)(j-2)}T_{\gamma_n}T_{\varepsilon_{j-1}}T_\nu = q^{(j-1)(j-2)}(T_{s_1}\cdots T_{s_{j-1}})(T_{s_j}\cdots T_{s_{n-1}})T_{\varepsilon_{j-1}}T_\nu \\ &= q^{(j-1)(j-2)+\ell(\sigma)-\ell(\varepsilon_j)}(T_{s_1}\cdots T_{s_{j-1}})T_{\varepsilon_j}T_\nu \\ &= q^{(j-1)^2}(T_{s_1}\cdots T_{s_{j-1}})T_{\varepsilon_j}T_\nu. \end{aligned}$$

Now by (1.6), $T_{s_i}T_{\varepsilon_j} = qT_{\varepsilon_j}$ for $i < j$, and the result follows. $\square$

2. A new presentation for the q-rook monoid

Let q be an indeterminate. For integers $n \geqslant 2$, define $A_n(q)$ to be the associative $\mathbb{C}(q)$-algebra with 1 generated by $T_1, \ldots, T_{n-1}$ and $P_1, \ldots, P_n$ subject to the relations

$$\begin{aligned} &\text{(A1)} \quad T_i^2 = q\cdot 1 + (q-1)T_i, \quad \text{for } 1 \leqslant i \leqslant n-1, \\ &\text{(A2)} \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad \text{for } 1 \leqslant i \leqslant n-2, \\ &\text{(A3)} \quad T_iT_j = T_jT_i, \quad \text{when } |i-j| \geqslant 2, \\ &\text{(A4)} \quad T_iP_j = P_jT_i = qP_j, \quad \text{for } 1 \leqslant i < j \leqslant n, \\ &\text{(A5)} \quad T_iP_j = P_jT_i, \quad \text{for } 1 \leqslant j < i \leqslant n-1, \end{aligned} \tag{2.1}$$

(A6) $P_i^2 = P_i$, for $1 \leqslant i \leqslant n$,

(A7) $P_{i+1} = q P_i T_i^{-1} P_i$, for $2 \leqslant i \leqslant n$.

Let $A_0(q) = \mathbb{C}(q)$, and let $A_1(q)$ be the $\mathbb{C}(q)$-span of 1 and P_1 subject to $P_1^2 = P_1$. From (A1) we have

$$T_i^{-1} = \left(q^{-1} - 1\right) \cdot 1 + q^{-1} T_i. \tag{2.2}$$

It follows that (A7) is equivalent to

$$P_{i+1} = P_i T_i P_i - (q-1) P_i. \tag{2.3}$$

From Lemmas 1.1 and 1.4, we see that the T_i and the P_i satisfy the same relations in both $\mathcal{I}_n(q)$ and $A_n(q)$. Furthermore, $T_1, \ldots, T_{n-1}$ and P_1 generate $A_n(q)$, so there is a surjection from $A_n(q)$ to $\mathcal{I}_n(q)$. In this section, we will show that they have the same dimension and are isomorphic. For this reason, we choose to use the same notation T_i and P_i in both algebras.

For $w \in S_n$ with reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ define $T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell}$. Since the T_i satisfy the braid relations (A2) and (A3), T_w is independent of the choice of reduced word for w. Furthermore, the T_i satisfy the same relations as they do in $\mathcal{H}_n(q)$, so the subalgebra spanned by $T_1, \ldots, T_{n-1}$ is a homomorphic image of $\mathcal{H}_n(q)$ and the T_w, $w \in S_n$ span this subalgebra. In Section 3 we will show that this subalgebra is isomorphic to $\mathcal{H}_n(q)$.

If $K \subseteq \{1, 2, \ldots, n\}$ define the subgroup $S_K \subseteq S_n$ to be the group of permutations on the elements of K. For $1 \leqslant i \leqslant n$, define $T_{i,i} = 1$, and define

$$T_{i,j} = T_{j-1} T_{j-2} \cdots T_i, \quad \text{for } 1 \leqslant i < j \leqslant n.$$

Let $A = \{a_1, a_2, \ldots, a_k\} \subseteq \{1, 2, \ldots, n\}$, and assume that $a_1 < a_2 < \cdots < a_k$. Define

$$T_A = T_{1,a_1} T_{2,a_2} \cdots T_{k,a_k}. \tag{2.4}$$

Now for $0 \leqslant k \leqslant n$, let Ω_k be the following set of triples,

$$\Omega_k = \left\{ (A, B, w) \;\middle|\; \begin{array}{l} A, B \subseteq \{1, 2, \ldots, n\}, \\ |A| = |B| = k, \\ w \in S_{\{k+1, \ldots, n\}}, \end{array} \right\}, \tag{2.5}$$

and let

$$\Omega = \bigcup_{k=0}^{n} \Omega_k. \tag{2.6}$$

Define the following *standard words*

$$T_{(A,B,w)} = T_A T_w P_k T_B^{-1}, \quad (A, B, w) \in \Omega_k. \tag{2.7}$$

Note that $T_w P_k = P_k T_w$ by (A5). Furthermore, there are $\binom{n}{k}^2$ ways to choose A and B, so

$$|\Omega_k| = \binom{n}{k}^2 (n-k)! \quad \text{and} \quad |\Omega| = \sum_{k=0}^{n} \binom{n}{k}^2 (n-k)! = |R_n|. \tag{2.8}$$

Theorem 2.1. *The standard words* $\{T_{(A,B,w)} \mid (A,B,w) \in \Omega\}$ *span* $A_n(q)$. *In particular,* $\dim(A_n(q)) \leqslant |R_n|$.

Proof. From (A7) we know that T_i, $1 \leqslant i \leqslant n-1$, and P_1 generate $A_n(q)$. Furthermore, T_i and P_1 are standard words. It suffices to show that for all $(A,B,w) \in \Omega$, we can write $T_{(A,B,w)} T_i$ and $T_{(A,B,w)} P_1$ as a linear combination of standard words. Since $T_i = q T_i^{-1} + (q-1) \cdot 1$, it is equivalent to show that $T_{(A,B,w)} T_i^{-1}$ and $T_{(A,B,w)} P_1$ can be written as linear combinations of standard words.

Case 1. $T_{(A,B,w)} T_i^{-1}$ *is a linear combination of standard words.*

Suppose $i, i+1 \in B$. We use (A2) and (A3) to verify that

$$\big(T_{j+1,i+1}^{-1} T_{j,i}^{-1}\big) T_i^{-1} = T_j^{-1} \big(T_{j+1,i+1}^{-1} T_{j,i}^{-1}\big).$$

Then since $i, i+1 \in B$, we can write $T_B^{-1} = X T_{j+1,i+1}^{-1} T_{j,i}^{-1} Y$ so that X commutes with T_j^{-1} and Y commutes with T_i^{-1}. Thus,

$$P_k T_B^{-1} = P_k X T_{j+1,i+1}^{-1} T_{j,i}^{-1} T_i^{-1} Y = P_k X T_j^{-1} T_{j+1,i+1}^{-1} T_{j,i}^{-1} Y = P_k T_j^{-1} T_B^{-1} = q^{-1} P_k T_B^{-1}$$

proving the result in this case.

Now suppose $i, i+1 \in B^c$. In this case $T_B = XY$ where Y consists of elements of the form $T_{\ell,j}^{-1}$ with $j < i$ and X consists of elements of the form $T_{\ell,j}^{-1}$ with $j > i$. It follows that T_i^{-1} commutes with Y, and $X = T_{t,j_t}^{-1} T_{t-1,j_{t-1}}^{-1} \cdots T_{\ell,j_1}^{-1}$ with $i < j_1 < j_2 < \cdots < j_t$ and $i \geqslant \ell$. If $\ell \leqslant i < j-2$, then $T_{k,j}^{-1} T_i^{-1} = T_{i+1}^{-1} T_{k,j}^{-1}$. Thus $T_w T_B^{-1} T_i^{-1} = T_w T_j^{-1} T_B^{-1}$ with $j > k$. We now can express $T_w T_j^{-1}$ as a linear combination of $T_{w'}$ with $w' \in S_{\{k+1,\ldots,n\}}$.

Now suppose $i \in B,\ i+1 \in B^c$. We write $T_B = X T_{\ell,i}^{-1} Y$ where Y consists of elements of the form $T_{s,j}^{-1}$ with $j < i$ and X consists of elements of the form $T_{t,j}^{-1}$ with $j > i$. It follows that

$$T_B T_i^{-1} = X T_{\ell,i}^{-1} T_i^{-1} Y = X T_{\ell,i+1}^{-1} Y = T_{B'}^{-1},$$

where B' is the same set as B except with i replaced by $i+1$.

Finally, let $i \in B,\ i+1 \in B^c$. We write $T_B = X T_{\ell,i+1}^{-1} Y$ where where Y consists of elements of the form $T_{s,j}^{-1}$ with $j < i$ and X consists of elements of the form $T_{t,j}^{-1}$ with $j > i$. It follows that

$$\begin{aligned}T_B T_i^{-1} &= X T_{\ell,i+1}^{-1} T_i^{-1} Y = \left(q^{-1}-1\right) X T_{\ell,i+1}^{-1} Y + q^{-1} X T_{\ell,i}^{-1} Y \\ &= \left(q^{-1}-1\right) T_B^{-1} + q^{-1} T_{B'}^{-1},\end{aligned}$$

where B' is the same set as B except with $i+1$ replaced by i.

Case 2. *$T_{(A,B,w)} P_1$ is a linear combination of standard words.*

Suppose $1 \in B$. In this case T_B^{-1} contains only T_i^{-1} with $i > 1$, so by (A5), T_B^{-1} commutes with P_1. Thus, $P_k T_B^{-1} P_1 = P_k P_1 T_B^{-1} = P_k T_B^{-1}$.

Now suppose $1 \in B^c$ and $B \neq \emptyset$. We have

$$P_i T_{i,b}^{-1} P_i = P_i \left(T_i^{-1} \cdots T_{b-1}^{-1}\right) P_i = P_i T_i^{-1} P_i \left(T_{i+1}^{-1} \cdots T_{b-1}^{-1}\right) = q^{-1} P_{i+1} T_{i+1,b}^{-1}. \qquad (*)$$

In the following calculation, we use $(*)$ and fact that $P_k = P_k P_i$ for $i \leqslant k$ (see (A6) and (A7)):

$$\begin{aligned}P_k T_B^{-1} P_1 &= P_k P_1 T_B^{-1} P_1 = P_k \left(T_{k,b_k}^{-1} \cdots T_{2,b_2}^{-1}\right)\left(P_1 T_{1,b_1}^{-1} P_1\right) \\ &= q^{-1} P_k \left(T_{k,b_k}^{-1} \cdots T_{2,b_2}^{-1}\right) P_2 T_{2,b_1}^{-1} = q^{-1} P_k P_2 \left(T_{k,b_k}^{-1} \cdots T_{2,b_2}^{-1}\right) P_2 T_{2,b_1}^{-1} \\ &= q^{-1} P_k \left(T_{k,b_k}^{-1} \cdots T_{3,b_3}^{-1}\right)\left(P_2 T_{2,b_2}^{-1} P_2\right) T_{2,b_1}^{-1} \\ &= q^{-2} P_k \left(T_{k,b_k}^{-1} \cdots T_{3,b_3}^{-1}\right) P_3 T_{3,b_2}^{-1} T_{2,b_1}^{-1} \\ &\;\;\vdots \\ &= q^{-k} P_{k+1} \left(T_{k+1,b_k}^{-1} \cdots T_{3,b_2}^{-1} T_{2,b_1}^{-1}\right) = q^{-k} P_{k+1} T_{B'}^{-1},\end{aligned}$$

where $B' = \{1, b_1, \ldots, b_k\}$.

Finally, suppose $B = \emptyset$. We prove that

$$T_w P_1 = (T_k T_{k-1} \cdots T_1) P_1 T_{w'}, \quad \text{with } w' \in S_{\{2,\ldots,n\}}. \qquad (**)$$

This finishes the proof since $T_w P_1$ is a standard word with $A = \{k+1\}$ and $B = \{1\}$.

We prove $(**)$ by induction on $\ell(w)$. If $\ell(w) = 1$, then $T_i P_1$ is a standard word. If $i = 1$ then $T_1 P_1 = T_A P_1 T_B^{-1}$ where $A = \{2\}$ and $B = \{1\}$. If $i > 1$, then $T_i P_1 = T_A P_1 T_w T_B$ with $T_w = T_i$, $A = \{1\}$, and $B = \{1\}$.

If $\ell(w) = t > 1$, then let $T_w = T_{i_1} T_{i_2} \cdots T_{i_t}$. Suppose $i_t > 1$. Then we can apply induction

$$T_w P_1 = (T_{i_1} \cdots T_{i_{t-1}}) P_1 T_{i_t} = (T_k T_{k-1} \cdots T_1) P_1 T_w T_{i_t}.$$

We then re-express $T_w T_{i_t}$ as a linear combination of $T_{w'}$ with $w' \in S_{\{2,\ldots,n\}}$.

If $i_t = 1$, then there exists an $r \geqslant 1$ so that $T_w P_1 = T_{i_1} \cdots T_j T_r T_{r-1} \cdots T_1 P_1$ and $j \neq r+1$. We know that $j \neq r$, or w is not minimal. If $j > r+1$, then T_j commutes with all the elements to its right, and we can apply induction as in the previous case. If $j < r + 1$, then

$$T_j T_r T_{r-1} \cdots T_1 P_1 = T_r T_{r-1} \cdots T_1 P_1 T_{j+1}$$

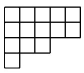
and we can apply induction. □

We have a surjection from $A_n(q)$ to $\mathcal{I}_n(q)$ and we have a set of $|R_n|$ words which span $A_n(q)$, so $\dim(\mathcal{I}_n(q)) \leqslant \dim(A_n(q)) \leqslant |R_n|$. Solomon [21] has proved the lower bound $\dim(\mathcal{I}_n(q)) = |R_n|$. We also will obtain this lower bound in the next section by producing sufficiently many irreducible representations. Thus,

Corollary 2.2. $A_n(q) \cong \mathcal{I}_n(q)$.

3. Irreducible representations for $\mathcal{I}_n(q)$

We use the notation for partitions and tableaux found in [11]. In particular, we let $\lambda \vdash k$ denote the fact that λ is a partition of the nonnegative integer k, and we write $|\lambda| = k$. The length $\ell(\lambda)$ of λ is the number of nonzero parts of λ. We identify λ with its Young diagram. Thus,

$\lambda = (5, 5, 3, 1) =$ , $\quad \ell(\lambda) = 4, \quad$ and $\quad |\lambda| = 14.$

For integers $n \geqslant 0$ define

$$\Lambda_n = \{\lambda \vdash k \mid 0 \leqslant k \leqslant n\}. \tag{3.1}$$

For $\lambda \in \Lambda_n$, an *n-standard tableau of shape* λ is a filling of the diagram of λ with numbers from $\{1, 2, \ldots, n\}$ such that

(1) each number appears at most 1 time,
(2) the entries in each column strictly increase from top to bottom, and
(3) the entries in each row strictly increase from left to right.

We let $\mathcal{T}_n^\lambda$ denote the set of standard tableaux of shape λ. If $\lambda \vdash k$, the number of k-standard tableaux of shape λ is given by

$$f_\lambda = \frac{n!}{\prod_{b \in \lambda} h_b}, \tag{3.2}$$

where the product is over all the boxes b in λ, and h_b is the hook length of b given by $h_b = \lambda_i + \lambda'_j - i - j + 1$ if b is in position (i, j) and λ' is the conjugate (transposed) partition. If $\lambda \vdash k$ and $n \geqslant k$ then there are $\binom{n}{k}$ ways to choose the entries of a tableau of shape λ so the number of n-standard tableaux of shape λ is $\binom{n}{k} f_\lambda$.

The symmetric group S_n acts on tableaux by permuting their entries. If $L \in \mathcal{T}_n^\lambda$, then $s_i L$ is the tableau that is obtained from L by replacing i (if $i \in L$) by $i+1$ and replacing $i+1$ (if $i+1 \in L$) by i. Note that $s_i L$ may be non-standard, since condition (2) or (3) may fail, and $s_i L = L$ if and only if $i, i+1 \notin L$.

Let $v_L,\ L \in \mathcal{T}_n^\lambda$, denote a set of vectors indexed by the n-standard tableaux of shape λ. Let

$$V^\lambda = \mathbb{C}(q^{1/2})\text{-span}\{v_L \mid L \in \mathcal{T}_n^\lambda\} \tag{3.3}$$

In this way the symbols $v_L, L \in \mathcal{T}_n^\lambda$ are a basis of the vector space V^λ. It follows that if $\lambda \vdash k$, then

$$\dim(V^\lambda) = \#(n\text{-standard tableaux of shape } \lambda) = \binom{n}{k} f_\lambda. \tag{3.4}$$

If b is a box in position (i, j) of λ, then the *content* of b is

$$\mathrm{ct}(b) = j - i. \tag{3.5}$$

Let $L \in \mathcal{T}_n^\lambda$. If $i, i+1 \in L$, then let $L(i)$ and $L(i+1)$ denote the box in L containing i and $i+1$, respectively. Define

$$a_L(i) = \frac{q-1}{1 - q^{\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))}}. \tag{3.6}$$

Define an action of $T_i,\ 1 \leqslant i \leqslant n-1$, on V^λ as follows:

$$T_i v_L = \begin{cases} a_L(i)\, v_L + (1 + a_L(i))\, v_{L'}, & \text{if } i, i+1 \in L, \\ (q-1)\, v_L + q^{1/2}\, v_{s_i L}, & \text{if } i \notin L,\ i+1 \in L, \\ q^{1/2}\, v_{s_i L}, & \text{if } i \in L,\ i+1 \notin L, \\ q\, v_L, & \text{if } i, i+1 \notin L, \end{cases} \tag{3.7}$$

where

$$v_{L'} = \begin{cases} v_{s_i L}, & \text{if } s_i L \text{ is } n\text{-standard}, \\ 0, & \text{otherwise}. \end{cases}$$

Define an action of $P_i,\ 1 \leqslant i \leqslant n$, on V^λ by

$$P_i\, v_L = \begin{cases} v_L, & \text{if } 1, 2, \ldots, i \notin L, \\ 0, & \text{otherwise}. \end{cases} \tag{3.8}$$

Remark 3.1. If $i, i+1 \in L$ then the action of T_i on v_L is the same as the action in Hoefsmit's [7] seminormal representation of $\mathcal{H}_n(q)$.

Theorem 3.2. *For each $\lambda \in \Lambda_n$, the actions of the generators of $\mathcal{I}_n(q)$ on the vector space V^λ afford an irreducible representation of $\mathcal{I}_n(q)$. Moreover, the set V^λ, $\lambda \in \Lambda_n$, is a complete set of irreducible, pairwise non-isomorphic $\mathcal{I}_n(q)$-modules.*

Proof. First we check relations (A1)–(A7) in the presentation (2.1).

(A1) Let L be a standard tableaux. Then T_i acts on the subspace spanned by v_L and $v_{L'}$. Let M be the matrix of T_i with respect to $\{v_L, v_{L'}\}$. If $i, i+1 \in L$, then this is the same matrix as in the seminormal action of $\mathcal{H}_n(q)$, so we know from [7] that $M^2 = (q-1)M + qI_2$, where I_2 is the 2×2 identity matrix. If $i \notin L$ and $i+1 \in L$, then

$$M = \begin{pmatrix} q-1 & q^{1/2} \\ q^{1/2} & 0 \end{pmatrix}.$$

Since $\det(M) = -q$ and $\operatorname{trace}(M) = q-1$, we have $M^2 = (q-1)M + qI_2$. The case $i \in L,\ i+1 \notin L$ is proved by exchanging the rows and columns of M in the previous case. If $i, i+1 \notin L$, then $M = \operatorname{diag}(q, q)$ which trivially satisfies $M^2 = (q-1)M + qI_2$.

(A3) We see from $T_i v_L = a v_L + b v_{s_i L}$ that the action of T_i affects only positions i and $i+1$ in L. Since $|i-j| > 1$, the sets $\{i, i+1\}$ and $\{j, j+1\}$ are disjoint and thus the actions of T_i and T_j commute.

(A4)–(A5) If $i \neq j$, then $1, \ldots, j \notin L$ if and only if $1, \ldots, j \notin s_i L$. Thus, $i \neq j$ and $1, \ldots, j \notin L$ imply that $T_i P_j v_L$ and $P_j T_i v_L$ are both equal to $T_i v_L$. If $i \neq j$ and it is not the case that $1, \ldots, j \notin L$, then $T_i P_j v_L = 0$ and $P_j T_i v_L = 0$. If $i < j$, and $1, \ldots, j \notin L$, then $T_i v_L = q v_L$, so $T_i P_j$ acts the same as $q P_j$.

(A6) is immediate from (3.8).

(A7) We verify the equivalent condition (2.3): $P_{j+1} = P_j T_j P_j + (1-q) P_j$. If it is *not* the case that $1, \ldots, j \notin L$, then both $P_j v_L = 0$ and $P_{j+1} v_L = 0$, and the result holds.

If $1, \ldots, j+1 \notin L$, then $P_j v_L = P_{j+1} v_L = v_L$, and $T_j v_L = q v_L$. Thus,

$$P_j T_j P_j v_L + (1-q) P_j v_L = q v_L + (1-q) v_L = v_L = P_{j+1} v_L.$$

If $1, \ldots, j \notin L$ and $j+1 \in L$, then $P_j v_L = v_L$, $P_j v_{s_j L} = 0$, $P_{j+1} v_L = 0$, and $T_j v_L = (q-1) v_L + q^{1/2} v_{s_j L}$. Thus,

$$P_j T_j P_j v_L + (1-q) P_j v_L = (q-1) v_L + (1-q) v_L = 0 = P_{j+1} v_L.$$

(A2) depends on the positions of $i,\ i+1$, and $i+2$. When $i, i+1, i+2 \in L$, we know that the relation holds, since the action is exactly the same as $\mathcal{H}_n(q)$ (see [7]). If $i, i+1, i+2 \notin L$, then both T_i and T_{i+1} act by multiplication by q, and (A2) holds. We then consider, separately, the cases when one of $i, i+1, i+2$ is in T and when two of $i, i+1, i+2$ are in T.

Let L_i be an n-standard tableau with $i \in L_i$ and $i+1, i+2 \notin L_i$. Let $L_{i+1} = s_i L_i$ and $L_{i+2} = s_{i+1} L_{i+1}$. Note that L_{i+1} contains $i+1$ and not i or $i+2$ and L_{i+2} contains $i+2$

and not i or $i+1$. For $k = i, i+1$ let M_k denote the matrix of T_k acting on $\{L_i, L_{i+1}, L_{i+2}\}$. Then

$$M_i = \begin{pmatrix} 0 & q^{1/2} & 0 \\ q^{1/2} & q-1 & 0 \\ 0 & 0 & q \end{pmatrix}, \qquad M_{i+1} = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & q^{1/2} \\ 0 & q^{1/2} & q-1 \end{pmatrix}.$$

It is a straight-forward calculation to check that $M_i M_{i+1} M_i = M_{i+1} M_i M_{i+1}$.

Suppose that $i,\ i+1$ are in the same row (or column) in an n-standard tableau L_a and that $i+2 \notin L_a$. Let $L_b = s_{i+1} L_a$ and $T_c = s_i L_b$. Note that $i,\ i+2$ are in the same row (column) in L_b and $i+1, i+2$ are in the same row (column) in T_c. For $k = i, i+1$ let M_k denote the matrix of T_k acting on $\{L_a, L_b, L_c\}$. Then

$$M_i = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & q^{1/2} \\ 0 & q^{1/2} & q-1 \end{pmatrix}, \qquad M_{i+1} = \begin{pmatrix} 0 & q^{1/2} & 0 \\ q^{1/2} & q-1 & 0 \\ 0 & 0 & x \end{pmatrix},$$

where $x = q$ if $i, i+1$ are in the same row of T_a and $x = -1$ if $i,\ i+1$ are in the same column of L_a. Again it is straight-forward to check that $M_i M_{i+1} M_i = M_{i+1} M_i M_{i+1}$.

Finally, let $i, i+1 \in L_a$ with $i,\ i+1$ not adjacent, and let $L_b = s_i L_a$, $L_c = s_{i+1} L_b$, $L_d = s_i L_c$, $L_e = s_{i+1} L_a$, and $L_f = s_i L_e$. Then if α is the box containing i in L_a and β is the box containing $i+1$ in L_b, we have

$$\begin{aligned} &L_a \text{ has } i \text{ in } \alpha \text{ and } i+1 \text{ in } \beta, && L_b \text{ has } i+1 \text{ in } \alpha \text{ and } i \text{ in } \beta, \\ &L_c \text{ has } i+2 \text{ in } \alpha \text{ and } i \text{ in } \beta, && L_d \text{ has } i+2 \text{ in } \alpha \text{ and } i+1 \text{ in } \beta, \\ &L_e \text{ has } i \text{ in } \alpha \text{ and } i+2 \text{ in } \beta, && L_f \text{ has } i+1 \text{ in } \alpha \text{ and } i+2 \text{ in } \beta. \end{aligned}$$

For $k = i, i+1$ let M_k denote the matrix of T_k acting on $\{L_a, L_b, L_c, L_d, L_e, L_f\}$. Then

$$M_i = \begin{pmatrix} \delta(k) & 1+\delta(k) & 0 & 0 & 0 & 0 \\ 1+\delta(-k) & \delta(-k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & q^{1/2} & q-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 & q^{1/2} & q-1 \end{pmatrix}$$

and

$$M_{i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & q^{1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & q-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta(-k) & 0 & 1+\delta(-k) \\ q^{1/2} & 0 & 0 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 1+\delta(k) & 0 & \delta(k) \end{pmatrix},$$

where $k = \mathrm{ct}(\alpha) - \mathrm{ct}(\beta)$ and $\delta(k) = (q-1)/(1-q^k)$. After multiplying out $M_i M_{i+1} M_i$ and $M_{i+1} M_i M_{i+1}$, the only non-trivial relations to check are

(1) $\delta(k)+\delta(-k)=q-1$, and
(2) $q+(q-1)\delta(k)=\delta(k)^2+[1+\delta(k)][1+\delta(-k)]$.

They both follow quite easily from the relation $\delta(-k)=-q^k\delta(k)$.

Let B_k be the subalgebra of $\mathcal{I}_n(q)$ spanned by $T_1,\ldots,T_{k-1},P_1,\ldots,P_k$ so that $B_1\subseteq B_2\subseteq\cdots\subseteq B_n=\mathcal{I}_n(q)$. Clearly, there is a surjection from $\mathcal{I}_k(q)$ to B_k. We will see that they are isomorphic by producing sufficiently many irreducible representations.

Let $1\leqslant k\leqslant n$ and $\lambda\in\Lambda_k\subseteq\Lambda_n$. Then V^λ is spanned by v_L, $L\in\mathcal{T}_n^\lambda$, and is a module for the subalgebra B_k. Let $V^{\lambda,k}\subseteq V^\lambda$ be the subspace spanned by v_L, $L\in\mathcal{T}_k^\lambda$. From (3.7) and (3.8), we see that $V^{\lambda,k}$ is a B_k-submodule of the B_n-module V^λ. We use induction on k to prove that the modules $V^{\lambda,k}$, $\lambda\in\Lambda_k$, are irreducible modules for B_k. In particular, this shows that the modules $V^\lambda=V^{\lambda,n}$, $\lambda\in\Lambda_n$, are irreducible for $\mathcal{I}_n(q)=B_n$.

If $k=1$, then the result is true since the modules, which correspond to $\lambda=\emptyset$ and $\lambda=(1)$, are 1-dimensional. Now, assume that $k>1$ and that the property holds for B_{k-1}. Fix $\lambda\in\Lambda_k$, and consider the restriction of $V^{\lambda,k}$ to B_{k-1}. We partition the standard tableaux $\mathcal{T}_k^\lambda$ into subsets as follows. Let $c_1,\ldots,c_\ell$ denote the "corners" of the partition λ. These are boxes c_i in λ such that λ contains no box to the right or below c_i (i.e., these are the possible locations of k in L). Define

$$\mathcal{T}_k^\lambda(0)=\left\{L\in\mathcal{T}_k^\lambda\;\middle|\;n\notin L\right\}\quad\text{and}\quad\mathcal{T}_k^\lambda(i)=\left\{L\in\mathcal{T}_k^\lambda\;\middle|\;n\in c_i\right\},\quad 1\leqslant i\leqslant k.$$

If $|\lambda|=k$, then L must contain k. In this case we omit the possibility that $i=0$. Now define

$$V_i^{\lambda,k}=\mathbb{C}\left(q^{1/2}\right)\text{-span}\left\{v_L\;\middle|\;L\in\mathcal{T}_k^\lambda(i)\right\},\quad 0\leqslant i\leqslant k.$$

By the definition of the action of T_i, $1\leqslant i\leqslant k-2$, and P_j, $1\leqslant j\leqslant k-1$, we see that $V_i^{\lambda,k}$ is a module for B_{k-1}. In fact $V_i^{\lambda,k}\cong V^{\mu,k-1}$, where μ is obtained from λ by removing c_i, for $1\leqslant i\leqslant n$, and $\mu=\lambda$ when $i=0$. The induction hypothesis shows that $V_i^{\lambda,k}$, $0\leqslant i\leqslant k$, is a set of irreducible, non-isomorphic B_{k-1}-modules (again omit $i=0$ if $|\lambda|=k$).

Suppose $W\subseteq V^{\lambda,k}$ is a nonzero B_k-submodule of $V^{\lambda,k}$. If we consider W to be a B_{k-1}-module, then W contains some irreducible component $V_i^{\lambda,k}$. For each $j\notin\{i,0\}$, we can choose $L\in\mathcal{T}^{\lambda,k}(i)$ with $k-1$ in corner c_j. Then k and $k-1$ are not adjacent in L, so $T_{k-1}v_L=av_L+bv_{s_{k-1}L}$ with $b\neq 0$. Thus $v_{s_{k-1}L}\in W$ and $s_{k-1}L\in\mathcal{T}_k^\lambda(j)$. Furthermore, if $|\lambda|<k$, then we can find $L\in\mathcal{T}^{\lambda,k}(i)$ so that L does not contain $k-1$. Then $T_{k-1}v_L=(q-1)v_L+q^{1/2}v_{s_{k-1}L}$. Thus $v_{s_{k-1}L}\in W$ and $s_{k-1}L\in\mathcal{T}_k^\lambda(0)$. This tells us that $V_j^{\lambda,k}\subseteq W$ for each j and so $W=V^{\lambda,k}$, proving that $V^{\lambda,k}$ is irreducible.

If $\lambda\neq\mu\in\Lambda_k$, then $V^{\lambda,k}$ and $V^{\mu,k}$ are non-isomorphic, because they have different decompositions as B_{k-1}-modules.

The fact that $V^{\lambda,k}$, $\lambda\in\Lambda_k$, is a complete set of irreducible B_k-representations comes from summing the squared dimensions of these representations and comparing with the dimension of B_k. Indeed,

$$\sum_{\ell=0}^{k}\sum_{\lambda\vdash\ell}\binom{k}{\ell}^2 f_\lambda=\sum_{\ell=0}^{k}\binom{k}{\ell}^2\sum_{\lambda\vdash\ell}f_\lambda=\sum_{\ell=0}^{k}\binom{k}{\ell}^2\ell!,$$

where $\sum_{\lambda\vdash\ell} f_\lambda = \ell!$ comes from the representation theory of S_ℓ. We know that B_k is a homomorphic image of $\mathcal{I}_k(q)$ and now we have shown that they have the same dimension. Thus, $B_k \cong \mathcal{I}_k(q)$ and the $V^{\lambda,k}$ form a complete set of irreducible B_k-modules. In particular, V^λ, $\lambda \in \Lambda_n$, is a complete set of irreducible $\mathcal{I}_n(q)$-modules. □

The following is a corollary of the proof of Theorem 3.2.

Corollary 3.3. *The subalgebra of $\mathcal{I}_n(q)$ spanned by $T_1, \ldots, T_{k-1}, P_1, \ldots, P_k$ is isomorphic to $\mathcal{I}_k(q)$. Furthermore, for $\lambda \in \Lambda_n$, the decomposition of V^λ into irreducible modules for $\mathcal{I}_{n-1}(q)$ is given by*

$$V^\lambda \cong \bigoplus_{\mu\in\lambda^{-,=}} V^\mu,$$

where $\lambda^{-,=}$ is the set of all partitions $\mu \in \Lambda_{n-1}$ such that μ equals λ or μ is obtained from λ by removing a box.

From Corollary 3.3 we see that the Bratteli diagram of $\mathcal{I}_n(q)$ is given in Fig. 1. The vertices on row n are given by Λ_n and the edges are determined by restriction rules from $\mathcal{I}_n(q)$ to $\mathcal{I}_{n-1}(q)$. The basis of V^λ partitions into subsets which explicitly realize the decomposition shown in Corollary 3.3 and Fig. 1.

Corollary 3.4. *The subalgebra of $\mathcal{I}_n(q)$ spanned by $T_1, \ldots, T_{n-1}$ is isomorphic to $\mathcal{H}_n(q)$.*

Proof. Let C_n be the subalgebra of $\mathcal{I}_n(q)$ spanned by $T_1, T_2, \ldots, T_{k-1}$. Since the T_i satisfy relations (A1)–(A3), we see that C_n is a homomorphic image of $\mathcal{H}_n(q)$. The

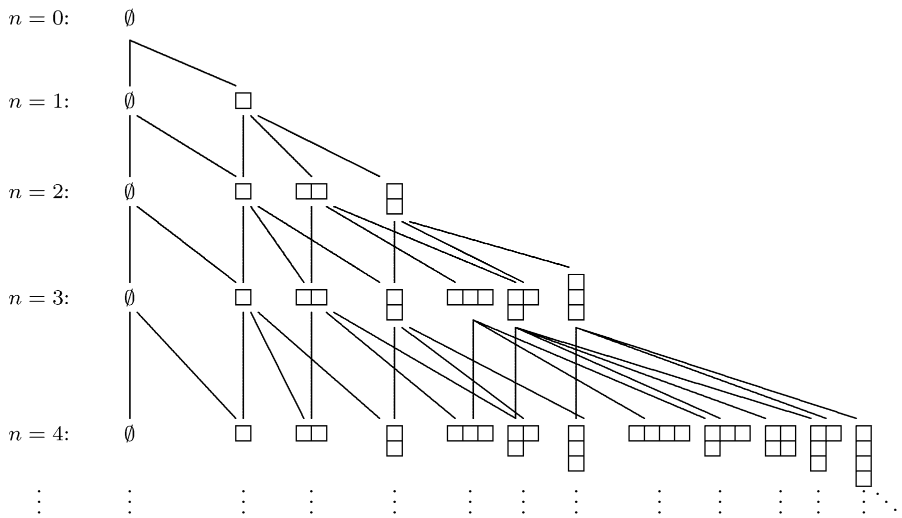


Fig. 1. Bratteli diagram for $\mathcal{I}_n(q)$.

set of $\mathcal{I}_n(q)$-representations $V^\lambda, \lambda \vdash n,$ are representations for the subalgebra C_n and thus are representations of $\mathcal{H}_n(q)$. Furthermore, they are isomorphic to Hoefsmit's [7] seminormal representations of $\mathcal{H}_n(q)$, which are a complete set of irreducible $\mathcal{H}_n(q)$-representations. Since these representations factor through C_n, it follows that C_n and $\mathcal{H}_n(q)$ are isomorphic. □

3.1. Jucys–Murphy elements

Hoefsmit [7] defines special elements in $\mathcal{H}_n(q)$ which act diagonally on the seminormal representations. The analogous elements in S_n later became known as Jucys–Murphy elements (see [17]). We now define analogous elements in $\mathcal{I}_n(q)$.

For $1 \leqslant i \leqslant n$, define

$$X_i = q^{-(i-1)}(T_{i-1}T_{i-2}\cdots T_1)(1-P_1)(T_1T_2\cdots T_{i-1}),$$

so that $X_i = q^{-1}T_{i-1}X_{i-1}T_{i-1}$, for $i \geqslant 2$.

Proposition 3.5. *For $1 \leqslant i \leqslant n$ we have*

$$X_i v_L = \begin{cases} q^{\mathrm{ct}(L(i))}v_L, & \text{if } i \in L,\\ 0, & \text{if } i \notin L,\end{cases}$$

Proof. We use induction on i. If $i = 1$, then $X_1 = P_1$ and the result holds by (3.8). Now we assume that the result is true for X_i and prove it for X_{i+1} by cases determined by the position of $i,\ i+1$ in L.

First assume $i+1 \notin L$. If $i \notin L$, then

$$X_{i+1}v_L = q^{-1}T_iX_iT_iv_L = T_iX_iv_L = 0.$$

If $i \in L$, then

$$X_{i+1}v_L = q^{-1}T_iX_iT_iv_L = q^{-1/2}T_iX_iv_{s_iL} = 0.$$

Now assume $i+1 \in L$. If $i \notin L$, then

$$\begin{aligned} X_{i+1}v_L &= q^{-1}T_iX_iT_iv_L = q^{-1}(q-1)T_iX_iv_L + q^{-1/2}T_iX_iv_{s_iL}\\ &= 0 + q^{-1/2}q^{\mathrm{ct}(L(i+1))}T_iv_{s_iL} = q^{\mathrm{ct}(L(i+1))}v_L. \end{aligned}$$

Finally, let $i, i+1 \in L$. As in the proof of Theorem 3.2, let $d = \mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))$ and let $\delta(d) = (q-1)/(1-q^d)$. Then

$$\begin{aligned} X_{i+1}v_L &= q^{-1}T_iX_iT_iv_L = q^{-1}T_iX_i\big[\delta(d)v_L + \big(1+\delta(d)\big)v_{L'}\big]\\ &= q^{-1}T_i\big[\delta(d)q^{\mathrm{ct}(L(i))}v_L + \big(1+\delta(d)\big)q^{\mathrm{ct}(L(i+1))}v_{L'}\big]\\ &= q^{-1}\Big[\delta(d)q^{\mathrm{ct}(L(i))}\big(\delta(d)v_L + \big(1+\delta(d)\big)v_{L'}\big) \end{aligned}$$

$$+ \big(1+\delta(d)\big)q^{\mathrm{ct}(L(i+1))}\big(\delta(-d)v_{L'} + \big(1+\delta(-d)\big)v_L\big)\Big]$$
$$= Av_L + Bv_{L'},$$

where

$$A = q^{-1}\big[\delta(d)^2 q^{\mathrm{ct}(L(i))} + \big(1+\delta(d)\big)\big(1+\delta(-d)\big)q^{\mathrm{ct}(L(i+1))}\big] \quad \text{and}$$
$$B = q^{-1}\big(1+\delta(d)\big)\big[\delta(d)q^{\mathrm{ct}(L(i))} + \delta(-d)q^{\mathrm{ct}(L(i+1))}\big].$$

Now, $B = 0$ follows quite easily from $\delta(-d) = -q^d\delta(d)$ and

$$\begin{aligned} A &= q^{-1}\big[\delta(d)^2 q^{\mathrm{ct}(L(i))} + \big(1+\delta(d)\big)\big(1+\delta(-d)\big)q^{\mathrm{ct}(L(i+1))}\big] \\ &= q^{-1}q^{\mathrm{ct}(L(i+1))}\big[\delta(d)^2 q^d + \big(1+\delta(d)\big)\big(1+\delta(-d)\big)\big] \\ &= q^{-1}q^{\mathrm{ct}(L(i+1))}\big[\delta(d)^2 q^d + q - q^d\delta(d)^2\big] \\ &= q^{\mathrm{ct}(L(i+1))}. \quad \square \end{aligned}$$

4. Schur–Weyl duality

In this section we show that $\mathcal{I}_n(q)$ and the quantum general linear group $U_q\mathfrak{gl}(r)$ are in Schur–Weyl duality on tensor space.

4.1. The quantum general linear group

Following Jimbo [9], we define the quantum $U_q\mathfrak{gl}(r)$ corresponding to the Lie algebra $\mathfrak{gl}(r)$. The algebra we define here is the same as in [9], except with his parameter q replaced by $q^{1/2}$. Let $U_q\mathfrak{gl}(r)$ be the $\mathbb{C}(q^{1/4})$-algebra given by generators

$$e_i, \quad f_i \quad (1 \leqslant i < r), \quad \text{and} \quad q^{\pm\varepsilon_i/2} \quad (1 \leqslant i \leqslant n),$$

with relations

$$q^{\varepsilon_i/2}q^{\varepsilon_j/2} = q^{\varepsilon_j/2}q^{\varepsilon_i/2}, \qquad q^{\varepsilon_i/2}q^{-\varepsilon_i/2} = q^{-\varepsilon_i/2}q^{\varepsilon_i/2} = 1,$$
$$q^{\varepsilon_i/2}e_jq^{-\varepsilon_i/2} = \begin{cases} q^{-1/2}e_j, & \text{if } j = i-1, \\ q^{1/2}e_j, & \text{if } j = i, \\ e_j, & \text{otherwise}, \end{cases}$$
$$q^{\varepsilon_i/2}f_jq^{-\varepsilon_i/2} = \begin{cases} q^{1/2}f_j, & \text{if } j = i-1, \\ q^{-1/2}f_j, & \text{if } j = i, \\ f_j, & \text{otherwise}, \end{cases}$$
$$e_if_j - f_je_i = \delta_{ij}\frac{q^{1/2(\varepsilon_i-\varepsilon_{i+1})} - q^{-1/2(\varepsilon_i-\varepsilon_{i+1})}}{q^{1/2}-q^{-1/2}},$$

$$e_{i\pm1}e_i^2-\left(q^{1/2}+q^{-1/2}\right)e_ie_{i\pm1}e_i+e_i^2e_{i\pm1}=0,$$
$$f_{i\pm1}f_i^2-\left(q^{1/2}+q^{-1/2}\right)f_if_{i\pm1}f_i+f_i^2f_{i\pm1}=0,$$
$$e_ie_j=e_je_i,\qquad f_if_j=f_jf_i,\quad \text{if } |i-j|>1.$$

Let

$$t_i=q^{\varepsilon_i/4}\quad (1\leqslant i\leqslant r),\qquad k_i=t_it_{i+1}^{-1}\quad (1\leqslant i\leqslant r-1).$$

There is a Hopf algebra structure (see [9, p. 248]) on $U_q\mathfrak{gl}(r)$ with comultiplication Δ and counit u given by

$$\begin{aligned}
\Delta(e_i)&=e_i\otimes k_i^{-1}+k_i\otimes e_i, && u(e_i)=0,\\
\Delta(f_i)&=f_i\otimes k_i^{-1}+k_i\otimes f_i, && u(f_i)=0,\\
\Delta(t_i)&=t_i\otimes t_i, && u(t_i)=1.
\end{aligned}\tag{4.1}$$

The "fundamental" r-dimensional $U_q\mathfrak{gl}(r)$-module V is the vector space

$$V=\mathbb{C}\left(q^{1/4}\right)\text{-span}\{v_1,\dots,v_r\}$$

(so that the symbols v_i form a basis of V) with $U_q\mathfrak{gl}(r)$-action given by (see [9, Proposition 1, Remark 1]),

$$e_iv_j=\begin{cases}v_{j+1}, & \text{if } j=i,\\ 0, & \text{if } j\neq i,\end{cases}\qquad f_iv_j=\begin{cases}v_{j-1}, & \text{if } j=i+1,\\ 0, & \text{if } j\neq i+1,\end{cases}\qquad\text{and}$$
$$t_iv_j=\begin{cases}q^{1/4}v_j, & \text{if } j=i,\\ v_j, & \text{if } j\neq i.\end{cases}$$

The "trivial" 1-dimensional $U_q\mathfrak{gl}(r)$-module W is the vector space

$$W=\mathbb{C}\left(q^{1/4}\right)\text{-span}\{v_0\}$$

(so that the symbol v_0 is a basis of W) with $U_q\mathfrak{gl}(r)$-action given by the counit u (see (4.1)),

$$e_iv_0=f_iv_0=0\quad\text{and}\quad t_iv_0=v_0.$$

Let λ be a partition with $\ell(\lambda)\leqslant r$, and let V^λ be an irreducible $U_q\mathfrak{gl}(r)$-module of highest weight λ. Then $W=V^{\emptyset}$ and $V=V^{(1)}$. The decomposition rules for tensoring by V and W are (see [1, Proposition 10.1.16]),

$$V^\lambda\otimes W\cong V^\lambda\quad\text{and}\quad V^\lambda\otimes V\cong\bigoplus_{\mu\in\lambda^+}V^\mu,\tag{4.2}$$

where λ^+ is the set of partitions that are obtained by adding a box to λ. Thus,

$$V^\lambda \otimes (W \oplus V) \cong \bigoplus_{\mu \in \lambda^{+,=}} V^\mu, \tag{4.3}$$

where $\lambda^{+,=}$ is the set of partitions that are obtained by adding 0 or 1 boxes to λ.

4.2. Centralizer algebra of the tensor power representation

The coproduct on $U_q\mathfrak{gl}(r)$ is coassociative, so it makes sense to consider the tensor product representation $(W \oplus V)^{\otimes n}$. It follows from (4.3) and induction that the n-fold tensor product $(W \oplus V)^{\otimes n}$ decomposes into irreducible $U_q\mathfrak{gl}(r)$-modules as

$$(W \oplus V)^{\otimes n} \cong \bigoplus_{k=0}^{n} \bigoplus_{\lambda \vdash k} \binom{n}{k} f_\lambda V^\lambda, \tag{4.4}$$

where f_λ is the number of standard tableaux of shape λ (see (2.1)). The Bratteli diagram for $U_q\mathfrak{gl}(r)$ is shown in Fig. 1. It has the partitions Λ_n on level n, and a vertex $\mu \in \Lambda_{n+1}$ is connected to a vertex $\lambda \in \Lambda_n$ if $\mu \in \lambda^{+,=}$.

The centralizer algebra

$$C_n = \mathrm{End}_{U_q\mathfrak{gl}(r)}\big((W \oplus V)^{\otimes n}\big)$$

is the set of transformations in $\mathrm{End}((W \oplus V)^{\otimes n})$ which commute with $U_q\mathfrak{gl}(r)$. By general results from double centralizer theory (see, for example, [2, §3D]), we have

(1) C_n is semisimple, and the irreducible representations of C_n are indexed by Λ_n, i.e., the same set that indexes the irreducible representations of $U_q\mathfrak{gl}(r)$ which appear in $(W \oplus V)^{\otimes n}$.
(2) For $\lambda \in \Lambda_n$ let M^λ denote the irreducible C_n-module indexed by λ. Then $\dim(M^\lambda) = m_\lambda$ is the multiplicity of V^λ in the decomposition of $(W \oplus V)^{\otimes n}$ as a $U_q\mathfrak{gl}(r)$-module, and $\dim(V^\lambda) = d_\lambda$ is the multiplicity of M^λ in the decomposition of $(W \oplus V)^{\otimes n}$ as a C_n-module. It follows that m_λ is the number of paths from $\emptyset$ to λ in Fig. 1. We choose $|\lambda|$ levels on which to add a box, and there are f_λ ways to add boxes to $\emptyset$ and reach λ. Thus,

$$m_\lambda = \#(\text{paths from } \emptyset \text{ to } \lambda) = \binom{n}{|\lambda|} f_\lambda.$$

(3) When $r \geqslant n$, all of the partitions in Λ_n appear in the Bratteli diagram, and

$$\dim(C_n) = \sum_{k=0}^{n} \sum_{\lambda \vdash k} \binom{n}{k}^2 f_\lambda^2 = \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{\lambda \vdash k} f_\lambda^2 = \sum_{k=0}^{n} \binom{n}{k}^2 k! = |R_n|. \tag{4.5}$$

4.3. R-matrices

We consider the embedding $U_q\mathfrak{gl}(r) \subset U_q\mathfrak{gl}(r+1)$ so that $U_q\mathfrak{gl}(r)$ is defined as in Section 4.1 and $U_q\mathfrak{gl}(r+1)$ is generated by e_i, f_i, $0 \leqslant i < r$, and t_i, $0 \leqslant i \leqslant r$, with the appropriately extended relations from Section 4.1. Then we define the fundamental representation of $U_q\mathfrak{gl}(r+1)$ as

$$U = \mathbb{C}\big(q^{1/4}\big)\text{-span}\{v_0, v_1, \ldots, v_r\},$$

where the symbols v_i are a basis for U such that $W = \mathbb{C}(q^{1/4})\text{-span}\{v_0\}$, $V = \mathbb{C}(q^{1/4})\text{-span}\{v_1, \ldots, v_r\}$, and thus we have the restriction rule

$$\operatorname{Res}^{U_q\mathfrak{gl}(r+1)}_{U_q\mathfrak{gl}(r)} U = W \oplus V.$$

The $\mathcal{R}$-matrix (see [9, §4]) for $U_q\mathfrak{gl}(r+1)$ provides a canonical $U_q\mathfrak{gl}(r+1)$-module isomorphism $\check{R}_{MN} : M \otimes N \to N \otimes M$ for any two $U_q\mathfrak{gl}(r+1)$-modules M and N. The $\mathcal{R}$-matrix for U, $\check{R}_{UU} : U \otimes U \to U \otimes U$, is given explicitly in [9, formula (7)]. We rescale it to the operator $\check{S} = q^{1/2}\check{R}_{UU}$. For each $0 \leqslant i, j \leqslant r$, we have

$$\check{S}(v_i \otimes v_j) = q^{1/2}\check{R}_{UU}(v_i \otimes v_j) = \begin{cases} q v_j \otimes v_j, & \text{if } i = j, \\ q^{1/2} v_j \otimes v_i, & \text{if } i > j, \\ q^{1/2} v_j \otimes v_i + (q-1)(v_i \otimes v_j), & \text{if } i < j. \end{cases}$$

For each $1 \leqslant i \leqslant n-1$ define

$$\check{S}_i = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \check{S} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}, \tag{4.6}$$

where $\check{S}$ appears as the transformation in the ith and $(i+1)$st factor. Jimbo [9, Proposition 3], shows that $\check{S}$ commutes with $U_q\mathfrak{gl}(r+1)$ and thus $\check{S} \in C_n$.

Define $\check{E} \in \mathrm{End}_{U_q\mathfrak{gl}(r)}(W \oplus V)$ to be projection onto the trivial module W, and let

$$\check{E}_i = \check{E} \otimes \cdots \otimes \check{E} \otimes \mathrm{id} \otimes \cdots \mathrm{id} \in C_n, \tag{4.7}$$

where the projection onto the trivial module $\check{E}$ appears in the first i tensor slots and the identity transformation id appears in the remaining $n-i$ tensor slots.

Proposition 4.1. *Let V be fundamental $U_q\mathfrak{gl}(r)$-module and let W be the trivial $U_q\mathfrak{gl}(r)$-module. The matrices $\check{S}_i$ and $\check{E}_i$ satisfy the following relations as transformations on $U^{\otimes n}$*

(1) $\check{S}_i^2 = (q-1)\check{S}_i^2 + q \cdot 1$, $1 \leqslant i \leqslant n-1$,
(2) $\check{S}_i\check{S}_{i+1}\check{S}_i$, $1 \leqslant i \leqslant n-2$,
(3) $\check{S}_i\check{S}_j = \check{S}_j\check{S}_i$, $|i-j| > 2$,
(4) $\check{S}_i\check{E}_j = \check{E}_j\check{S}_i = q\check{E}_j$, $1 \leqslant i < j \leqslant n$,
(5) $\check{S}_i\check{E}_j = \check{E}_j\check{S}_i$, $1 \leqslant j < i \leqslant n$,

(6) $\check{E}_i^2 = \check{E}_i,\ 1 \leqslant i \leqslant n$,
(7) $\check{E}_{i+1} = \check{E}_i \check{S}_i \check{E}_i + (1-q)\check{E}_i,\ 2 \leqslant i \leqslant n$.

Proof. Let $U_q\mathfrak{gl}(r)$ be embedded in $U_q\mathfrak{gl}(r+1)$ as discussed above so that $U = V \oplus W$ as a module for $U_q\mathfrak{gl}(r)$. From [9], we know that $\check{S}_i$ is in $\mathrm{End}_{U_q\mathfrak{gl}(r+1)}(U^{\otimes n}) \subseteq C_n$ and that the $\check{S}_i$ satisfy relations (1)–(3). These are not difficult to verify.

If $j < i$, then $\check{S}_i$ acts as the identity in tensor positions $1, \ldots, j$ and $\check{E}_j$ acts as identity in tensor positions $i, i+1$, so $\check{S}_i$ and $\check{E}_j$ commute and property (5) holds.

Property (6) follows immediately from the fact that $\check{E}_j$ is a projection.

For properties (4) and (7), we check the actions on the basis of simple tensors $v_{k_1} \otimes \cdots \otimes v_{k_n}$ with $0 \leqslant k_j \leqslant r+1$. Let $\mathbf{v} = v_{k_1} \otimes \cdots \otimes v_{k_n}$ and let $\mathbf{v}'$ be obtained from $\mathbf{v}$ by switching v_{k_i} with $v_{k_{i+1}}$. Thus $\check{S}_i\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}'$ with $\alpha, \beta \in \mathbb{C}(q^{1/2})$.

Assume that $j > i$. If $k_1 = \cdots = k_j = 0$, then $\check{E}_j\mathbf{v} = \mathbf{v}$ and $\check{S}_i\mathbf{v} = q\mathbf{v}$, so $\check{S}_i\check{E}_j\mathbf{v} = \check{S}_i\mathbf{v} = q\mathbf{v} = q\check{E}_j\mathbf{v} = \check{E}_j\check{S}_i\mathbf{v}$. If it is not the case that $k_1 = \cdots = k_j = 0$, then $\check{E}_j\mathbf{v} = \check{E}_j\mathbf{v}' = 0$, so $\check{E}_j\check{S}_i\mathbf{v} = \check{E}_j(\alpha\mathbf{v} + \beta\mathbf{v}') = 0 = q\check{E}_j\mathbf{v} = \check{S}_i\check{E}_j\mathbf{v}$, and property (4) holds.

If it is not the case that $k_1 = k_2 = \cdots = k_i = 0$ then $\check{E}_i\mathbf{v} = 0$ and $\check{E}_{i+1}\mathbf{v} = 0$, so

$$\check{E}_{i+1}\mathbf{v} = 0 = \big(\check{E}_i\check{S}_i\check{E}_i + (1-q)\check{E}_i\big)\mathbf{v}.$$

Now assume $k_1 = k_2 = \cdots = k_i = 0$. If $k_{i+1} = 0$, then $\check{E}_i\mathbf{v} = \mathbf{v}$, $\check{E}_{i+1}\mathbf{v} = \mathbf{v}$, and $\check{S}_i\mathbf{v} = q\mathbf{v}$, so

$$\big(\check{E}_i\check{S}_i\check{E}_i + (1-q)\check{E}_i\big)\mathbf{v} = q\mathbf{v} + (1-q)\mathbf{v} = \mathbf{v} = \check{E}_i\mathbf{v} = \mathbf{v}.$$

If $k_{i+1} > 0$, then $\check{E}_i\mathbf{v} = \mathbf{v}$, $\check{E}_i\mathbf{v}' = 0$, $\check{E}_{i+1}\mathbf{v} = 0$, and $\check{S}_i\mathbf{v} = (q-1)\mathbf{v} + q^{1/2}\mathbf{v}'$, so

$$\begin{aligned}\big(\check{E}_i\check{S}_i\check{E}_i + (1-q)\check{E}_i\big)\mathbf{v} &= \check{E}_i\big((q-1)\mathbf{v} + +q^{1/2}\mathbf{v}'\big) + (1-q)\mathbf{v} \\ &= (q-1)\mathbf{v} + (1-q)\mathbf{v} = 0 = \check{E}_{i+1}\mathbf{v}.\end{aligned}$$

Thus, (7) holds and the proposition is proved. □

Corollary 4.2. *The elements $\check{E}_1$ and $\check{S}_i$, $1 \leqslant i \leqslant n-1$, generate C_n.*

Proof. Let D_n denote the subalgebra generated by $\check{E}_1$ and $\check{S}_i$, $1 \leqslant i \leqslant n-1$. From [20], we know that, under the specialization $q \to 1$, $\check{E}_1$ and $\check{S}_i$ specialize to generators of $\mathrm{End}_{GL(r,\mathbb{C})}((W \oplus V)^{\otimes n})$, which has the same dimension as C_n. Under such a specialization the dimension cannot go up. This follows from [3, §68.A], since there is a basis for D_n consisting of words in the generators E_1, S_i and the structure constants for this basis are well-defined (do not have poles) at $q = 1$. Thus, D_n is a subalgebra of C_n with the same dimension as C_n, and so they are equal. □

Corollary 4.3. *The map* $\phi : A_n(q) \to \mathrm{End}_{U_q\mathfrak{gl}(r)}((W \oplus V)^{\otimes n})$ *given by*

$$\phi(T_i) = \check{S}_i \quad \text{and} \quad \phi(P_i) = \check{E}_i$$

is a surjective algebra homomorphism, and if $r \geqslant n$, *then* ϕ *is an isomorphism. The action of* T_i, $1 \leqslant i \leqslant n-1$ *and* P_j, $1 \leqslant j \leqslant n$ *on simple tensors* $\mathbf{v} = v_{k_1} \otimes \cdots \otimes v_{k_n}$ *is given by*

$$T_i\mathbf{v} = \begin{cases} (q-1)\mathbf{v} + q^{1/2}\mathbf{v}', & \text{if } k_i < k_{i+1}, \\ q^{1/2}\mathbf{v}', & \text{if } k_i > k_{i+1}, \\ q\mathbf{v}, & \text{if } k_i = k_{i+1}, \end{cases} \qquad P_j\mathbf{v} = \begin{cases} \mathbf{v}, & \text{if } k_1 = \cdots = k_j = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4.8}$$

where $\mathbf{v}'$ *is the simple tensor obtained from* $\mathbf{v}$ *by switching* v_{k_i} *with* $v_{k_{i+1}}$.

Proof. Proposition 4.1 and Corollary 4.2 tell us that ϕ is a surjective homomorphism. By comparing dimensions when $r \geqslant n$, we see that ϕ is an isomorphism. The action of the generators follows from (4.7) and (4.8). Note: one can also verify the relations (2.1). □

Remark 4.4. It is natural to look for a presentation of $\mathcal{I}_n(q)$ using generators Π_i which project onto the trivial module W in only the ith tensor slot. At $q \to 1$, these correspond to the idempotents $\pi_i = 1 - E_{i,i} \in R_n$. Furthermore, we have $P_i = \Pi_1\Pi_2\cdots\Pi_i$. However, the Π_i appear to have a complicated relation with the T_i. Using a computer, M. Dieng found that in $\mathcal{I}_3(q)$,

$$\begin{aligned} \Pi_2 &= T_1^{-1}\Pi_1T_1 + \frac{(q-1)}{q^3}\big(T_1^{-1}P_1 + T_1^{-1}P_2\big), \\ \Pi_3 &= T_2^{-1}\Pi_2T_2 + (q-1)^2T_2^{-1}T_1^{-1}P_1 + (q-1)T_2^{-1}T_1^{-1}P_1T_1 \\ &\quad - \frac{(q-1)^2}{q}\big(T_1T_2^{-1}P_2 + T_2^{-1}P_2 + T_1^{-1}T_2^{-1}P_2T_2\big) \\ &\quad + \frac{(q-1)}{q}T_1^{-1}T_2^{-1}P_2T_2T_1 + \frac{(q-1)^2(q+1)}{q^3}P_3. \end{aligned}$$

Acknowledgments

I thank Arun Ram and Louis Solomon for numerous enlightening conversations and helpful suggestions and for suggesting improvements on early versions of this paper. I also thank Momar Dieng, whose work on the characters of $\mathcal{I}_n(q)$ in [4] helped lead to the presentation (2.1) and to the calculations in Remark 4.4.

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