FINITE POSETS AND THEIR REPRESENTATION ALGEBRAS

MURRAY GERSTENHABER AND MARY SCHAPS

ABSTRACT. A finite poset $(P \preceq S)$ determines a finite dimensional algebra T_P over the field over the field $\mathbb{F}$ of two elements, with an upper triangular representation. We determine the structure of the radical of the representation algebra A of the monoid $(T_P, \cdot)$ over a field of characteristic different from 2. We consider algebra deformations of T_P, using the cohomology comparison theorem. We also consider degenerations of A over a complete discrete valuation ring with residue field of characteristic 2.

1. INTRODUCTION

The poset algebras form an important class of finite dimensional algebras, which have been intensively studied in connection with the representations of posets [ARS]. In this paper we consider the multiplicative monoid of the poset algebra, and in order to get the simplest possible monoid, we restrict ourselves to considering the poset algebra over a field of two elements. We then study representations of the multiplicative monoid. The structure is much richer than that for the poset algebra itself.

Let $(P, \preceq)$ be a finite poset. We may identify P with $\{1, \dots, n\}$ in such a way that $i \preceq j$ implies that $i \leq j$ in the ordinary linear order on the integers. Let $\{E_{ij}\}$ be the matrix units of $M_n(\mathbb{F}_2)$, the $n \times n$ matrices over the field of two elements. Then the poset $(P, \preceq)$ determines a linear subspace $T_P = \langle E_{ij} \bigm| i \preceq j \rangle$. By transitivity, T_P is actually a semigroup under multiplication. By reflexivity, $(T_P, \cdot)$ is a monoid. Since the additive structure is clearly compatible with the multiplication, $(T_P, +, \cdot)$ has the structure of a unitary algebra over $\mathbb{F}_2$. By anti-symmetry, it is embedded as a subalgebra in $T_n(\mathbb{F}_2)$, the upper triangular $n \times n$ matrices over $\mathbb{F}_2$. Letting M_P be the multiplicative monoid $(T_P, \cdot)$, we note that M_P is a monoid with an absorbing zero element, which we will denote by Z.

A representation of a monoid M with a zero element Z over a field k is a monoid homomorphism into $(M_m(k), \cdot)$ which sends Z to the zero matrix. By standard semigroup representation theory, the monoid representations of M corresponds to the algebra representations of the reduced monoid algebra

$$A = kM/kZ.$$

When $M = M_P$, we will refer to A as the representation algebra of P over k.

There is a great deal known about $A/\operatorname{Rad}(A)$ in terms of the structure of the semigroup, and we will introduce these results in their place.

In §2, we will discuss the structure of $\operatorname{Rad}(A)$. In §3 and §4, we consider deformation of T_P. In §5, we consider the result of replacing k by a complete discrete valuation ring $\mathcal{O}$ of characteristic 0, with residue field of characteristic 2, and describe the resulting changes in the radical under degeneration to the closed point.

2. The radical of the representation algebra

The standard tool for studying the representations of a finite dimensional algebra is its quiver, as in [G]. However, as we showed in [Sc2] when one considers deformations and degenerations, the appropriate object is not the quiver but the basis graph, a directed graph whose vertices and arrows correspond to a vector space basis of A, weighted according to the highest power of the radical containing the basis element.

Definition. Let $\{e_i, \dots, e_r\}$ be a complete set of orthogonal idempotents of a k-algebra A, i.e., $\Sigma e_i = 1_A$ and $e_i \cdot e_j = 0$ for $i \neq j$. A basis graph of A adapted to $\{e_i\}$ is a directed graph with r vertices, labelled by the e_i, with $(\dim e_i A e_j) - 1$ loops at e_i, and $\dim e_i A e_j$ arrows from e_i to e_j. The *weighted* basis graph is given by choosing bases filtered by powers of the radical and weighting each arrow by the minimal radical power containing the corresponding basis element. If all the idempotents are primitive, the graph is independent of the choice of idempotents and is called the *basis graph of* A [Sc2].

Let us establish some notation. Given the poset $P = \{1, \dots, n\}$, we let $G_P \subseteq P \times P$ be the graph of the poset, i.e.,

$$G_P = \{(i,j) \mid i \preceq j\}.$$

The elements of M_P are in one-to-one correspondence with the power set $\mathcal{P}(G_P)$; and the zero element Z corresponds to the empty set $\emptyset$. For each $B \in \mathcal{P}(G_P)$, let $S(B)$ be the corresponding element of M_P, a non-zero matrix with "1" in the (i,j) position for each $(i,j) \in B$. Let

$$\Delta = \{(1,1), \dots, (n,n)\}$$

be the diagonal of $P \times P$. The identity of M_P is $S(\Delta)$.

For any $D \subseteq \Delta$, the element $S(D)$ is an idempotent, and in fact there is a commutative idempotent submonoid E of the monoid M_P given by

$$E = \{S(D) \mid D \subseteq \Delta\},$$

[St].

The representation algebra of the commutative idempotent monoid E is a product of $|E| - 1$ copies of k. The idempotents corresponding to the distinct copies are given for $D \neq \emptyset$, by

$$\begin{aligned} \hat{E}(D) = S(D) &- \sum_{(i,i) \in D} S(D - \{(i,i)\}) \\ &+ \sum_{\{(i,i),(j,j)\} \in D} S(D - \{(i,i),(j,j)\}) \dots (-1)^{|D|-1} \sum_{(i,i) \in D} S(\{(i,i)\}), \end{aligned}$$

where the sum is formed by the standard exclusion-inclusion principle [St]. Note that this basis of primitive idempotents is much more appropriate for working with the representation algebra of the monoid E than the original monoid basis E. In particular, this set of idempotents is complete and orthogonal, though not necessarily primitive.

Since E is a submonoid of M_P, the representation algebra of E is a subalgebra of A. Thus $E = \{\hat{E}(D) \mid D \subseteq \Delta,\ D \neq \emptyset\}$ is a complete orthogonal set of idempotents for the representation algebra A, and we can try to find a basis graph w.r.t. this set of idempotents $\hat{E}$.

Let $\mathcal{P}^* = \mathcal{P}(G_P) - \{0\}$. For any $B \in \mathcal{P}^*$, we define

$$L(B) = \{(i,i) \mid \exists j, (i,j) \in B\} \subseteq \Delta,$$
$$R(B) = \{(j,j) \mid \exists i, (i,j) \in B\} \subseteq \Delta.$$

Definition. For any $B \in \mathcal{P}^*$, the element of A adapted to $\hat{E}$ which corresponds to B is

$$W(B) = \hat{E}(L(B))S(B)\hat{E}(R(B)).$$

Lemma 2.1. *The set $\{W(B) \mid B \in \mathcal{P}^*\}$ is a basis for A over k, with a base change matrix which is upper triangular with unit diagonal relative to an ordering of $\mathcal{P}^*$ compatible with inclusion. Each $S(B) = W(B) + \sum_{B' \subseteq B} \mu_{B'} W(B')$ for some coefficients $\mu_{B'} \in k$.*

Proof. $S(L(B)) \cdot S(B) = S(B)$, since $E_{ii}E_{ij} = E_{ij}$, and similarly

$$S(B) \cdot S(R(B)) = S(B)$$

since $E_{ij}E_{jj} = E_{ij}$. In $\hat{E}(L(B))$, a matrix $S(D')$ occurs with nonzero coefficient only if $D' \subseteq L(B)$. For any $D' \subsetneq L(B)$, $S(D') \cdot S(B) = S(B')$, with $B' \subsetneq B$ and for any $D'' \subsetneq R(B)$, $S(B)S(D'') = S(B'')$, with $B'' \subsetneq B$. Thus in $W(B)$, $S(B)$ appears with coefficient 1 and all remaining $S(B')$ which appear satisfy $B' \subsetneq B$, giving an invertible upper triangular base change matrix. Inverting the base change matrix gives the desired representation for $S(B)$. $\square$

Our next theoretical result will be to construct a subideal of the radical with no directed cycle, but first we will work out the simplest nontrivial example in order to illustrate all the definitions.

Example 1. Let $(P, \preceq)$ be the poset with two elements $P = \{1,2\}$, $1 \preceq 2$. The algebra T_P has dimension 3 over $\mathbb{F}_2$, being simply $T_2(\mathbb{F}_2)$, with $|T_P| = 2^3 = 8$. The graph G_P of the poset is $\{(1,1),(1,2),(2,2)\}$, and $\mathcal{P}(G_P)$ thus has eight elements, corresponding to the eight elements of $M_P = (T_P, \cdot)$. The basis graph of T_P is just $\cdot \longrightarrow \cdot$, which in this simple case is the Hasse diagram of the poset.

The representation algebra A is of dimension 7, since we have divided out by the zero element Z of the monoid M_P. The diagonal Δ is $\{(1,1),(2,2)\}$. The idempotents in the submonoid E of M_P are

$$E = \{S(\Delta), S(\{(1,1)\}), S(\{(2,2)\}), S(\emptyset)\}.$$

Since

$$W(\{(1,1)\}) = S(\{(1,1)\}) = E_{11},$$
$$W(\{(2,2)\}) = S(\{(2,2)\}) = E_{22},$$

and

$$\begin{aligned} W(\Delta) &= S(\Delta) - S(\{(1,1)\}) - S(\{(2,2)\}) \\ &= I_2 - E_{11} - E_{22}, \end{aligned}$$

we have

$$\hat{E} = \{W(\Delta), E_{11}, E_{22}\}.$$

The four remaining elements in the algebra basis are as follows:

$$B_0 = \{(1,2)\}, \quad L(B_0) = \{(1,1)\}, \quad R(B_0) = \{2,2\},$$
$$W(B_0) = S(B_0) = S(\{(1,2)\}),$$
$$B_1 = \{(1,1),(1,2)\}, \quad L(B_1) = \{(1,1)\}, \quad R(B_1) = \Delta,$$
$$\begin{aligned} W(B_1) &= \hat{E}(L(B_1))S(B_1)\hat{E}(\Delta) \\ &= S(B_1)\hat{E}(\Delta) \\ &= S(B_1) - S(\{(1,1)\}) - S(\{(1,2)\}), \end{aligned}$$
$$B_2 = \{(1,2),(2,2)\}, \quad L(B_2) = \Delta, \quad R(B_2) = \{(2,2)\}$$
$$\begin{aligned} W(B_2) &= (\hat{E}(\Delta)S(B_2)\hat{E}(R(B_2)) \\ &= \hat{E}(\Delta)S(B_2) \\ &= S(B_2) - S(\{(1,2)\}) - S(\{(2,2)\}), \end{aligned}$$
$$B_3 = \{(1,1),(1,2),(2,2)\}, \quad L(B_3) = \Delta, \quad R(B_3) = \Delta$$
$$\begin{aligned} W(B_3) &= \hat{E}(\Delta)S(B_3)\hat{E}(\Delta) \\ &= S(B_3) - S(B_1) - S(B_2) + S(B_0). \end{aligned}$$

Besides the multiplication by idempotents, the only possible non-zero products are:

$$\begin{aligned} W(B_1) \cdot W(B_2) = [S(B_1) - S(\{(1,1)\}) - S(\{(1,2)\})] \\ \cdot [S(B_2) - S(\{(1,2)\}) - S(\{(2,2)\}] \\ = -2S(\{(1,2)\}) \end{aligned}$$
$$W(B_1)W(B_3) = -W(B_1)$$
$$W(B_3)W(B_2) = -W(B_2)$$
$$W(B_3)W(B_3) = \hat{E}(\Delta).$$

Since $\operatorname{char} k \neq 2$, $S(\{(1,2)\} = (-\frac{1}{2})W(B_1)W(B_2)$ lies in $\operatorname{Rad}^2(A)$. Let $W = W(B_3)$. Since $W^2 = \hat{E}(\Delta)$, and

$$\hat{E}(\Delta)W = W\hat{E}(\Delta),$$

we see that $\hat{E}(\Delta)$ splits into two primitive idempotents

$$E_3' = \frac{1}{2}(\hat{E}(\Delta) - W)$$
$$E_3'' = \frac{1}{2}(\hat{E}(\Delta) + W).$$

We then have

$$W(B_1) \cdot E_3' = \frac{1}{2}(W(B_1) - (-W(B_1))) = W(B_1)$$
$$W(B_1) \cdot E_3'' = \frac{1}{2}(W(B_1) + (-W(B_1))) = 0,$$

and similarly for $W(B_2)$. Thus the weighted basis graph, after we substitute E_3', E_3'' for $\hat{E}(\Delta), W(B_3)$, is

$$\begin{tikzcd}
 & \cdot E_3'' & \\
 & \cdot E_3' \arrow[dr, "W(B_2)"] & \\
E_{11}\cdot \arrow[ur, "W(B_1)"] \arrow[rr, two heads] & & \cdot E_{22}
\end{tikzcd}$$

Note that the double arrowhead indicates weight 2 (not surjectivity).

For an algebra which is basic, i.e., whose semisimple part is a direct product of copies of the base field, the quiver is obtained from the weighted basis graph by omitting all arrows of weight greater than 1. In this case the quiver of A is

$$\begin{array}{c} \cdot E_3'' \\ E_{11}\cdot \longrightarrow \cdot E_3' \longrightarrow \cdot E_{22} \end{array}$$

which is "almost" a barycentric subdivision of the quiver $E_{11}\cdot \longrightarrow E_{22}$ of T_P.

We now return to the general case. The semisimple part of the representation algebra has been studied in depth in terms of the J-classes of M. Two elements $a, b \in M$ are in the same J-class if

$$MaM = MbM.$$

Two idempotents e, f are in the same J-class if there are elements x and y of M such that $e = xy$ and $f = yx$, and in this case we consider them equivalent, $e \sim f$. A J-class is regular if it contains a non-zero idempotent. If we choose representatives e_i, $i \in I$, of the different equivalence classes of idempotents, then the semisimple part S of A is a direct sum of semisimple algebras derived from group algebras of the groups of units of the subsemigroup e_iMe_i

$$k(e_iMe_i)^*.$$

In the particular case of the poset monoid M_P, the idempotents $E = \{S(D) \bigm| D \in \mathcal{P}(\Delta)\}$, form a complete set of representatives of the regular non-zero J-classes. The invertible elements in $S(D)M_PS(D)$ correspond to the subsets $B \in \mathcal{P}^*$ for which $L(B) = R(B) = D$ and $B \cap \Delta = D$. We let

$$H(D) = \{S(B) \bigm| L(B) = R(B) = D,\ B \cap \Delta = D\}.$$

If $\pi : \Delta \to P$ is the projection, set

$$Y(D) = G_P \cap (\pi(D) \times \pi(D)),$$

then the entry in every off diagonal position in $Y(D)$ can be either 0 or 1, giving us that $H(D)$ is a 2-group

$$|H(D)| = 2^{|Y(D)|-|D|}.$$

In Example 1, $|Y(\Delta)| = 3$, $|\Delta| = 2$, so $|H(\Delta)| = 2$. The group $H(\Delta) = \{S(\Delta), S(B_3)\}$ and the semisimple part of the representation algebra Δ is

$$S = kC_1 \oplus kC_1 \oplus kC_2,$$

where C_m denotes a cyclic group of order m.

Definition. For any subset of $L \leq P$, let the spanned subset $I(L)$ be the smallest interval in P, in the linear ordering, containing L. Let $\pi = \pi_1 = \pi_2$ be the projection of Δ on P. We say that two subsets D_1, D_2 of Δ are equivalent if $I(\pi(D_1)) = I(\pi(D_2))$. This equivalence relation generates a corresponding equivalence relation on $\hat{E}$. There are $\binom{n}{2}+n$ equivalence classes in $\hat{E}$. We then define a partial ordering of the equivalence classes of $\hat{E}$ by setting $[\hat{E}(D)] \leq [\hat{E}(D'))]$ if for $I(\pi(D)) = [a, \dots b]$ and $I(\pi(D')] = [c, \dots, d]$, we have $a \leq c$ and $b \leq d$.

Lemma 2.2. *If P is the linear poset on n elements, then there is a subset $B \subseteq G_P$ such that $L(B) = D$ and $R(B) = D'$ iff $[\hat{E}(D)] \leq [\hat{E}(D')]$.*

Proof. Let $I(\pi(D)) = [a, \dots, b]$ and $I(\pi(D')) = [c, \dots, d]$.

($\Rightarrow$) Suppose that there is a $B \subseteq G_P$ such that $L(B) = D$ and $R(B) = D'$. If (i, j) is a pair in B with $j = c$, then $a \leq i \leq j = c$, so $a \leq c$. Similarly, if (i, j) is a pair with $i = b$, then $b = i \leq j \leq d$, thus $[\hat{E}(D)] \leq [\hat{E}(D')]$.

($\Leftarrow$) If $a \leq c$ and $b \leq d$, then we can take B to be the set containing (a, j) for every $(j, j) \in D'$ and (i, d) for every $i \in D$, which is a well-defined subset of G_P. $\square$

We now prove a lemma which will simplify calculations.

Lemma 2.3. *If $R(B) \subsetneq D$, then $S(B) \cdot \hat{E}(D) = 0$. Similarly, if $L(B) \subsetneq D$, the $\hat{E}(D)S(B) = 0$.*

Proof. By inverting the upper triangular matrix expressing the $\hat{E}(D_j)$ as linear combinations of the $E(D_i)$, we get $E(D_j) = \sum \hat{E}(D_i)$ [St]. Let $D_j = R(B)$. Then $S(B) \cdot E(D_j) = S(B)$. Thus

$$S(B) \cdot \hat{E}(D) = S(B) \cdot E(D_j)\hat{E}(D) = S(B)\left(\sum_{D_i \subseteq D} \hat{E}(D_i)\right) \cdot \hat{E}(D) = 0$$

since all $\hat{E}(D_i)\hat{E}(D) = 0$. $\square$

For any two subsets of B_1, B_2 of G_p, we denote by $B_1 \cdot B_2$ the subset of G_p such that $S(B_1) \cdot S(B_2) = S(B_1 \cdot B_2)$ in the monoid. Note that by standard properties of matrix units, we always have $L(B_1) \cdot B_2 \subseteq L(B_1)$ and $R(B_1 \cdot B_2) \subseteq R(B_2)$.

Proposition 2.1. *The vector space V spanned by the set of all $W(B)$ such that $[\hat{E}(L(B))] \lneq [\hat{E}(R(B))]$ is a two-sided ideal contained in* Rad(A)*, with no directed cycles.*

Proof. <u>V is an ideal</u>. Since the $W(B)$ form a basis for A, it suffices to show that $W(B')W(B) \in V$ and $W(B)W(B'') \in V$ for any $B', B'' \in G_P$. Since $\hat{E}$ is an orthogonal set of idempotents, we have $W(B')W(B) = 0$ unless $R(B') = L(B)$. However, in that case, Lemma 2.2, we have

$$[\hat{E}(L(B')] \leq [\hat{E}(R(B')] \lneq [\hat{E}(R(B)].$$

By Lemma 2.3 and the orthogonality of $\hat{E}$, either $L(B' \cdot B) = L(B')$ and $R(B'B) = R(B)$ or else

$$\begin{aligned} W(B') \cdot W(B) &= \hat{E}(L(B')) \cdot W(B') \cdot W(B) \cdot \hat{E}(R(B)) \\ &= 0. \end{aligned}$$

Thus, in either event, $W(B') \cdot W(B) \in V$. The proof for $W(B) \cdot W(B'')$ is the same with the sides switched.

V is nilpotent. The ideal V is nilpotent because the chain length of a strictly decreasing chain of equivalence classes $[\hat{E}(D)]$ is $2(n-1)$. This poset $(\hat{E}/\sim, \leq)$ has a minimal element $\hat{E}(\{(1,1)\})$ and a maximal element $\hat{E}(\{n, n, \})$. Any maximal chain connecting them has end of the interval $I(D)$ moving step-by-step from 1 to n and the beginning of the interval moving step-by-step from 1 to n, in such a way that the end is always greater than or equal to the beginning, altogether $(n-1)+(n-1)$ steps.

Thus, viewed as an ideal, V to the power $2n-1$ must be zero, so it is nilpotent. As a nilpotent ideal, V is contained in the radical. $\square$

If we divide out by V, the quotient A/V is a direct sum of algebras of the following type:

Definition. For any natural number m, the gap algebra R_m of m is the representation algebra of the monoid of the upper triangular matrices $T_m(\mathbb{F}_2)$, for which the first and last element of the diagonal are non-zero.

The algebra R_m has dimension $\binom{m}{2} - 2$ over k, and its invertible subgroup R_m^* has order $2^{\binom{m-1}{2}}$. The monoid $(R_m, \cdot)$ has a central element of order 2, given by

$$C' = \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ & 1 & & & 0 \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

Thus, if we construct a representation algebra A_m for the monoid over a sufficiently large field k of characteristic different than 2, we get that A_m is a direct product of two algebras with identities

$$e' = \frac{1}{2}(I_m + C), \quad e'' = \frac{1}{2}(I_m - C).$$

We have already calculated these gap algebras in the first cases. For $m = 2$, the two algebras are copies of k.

Every representation algebra for a poset on n elements is a subalgebra of the representation algebra of the linear poset of n elements, for which $T_P = T_n(\overline{F_2})$. Furthermore, the set of idempotents for both monoids is the same.

Example 2. We consider the linear poset with three elements. The representation algebra has dimension 63. We first give a matrix whose entries are dimension of $\hat{E}(D_i)A(\hat{E}(D_j)$, where the seven elements D_i are ordered as follows: $D_1 = \{(1,1)\}$,

$D_2 = \{(1,1),(2,2)\}$, $D_3 = \{(2,2)\}$, $D_4 = \{(1,1),(2,2),(3,3)\}$, $D_5 = \{(1,1),(3,3)\}$, $D_6 = \{(2,2),(3,3)\}$, $D_7 = \{(3,3)\}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 8 & 2 & 7 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 10 & 2 & 8 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix was obtained by considering possible pairs $(L(B), R(B))$ and counting the elements $B \in \mathcal{P}^*$ with the corresponding pair. For example, the pair (D_5, D_4) corresponds to the two matrices $\begin{bmatrix} 1 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Although we were able to choose an ordering in which the above matrix is almost triangular, there is an oriented cycle between D_4 and D_5, not to mention all the loops in the basis graph w.r.t. $\{\hat{E}(D_i)\}$ corresponding to non idempotent elements of the $H(D_i)$.

By Proposition 2.1, the representation algebra of the representation group of the linear poset of n elements has no directed cycles if and only if this is true for every gap algebra $(R_m, \cdot)$ for $m \leq n$.

In this case, $H(D_1)$, $H(D_3)$ and $H(D_7)$ are trivial, $H(D_2)$, $H(D_5)$ and $H(D_6)$ are isomorphic to C_2, as in Example 1, and $H(D_4)$ is the dihedral group of order 8. In addition to the eight elements corresponding to elements of $H(D_4)$, $\hat{E}(D_4)A\hat{E}(D_4)$ contains two elements corresponding to $W(B)$ for

$$S(B) = \begin{bmatrix} 1 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The 16 dimensional algebra involving $\hat{E}(D_4)$ and $\hat{E}(D_5)$ is actually the sum of two 8 dimensional algebras cut out by the central idempotents e', e'' determined by the central element of order 2 in $H(D_4)$, corresponding to $W(C)$, where $C = \{(1,1),(1,3),(3,3)\}$. If

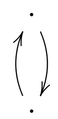
$$e' = \frac{1}{2}(\hat{E}(D_4) + W(C)),$$

Then the corresponding 8 dimensional algebra has a basis graph

$$\begin{array}{ccccc} & & \cdot & & \\ \cdot & & \twoheadrightarrow & & \cdot \\ & \searrow & \cdot & \nearrow & \\ & & \cdot & & \end{array}$$

Let $B_1 = \{(1,1),(2,3),(3,3)\}$ and let $B_2 = \{(1,1),(1,2),(3,3)\}$. The lowest idempotent is $e'\hat{E}(D_5)$. The diagonal arrows are $e'W(B_1)$ and $e'W(B_2)$.

If $e'' = \frac{1}{2}(\hat{E}(D_4) - W(C))$, then we get two copies of $M_2(k)$. One comes from the group algebra of the dihedral group. The second can be represented by a basis graph



The bottom idempotent is $e''\hat{E}(D_5)$, and the idempotent at the top is $-e'' \cdot W(B)$ for

$$B = \{(1,1),(1,2),(2,3),(3,3)\}.$$

The two matrix units are $e''W(B_1)$ and $e''W(B_2)$. In fact,

$$W(B) = \begin{bmatrix}1&1&0\\0&0&1\\0&0&1\end{bmatrix} - \begin{bmatrix}1&1&0\\0&0&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&1\\0&0&1\end{bmatrix} + \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

and by multiplying out we get

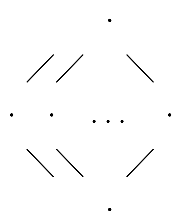
$$W(B)^2 = -e''W(B).$$

Thus $e'W(B)$ is in the radical and $-e''W(B)$ is an idempotent. In the diagram above, $e'W(B)$ is the horizontal arrow which is the composition of the two diagonal arrows. The representation algebra of $T_3(\mathbb{F}_2)$ has no oriented cycles in the quiver, since the matrix algebras appear in the quiver as isolated points.

In [DI], the representations of the triangular group are described as a "well-known nightmare." They are known explicitly only up to $n = 7$, and there does not seem to be a generic formula for the representations. Thus there is little hope for finding the quiver explicitly for general n.

For specific families of posets, it is possible to get a formula. We consider the particular case of the two-point suspension of the discrete poset on $n-2$ points.

Example 3. Let $(P, \preceq)$ be the following poset



The groups $H(D)$ are of three kinds: $(1,1),(n,n) \notin D$, then we get $H(D) \xrightarrow{\sim} C_1$. If one of $(1,1)$ or (n,n) is in D, then $H(D) \xrightarrow{\sim} (C_2)^{|D|-1}$, being isomorphic the set of $|D| \times |D|$ matrices over $\mathbb{F}_2$, with "1" on the diagonal and nonzero entries in the first row or the last column. Finally, for $D = \Delta$, we get a bouquet of $|D| - 2$ copies of the dihedral group D_4, identified at their centers. The representation are easily described: All characters which take a positive value on the central element C are linear, giving all the representations of $(C_2)^{2(n-2)}$.

The J-class of the idempotent $S(D)$, $D = \{(1,1),(n,n)\}$, generates, using the Rees theorem [R], a second copy of $M_{4^{n-2}}(\mathbb{F}_2)$, in that part of the gap algebra R_n, corresponding to the idempotent $\frac{1}{2}(I - C)$. In the portion of the gap algebra corresponding to $\frac{1}{2}(I + C)$, there are a number of pairs of idempotents connected by a single arrow, corresponding to nonzero vectors $v = (v_2, \ldots, v_{n-1}) \in \mathbb{F}_2^{n-2}$.

The linear characters correspond to a dual basis of $(C_2)^{2(n-2)}$. The value of a character $(x', y') \in (C_2)^{n-2} \times (C_2)^{n-2}$ on a matrix with $x = (a_{12}, \ldots, a_{1n-1})$ and $y = (a_{2n}, \ldots, a_{n-1,n})$ is $(-1)^{x \cdot x' + y \cdot y'}$. The pairs of characters are those for which $x' = v$ and $y = 0$ or $x = 0$ and $y' = v$. The arrow corresponds to a matrix with $x = y = v$. Example 3 above is the case $n = 3$.

3. Simplicial cohomology is a special case of Hochschild cohomology

We now turn to deformation theory. There are actually two algebras under consideration, the poset algebra T_P and the representation algebra $A = kM_p/kZ$. We defined T_P over the field $\bar{\mathbb{F}}_2$, but for purposes of deformation theory we may consider the poset algebra over a more general commutative, unital coefficient ring $\mathcal{O}$, which we will denote by $A_{\mathcal{O}}(P)$. The proof that $A_{\mathcal{O}}(P)$ is a unitary algebra embeddible in the upper triangular matrices is exactly as for T_P.

Let R be an integral domain with a maximal ideal m such that R/M is a field k . A *deformation* of an F-algebra A over (R, M) is a flat R-algebra B together with an isomorphism $A \xrightarrow{\sim} B \otimes_R R/M$. If R is a k-algebra of finite type, this is called a k-algebra deformation. If $R \xrightarrow{\sim} k[t]/(t)^2$, it is called a *first order* deformation. If $R \xrightarrow{\sim} k[[t]]$ and $M = (t)$, it is a *power series* deformation.

We will not go into the general theory of algebra deformations here. This material can be found in many places [G], [Sc1], [Sc2]. The point which interests us is that the first order deformations are parametrized by the second Hochschild cohomology group, and we will now show in detail how this can be constructed for the poset algebras $A_{\mathcal{O}}(P)$, via simplicial cohomology.

Simplicial and Hochschild cohomology, both of which we will describe briefly below, initially look as though they belong to entirely different realms. Simplical cohomology is defined for "nice" spaces. Here we will consider only finite polyhedra; for these the groups are finitely generated and vanish past the dimension of the space. Hochschild cohomology is defined for algebras. These may have an arbitrary commutative unital ring as coefficient ring, but even when the algebra is finite dimensional over a field, there may be an infinite sequence of non-trivial cohomology groups. What we will see, however, is that from any finite polyhedron Σ and coefficient ring k we can build an associative algebra A over k with the property that there is a canonical isomorphism from the Hochschild cohomology $H^*(A, A)$ of A with coefficients in itself to the simplicial cohomology $H^*(\Sigma, k)$ of Σ with coefficients in k. In fact, when we use the right cochain groups, those of the algebra become identical with those of the polyhedron. Much of this paper is drawn from [GS1]. The restriction here to finite polyhedra will, we hope, make the basic ideas clearer.

Simplicial cohomology has sometimes been described as an "algebraic snapshot" of a space. If two spaces S and T are homeomorphic, i.e., if there is a bijection $f : S \to T$

which is both continuous and has a continuous inverse, then the cohomology groups of S and T (if they are defined) will be identical. On the other hand, there is much that the cohomology groups miss; spaces with identical groups may be very different. Likewise, isomorphic algebras have identical cohomology groups, while the converse fails. The cohomology groups actually have richer structures, in particular, they are rings, but even this additional information may fail to distinguish between non-homeomorphic spaces or non-isomorphic algebras. Nevertheless they are very powerful tools for the understanding of spaces and algebras, respectively.

3.1. **Simplicial cohomology.** A geometric n-dimensional simplex is something which looks like the span of $n+1$ independent points in Euclidean n space. (To fix the ideas we may take the points to be the end points of the standard unit vectors in $\mathbb{R}^{n+1}$.) In particular, a 0-simplex is a point, a 1-simplex is a (closed) line segment, a 2-simplex is a (filled-in)triangle, etc.. A topological n-simplex is a space homeomorphic to a geometric one. An n simplex has $n+1$ faces of dimension n and these in turn have faces of lower dimension until one gets down to the vertices. In the following, "faces" will mean all of these. A polyhedron is a topological space which is a finite union of topological simplices where the intersection of any two of these simplices is either empty or a (full) common face of both. We impose the finiteness condition here for simplicity since otherwise we would have to make some restrictions on the topology. A space which is homeomorphic to a polyhedron is called triangulated once the homeomorphism is fixed; intuitively it has been divided into a finite number of geometric simplices. For example, a circle can be triangulated by marking three points, which divides it into the union of three 1-simplices. (Dividing it into only two won't do since the intersection of the two simplices then consists of two points and this is not a single common face.) A polyhedron can be triangulated in many ways. Sometimes a triangulated polyhedron is called a simplicial complex but that term is better reserved for a more abstract algebraic situation. From here on we will simply use polyhedron to mean a triangulated one. In addition we will assume that the vertices have been ordered so that for each simplex, we can say which is the first, which is the second, and so on. The actual choice of ordering will not be important. However, since an n-simplex has $n+1$ vertices, we will refer to them in order as vertices $0, 1, \dots, n$.

Now suppose that we have a polyhedron Σ. Choose a commutative unital coefficient ring k. An n-chain of Σ with coefficients in k is just a formal linear combination of n simplices of Σ with coefficients in k, i.e., an expression of the form $c_1\sigma_1+c_2\sigma_2+\cdots+c_r\sigma_r$ where the c_i are in k and the σ_i are n simplices. (Extensions of what we do here allow spaces which are "triangulated" into infinitely many simplices but chains are still just finite linear combinations of them.) The set of these linear combinations of n-simplices is a free module over k of rank equal to the number of n-simplices, but it is traditional to call it the "group" of n-chains of Σ with coefficients in k, denoted $C_n(\Sigma, k)$. Since the vertices of each of our n-simplices σ are ordered, we also have an order on its $n-1$ dimensional faces: the ith face, which we denote by $\partial_i\sigma$, is the one obtained by omitting the ith vertex. Here i runs from 0 to n. The boundary of σ, denoted $\partial\sigma$ is then defined by setting $\partial\sigma = \partial_0\sigma - \partial_1\sigma + \partial_2\sigma - \cdots + (-1)^n\partial_n\sigma$. This is an element of the group $C_{n-1}(\Sigma, k)$. (The boundary of a 0-simplex, i.e., of a vertex, is zero.) The definition of the boundary can be extended linearly to all of $C_n(\Sigma, k)$ by setting the boundary of a linear combination of simplices equal to the same linear combination of their boundaries, so we have a boundary map $\partial : C_n(\Sigma, k) \to C_{n-1}(\Sigma, k)$. Here ∂ should also have a subscript n, but for simplicity we may omit it. For simplicity we may also denote $C_n(\Sigma, k)$ by C_n. We then have a sequence of maps

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0.$$

At the left C_n is zero when n is greater than the dimension of the polyhedron, which by definition is the largest dimension of any simplex which it contains. This is a complex, i.e., the composite $\partial_{n-1}\partial_n$ of any two successive maps is zero. Denote the kernel, $\ker \partial_n$ by Z_n and call these the n-cycles, and the image, $\operatorname{im} \partial_{n+1} = \partial_{n+1} C_{n+1}$ by B_n and call these the n-boundaries. Then $B_n \subset Z_n$ and we can form the quotient $Z_n/B_n = H_n(\Sigma, k)$, called the nth homology group of Σ with coefficients in k (although it is actually a k-module). It is a remarkable fact (not easy to prove) that these homology groups depend only on the underlying topology of Σ and not on how it is triangulated (or the vertices numbered). This basic theorem depends on the fact that there are other ways to define homology groups for Σ. In particular, there are the singular homology groups which by their nature depend only on the topology, but are almost impossible to compute directly from the definition, while the above "simplicial" groups are relatively easy to compute. For a polyhedron, one can show that the various definitions of homology lead to the same result. A similar result will hold for the homology of algebras.

While the homology of a polyhedron Σ is a basic topological invariant of the space (i.e., it is the same for homeomorphic spaces), it is the cohomology, likewise an invariant, which mainly concerns us here. We define an n-cochain of Σ to be a map f from the set of all n-simplices of Σ into k. These can be added by adding their values, i.e., $(f + g)(\sigma) = f(\sigma) + g(\sigma)$ and can be multiplied by elements of $\mathcal{O}$, so they form a k-module, denoted $C^n(\Sigma, k)$ but again called abusively the "group" of n-cochains. An n-cochain f thus just assigns to every n-simplex σ an element $f(\sigma)$ in $\mathcal{O}$. This may not seem very different from an n-chain (which attaches a coefficient in $\mathcal{O}$ to every n-simplex of Σ) but that is an accident of the fact that we have limited ourselves to finite polyhedra. In the general case (think, e.g., of the entire plane triangulated into infinitely many triangles), an n-chain is still a formal *finite* linear combination of simplices, so the coefficient assigned to "almost all", i.e., all but a finite number of the simplices is zero. However, an n-cochain f may have a non-zero value on infinitely many simplices.

3.2. **Hochschild cohomology.** We turn now to algebras. While for spaces homology was defined first and cohomology followed, with algebras it was the reverse. Let A be an algebra over some commutative, unital coefficient ring $\mathcal{O}$. For simplicity we will generally assume that A has a unit element. Consider bimodules M over A. The most important A bimodule is A itself. The brilliant insight of Gerhard Hochschild (while in the U.S. army during World War II) was to observe that with these ingredients one could build a complex in the following way. Let $C^n(A, M)$ be the set of all $\mathcal{O}$-multilinear maps $F : A \times A \times \cdots \times A$ (n times) $\to M$, i.e., of maps which are $\mathcal{O}$-linear as a function of each individual argument. (Of course, this is the same as the set of all k-linear maps $A \otimes A \otimes \cdots \otimes A$ (n times) $\to M$.) The case $n = 0$ is allowed; $C^0(A, M)$ is understood to be just M itself. (A function of no variables with values in M is just an element of M.) The Hochschild coboundary $\delta : C^n \to C^{n+1}$ is then defined by

$$\begin{aligned}(\delta F)(a_1, a_2, \dots, a_{n+1}) = a_1 F(a_2, \dots, a_{n+1})& \\ + \sum_{i=1}^{n} (-1)^i F(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots a_{n+1})& \\ + (-1)^{n+1} F(a_1, \dots, a_n) a_{n+1}.&\end{aligned}$$

Notice in this formula that the first and last terms are well-defined because the bimodule M, in which F has its values, allows multiplication by elements of A from both left and right, while each intermediate term has only n arguments because in each two successive arguments are multiplied. Here are some examples in low dimensions. First, $\delta : C^0 \to C^1$ is defined by setting $(\delta m)(a) = am - ma$ for $a \in A, m \in M$. This makes sense because an element of C^0 is just an element of M and its coboundary is now a mapping of A into M. If $F \in C^n$ we may write F^n to indicate its dimension. Then

$$(\delta F^1)(a_1, a_2) = a_1 F^1(a_2) - F^1(a_1 a_2) + F^1(a_1) a_2$$
$$(\delta F^2)(a_1, a_2, a_3) = a_1 F^2(a_2, a_3) - F^2(a_1 a_2, a_3) + F^2(a_1, a_2 a_3) - F^2(a_1, a_2) a_3.$$

Here it is also an easy exercise to show that $\delta\delta = 0$, so we have the Hochschild cochain complex

$$C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^n \xrightarrow{\delta} C^{n+1} \xrightarrow{\delta} \cdots .$$

Just as before we define the n-cocycles $Z^n(A, M)$ by $Z^n = \ker \delta^n$, the n-coboundaries by $B^n = \delta^{n-1} C^{n-1}$ and the nth Hochschild cohomology group of the algebra A with coefficients in the bimodule M by $H^n(A, M) = Z^n / B^n$. (For any complex one similarly defines "homology" groups or "cohomology" groups according as the indices are descending or ascending.) The Hochschild "groups" are again actually $\mathcal{O}$-modules. We can now state more precisely the theorem which is the title of this section.

Theorem 3.1. *Let Σ be a polyhedron and k be an arbitrary commutative unital coefficient ring. Then there is a $\mathcal{O}$-algebra A which is free and of finite rank over $\mathcal{O}$ such that $H^n(A, A)$ is naturally isomorphic to $H^n(\Sigma, \mathcal{O})$ for all n.*

3.3. **Separable algebras.** Unlike the case for finite polyhedra where the homology and cohomology groups must vanish once one gets to a dimension greater than that of the polyhedron itself, the Hochschild groups need never vanish even when A and M are finite dimensional over a field. But like the topological case, they do vanish in positive dimensions for what are in some sense the simplest objects. In the topological case, the simplest object was a (solid) simplex. We could also take a disjoint union of a finite number of simplices of varying dimensions. The homology and cohomology groups of these vanish in all positive dimensions. For algebras over a ring $\mathcal{O}$ the "simplest" objects are a little more difficult to describe. They are the "separable" algebras S, the most basic example of which is the algebra of all $n \times n$ matrices $\mathcal{M}_n(\mathcal{O})$ with coefficients in k. For $n = 1$ this is just $\mathcal{O}$ itself. Any finite direct sum of separable $\mathcal{O}$-algebras is again separable. In particular, we can take $S = \mathcal{O} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}$ (any finite number of times). For all separable algebras S we have $H^n(S, M) = 0$ for all $n > 0$ and all S-bimodules M. The last example is the only case we shall need, but here is the full definition of separability for a $\mathcal{O}$-algebra S: Suppose we have a morphism $f : M \to N$ of left S-modules (these are, of course, therefore also $\mathcal{O}$-modules) and suppose that f "splits" as a morphism of $\mathcal{O}$-modules, i.e., that there is a $\mathcal{O}$-module morphism $g : N \to M$ such that $fgf = f$. (This will always be the case if $\mathcal{O}$ is a field.) Then it also splits as a morphism of S-modules, i.e., we can actually find such a g which is an S-module morphism. From the definition it is obvious that $\mathcal{O}$ itself has this property and it is not too difficult to show that $\mathcal{M}_n(\mathcal{O})$ also has it. (There are several other equivalent definitions of separability for an algebra, the most useful probably being that it has a "separability idempotent". For a discussion of these cf., e.g., [DI].) There are, as with topological spaces, different ways to define the cohomology of an algebra, all giving the same result, and for some of these the fact that the cohomology of a separable algebra is trivial is easy to see. When $\mathcal{O}$ is a field and the algebra A just a finite field extension then A is separable over $\mathcal{O}$ in the present sense if and only if it is separable as a field extension. The extension of the concept of separability to algebras over arbitrary coefficient rings is due to M. Auslander and O. Goldman [AG], building on groundwork laid by G. Azumaya [A]).

Hochschild cohomology in general is not easy to compute, but there are techniques which sometimes simplify it. First, call an n-cochain F in $C^n(A, M)$ normalized if it vanishes whenever any of its arguments is the unit element of $\mathcal{O}$. Denote the set of these by $\overline{C}^n(A, M)$. It is easy to check that the Hochschild coboundary $\delta C^n(A, M) \to C^{n+1}(A, M)$ in fact carries $\overline{C}^n(A, M)$ into $\overline{C}^{n+1}(A, M)$. The normalized cochains thus form a subcomplex of the full Hochschild complex. Now it is also easy to check that the inclusion of one complex into another induces a mapping from the homology or cohomology groups of the first into those of the second, but even though the map of complexes is one-to-one, that of the homology groups need not be. For example, the inclusion of the boundary, $\partial\sigma$, of an n-simplex σ into the solid simplex induces an inclusion of the complex $C^\bullet(\partial\sigma, \mathcal{O})$ of chain groups of the boundary into that of the solid simplex $C^\bullet(\sigma, \mathcal{O})$ but in the former there is a non-trivial homology class in dimension $n - 1$ which obviously becomes a boundary in the latter. (Here we have adopted the usual notational convention of denoting an entire complex simply by $C^\bullet$.) Nevertheless it is a theorem (not difficult) that the inclusion of $\overline{C}^\bullet(A, M)$ into $C^\bullet(A, M)$ induces an isomorphism of cohomology groups. We can therefore compute Hochschild cohomology using only normalized cochains. This makes it easy, for example, to compute the cohomology of the group ring kG when G is the group of two elements. (The most important module in this case is $\mathcal{O}$ itself, on which G acts trivially. As an exercise, compute the result when $\mathcal{O}$ is a field, first of characteristic not 2 and then of characteristic 2.)

Here is a deeper result which we will need. Suppose that S is a $\mathcal{O}$-subalgebra of A, arbitrary except that we will always assume that the unit element of A is contained in S. An S-relative cochain $F \in C^n(A, M)$ is one such that for all $s \in S$ we have

$$F(\dots, a_i s, a_{i+1}, \dots) = F(\dots, a_i, s a_{i+1}, \dots) \tag{1}$$

$$F(sa_1, \dots, a_n) = sF(a_1, \dots, a_n), \tag{2a}$$

$$F(a_1, \dots, a_n s) = F(a_1, \dots, a_n)s \tag{2b}$$

If F is normalized then it must vanish whenever any argument is in S. (Write s as $s \cdot 1$ or $1 \cdot s$ and use the above rules.) It is easy to check that the relative groups, denoted $C^n(A, S; M)$, also form a subcomplex of the Hochschild complex. The result we need is that when S is a separable algebra over k the inclusion of the complex of S-relative cochains into the full Hochschild complex induces an isomorphism of cohomology. Finally, the normalized relative cochain groups, denoted $\overline{C}^n(A, S; M)$ form a subcomplex of the relative groups and their inclusion into the full Hochschild cochain complex again induces an isomorphism of cohomology. (The proofs of these statements are relatively simple if one starts with the description of cohomology using projective resolutions.) It is this last subcomplex of normalized S-relative cochains which will be essential in the next section.

3.4. **Posets.** We now have the basic homological machinery we need and can start building the bridge between simplicial and algebraic cohomology. Suppose that P is a finite partially ordered set or poset. From P we can build both a polyhedron $\Sigma = \Sigma(P)$ and the $\mathcal{O}$-algebra $A_{\mathcal{O}} = A(P)$, which was described at the beginning of the section. The vertices of the polyhedron are just the elements $i, j, \dots$ of P. In this context, a (simplicial) n-simplex will be a linearly ordered sequence $\sigma = (i_0 \prec i_1 \prec \cdots \prec i_n)$. It should be intuitively clear how to glue together corresponding geometric simplices to form a polyhedron but it can also be done as follows. Map the elements of I to linearly independent points inside a Euclidean space of sufficiently high dimension. The convex hull of any set of points whose preimages were the vertices of a simplicial simplex is certainly a geometric simplex. The union of all of these is called the geometric realization of I; it is a polyhedron which automatically comes with a triangulation. We do not actually need it for calculation since abstractly the boundary of $\sigma = (i_0, \dots, i_n)$ is given by $\partial\sigma = \sum_{r=0}^{n}(i_0, \dots, \hat{\imath}_r, \dots, i_n)$, where $\hat{\imath}_r$ indicates that i_r is omitted.

Here are some simple examples of both constructions. If P is actually linearly ordered then Σ is just the (solid) N simplex and A is the algebra of all upper triangular matrices. We describe next how to construct a poset I whose geometric realization is a sphere of any dimension. For the N-sphere the elements of the poset will be the integers $\{1, 2, \dots, 2N+1, 2N+2\}$, with a partial ordering consistent with the natural order. A 0-sphere is just a pair of unconnected points which we may take to be the points $+1$ and -1 on the real line, so for this we can simply take $P = \{1, 2\}$ with no order relation between 1 and 2. Now add 3 and 4 with no order relation between them but with $\{1, 2 \prec 3, 4\}$ (meaning that 1 and 2 both precede 3 and both precede 4. Now Σ has four 1-simplices, namely $(1,3), (1,4), (2,3), (2,4)$ but no simplices of any higher dimension. These are joined in what is topologically a circle; it is as if we had taken two points in the plane, one at $+1$ on the y-axis and one at -1 and joined them by line segments to the two unconnected points we already had on the x-axis.

Now in 3-space take the points at $+1$ and at -1 on the z-axis and join them to all points on the (topological) circle we have already constructed. The result will be the surface of an octahedron; topologically this is a triangulated 2-sphere which we may therefore view as the geometric realization of the poset $\{1, 2 \prec 3, 4 \prec 5, 6\}$. Technically, we have taken the "two-point suspension" of the circle. Suspending again, the poset $\{1, 2 \prec 3, 4 \prec 5, 6 \prec 7, 8\}$ will give a three sphere, and so on. Here are the corresponding algebras of upper triangular matrices for $N = 0, 1, 2$, where a "$*$" indicates that the entry may be any element of k:

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

If P is any (finite) poset then the algebra $A_{\mathcal{O}}(P)$ (which, by our convention, always consists of upper triangular matrices) contains the algebra S of all diagonal matrices. As noted before, this subalgebra is separable over the coefficient ring k since it is a direct sum of N copies of $\mathcal{O}$, where N is the cardinality of P. We then have the following basic result.

Theorem 3.2. *Let P be a finite poset, $\mathcal{O}$ be a commutative unital coefficient ring, $A_{\mathcal{O}}(P)$ be the poset algebra with coefficients in $\mathcal{O}$, S be the separable subalgebra of diagonal matrices, and $\Sigma(P)$ be the simplicial complex built from P. Then there is a natural cochain isomorphism of complexes $\overline{C}^{\bullet}(A, S; A) \to C^{\bullet}(\Sigma, \mathcal{O})$. That is, for every $n \geq 0$ there is a natural isomorphism $f^n : \overline{C}^n(A, S; A) \to C^n(\Sigma, \mathcal{O})$ such that $\delta f^n = f^n \delta$ where on the left δ is the simplicial coboundary and on the right the Hochschild coboundary.*

Proof. A cochain $F \in \overline{C}^n(A, S; A)$ is completely determined by its values when the arguments are amongst the generators $E_{ij}, i \preceq j$, so we must examine the possible values of $F(E_{i_1 j_1}, E_{i_2 j_2}, \dots, E_{i_n j_n})$. Now $E_{i_r j_r} = E_{i_r j_r} E_{j_r j_r}$, so

$$\begin{aligned} F(\dots, E_{i_r j_r}, E_{i_{r+1} j_{r+1}}, \dots) &= F(\dots, E_{i_r j_r} E_{j_r j_r}, E_{i_{r+1} j_{r+1}}, \dots) \\ &= F(\dots, E_{i_r j_r}, E_{j_r j_r} E_{i_{r+1} j_{r+1}}, \dots) \end{aligned}$$

because every E_{jj} is in the separable subalgebra S. But $E_{j_r j_r} E_{i_{r+1} j_{r+1}} = 0$ unless $j_r = i_{r+1}$. Therefore, changing the numbering slightly, the only non-zero values of F are of the form $F(E_{i_0, i_1}, E_{i_1, i_2}, \dots, E_{i_{n-1}, i_n})$. Moreover, writing $E_{i_0, i_1} = E_{i_0, i_0} E_{i_0, i_1}$, $E_{i_{n-1}, i_n} = E_{i_{n-1}, i_n} E_{i_n, i_n}$, and using properties (2a) and (2b) of S-relative cochains, we see that $F(E_{i_0, i_1}, E_{i_1, i_2}, \dots, E_{i_{n-1}, i_n})$ lies in $E_{i_0, i_0} A E_{i_n, i_n}$. But this, as a $\mathcal{O}$-module, is just isomorphic to $\mathcal{O}$ itself. So F simply assigns an element of $\mathcal{O}$ to every linearly ordered "chain" $i_0 \prec i_1 \prec \cdots \prec i_n$ of elements of P. Moreover, the value will be zero if any $i_r = i_{r+1}$ for then the argument $E_{i_r, i_{r+1}}$ will be in the separable subalgebra S. So F assigns an element of $\mathcal{O}$ to every non-degenerate simplex $(i_0 \prec i_1 \cdots \prec i_n)$. That is, it is just a simplicial n-cochain. Thus $\overline{C}^n(A, S; A)$ is naturally identified with

$C^n(\Sigma, k)$. It is now an easy exercise to show that with this identification the Hochschild coboundary becomes the simplicial coboundary. □

Theorem 3.3. *Let P be a finite poset and k be a any commutative unital coefficient ring. Set $A = A_{\mathcal{O}}(P), \Sigma = \Sigma(P)$. Then for all n there is a isomorphism $H^n(A, A) \cong H^n(\Sigma, \mathcal{O})$.*

3.5. **Barycentric subdivision.** If a polyhedron Σ happens to be (the geometric realization of one) of the form $\Sigma(P)$ then the preceding is enough to prove Theorem 3.2, but that is not always the case. However, every polyhedron Σ gives rise in a natural way to a partially ordered set whose objects are just all the faces (of all dimensions) of Σ. The partial order is given by the face relation. That is, if σ, τ are simplices of Σ then $\sigma \prec \tau$ if σ is a face (of any dimension) of τ. We will denote this poset by $P(\Sigma)$. The geometric realization of this poset is the "barycentric subdivision" of Σ, denoted Σ'. It is more commonly pictured in the following way. If σ is a standard n-simplex then the vertices of its barycentric subdivision σ' are the barycenters, i.e., the centers of gravity, of σ and of all of its faces. Denoting the vertices of σ just by $\{0, 1, 2, \ldots, n\}$, the r-simplices of σ' are in 1-1 correspondence with the linearly ordered subsets $\{i_0 < i_1 < \cdots < i_r\}$. We may also view this as a chain of faces of Σ, namely $\{i_0\} \subset \{i_0, i_1\} \subset \{i_0, i_1, i_2\} \cdots \subset \{i_0, i_1, \ldots, i_r\}$, which is just an r-simplex of $\Sigma(P(\Sigma))$. Geometrically, every r-simplex of Σ has been broken into $(r + 1)!$ smaller simplices, but this does not change the topology of the geometric realization. In effect, we had a polyhedron which now has been triangulated by smaller simplices. Since the topology has not changed, neither has the cohomology. (There is, in fact, a simple purely algebraic way to show this using the concept of a chain homotopy between complexes which one can find in almost any standard text on algebraic topology; cf., e.g., [GH].) Since we now have a topological space homeomorphic to our original polyhedron and which is the geometric realization of $\Sigma(P)$ for some poset P, the proof of the main theorem is at an end.

3.6. **A peek at the Cohomology Comparison Theorem.** Theorem 3.2 is like the tip of an iceberg in that it is a very special case of a much deeper result (which, curiously, was discovered first). We have presented simplicial cohomology and Hochschild cohomology as though they came from entirely separate areas of mathematics. Originally they did but there is a generalization which combines the two. We will have to use some sophisticated concepts, but there are simple special cases. If we have a contravariant functor f from a small category $\mathcal{C}$, i.e., one whose objects form a set, to the category of unital associative algebras, then we can define cohomology groups which share features of both. (Such a contravariant functor is sometimes called a presheaf of algebras or a diagram of algebras.) When $\mathcal{C}$ is the trivial category, i.e., it consists of just a single object with the identity morphism, then all we have is a single algebra and the cohomology groups are just the Hochschild groups. At the other extreme, suppose that we have fixed a commutative, unital coefficient ring $\mathcal{O}$, that $\mathcal{C}$ is an arbitrary small category, and that the functor f is trivial in the sense that to every object of $\mathcal{C}$ it assigns this same $\mathcal{O}$ and to every morphism in $\mathcal{C}$ it assigns the identity map from $\mathcal{O}$ to itself. Then the associated cohomology is just the simplicial cohomology of the geometric realization of $\mathcal{C}$, with coefficients in $\mathcal{O}$. Presheaves of algebras occur very commonly. In fact, a unital morphism from one algebra to another is already an example, so to any such morphism we can assign, in a natural way, cohomology groups. The Cohomology Comparison Theorem says that for every such functor f one can construct in a natural way a single ring, called the diagram ring, whose Hochschild cohomology is that of the hybrid simplicial-Hochschild cohomology of the functor f. For some purposes this greatly simplifies the study of the hybrid cohomology since it shows, in particular, that the hybrid cohomology has the rich structure of the cohomology of a single ring. Every poset may be viewed as a category in which whenever $i \prec j$ there is a unique morphism from i to j; Theorem 3.2 is just the special case of the Cohomology Comparison Theorem when the poset is finite and the functor to algebras is the trivial one just described. Another important case is that where we have a group of automorphisms of an algebra. A group can be viewed as a category with but a single object and in which every morphism from that object to itself is an isomorphism. The hybrid cohomology is then equivariant cohomology and the Cohomology Comparison Theorem asserts that it, too, is just the Hochschild cohomology of a single ring. For an exposition of the Cohomology Comparison Theorem (in the context of algebraic deformation theory), see [GS2].

3.7. **Functoriality.** Those already familiar with algebraic topology and cohomology may have noticed one somewhat unsettling aspect of Theorem 3.2. A simplicial map $f : \Sigma_1 \to \Sigma_2$ between two polyhedra (i.e., which on each individual simplex is an affine transformation) induces a homomorphism (in the reverse direction) of cohomology groups $f^* : H^*(\Sigma_2) \to H^*(\Sigma_1)$, but the cohomology $H^*(A, A)$ of an algebra with coefficients in itself has no such functoriality. The problem here is that we should really have been taking as coefficient module not $A = A_{\mathcal{O}}(P)$ itself, but its dual, $A^* = \hom_k(A, k)$. This would give the correct functoriality, since $H^*(A, M)$ is contravariant as a functor of A and covariant as a a functor of M. Moreover, the dual of an algebra of the form $A_{\mathcal{O}}(P)$ is again an algebra; it consists of the transposes of all the matrices in $A_{\mathcal{O}}(P)$. We didn't do this in order to preserve the simplicity of Theorem 3.2, but it is a good exercise to verify that in replacing $A_{\mathcal{O}}(P)$ here by its dual nothing untoward happens.

4. Deformation of poset algebras over finite fields

We consider the deformations of poset algebras over a finite field F, where we now let $T_p(F)$ be the linear span of the matrix units $\langle E_{ij} \mid i < j \rangle$ over the field F, and let $M_p(F)$ be the corresponding monoid. If q is the order of the field, and m is the dimension of $T_p(F)$, with $n \le m \le \frac{n(n+1)}{2}$, then we naturally have q^m elements in $T_p(F)$, all of which can be represented by upper triangular matrices. The same set of diagonal idempotents E gives representatives of the regular J-classes, but now the local subgroups for an idempotent with k nonzero entries have a considerably more complicated structure, being a semidirect product of a "torus" isomorphic to C_{q-1}^k with a normal "unipotent" group of order q^r, with $0 \le r \le \frac{k(k-1)}{2}$. If we consider the representation algebra of $T_p(F)$ over a field k of characteristic prime to q and $q-1$, then the irreducible representations of these local groups would correspond to the simples.

For consideration of deformation theory, we look at deformations of the algebra $T_p(F)$, not its representation algebra. For $T_p(F)$, we have the Hochschild cohomology as defined in the previous chapter, taking $\mathcal{O}$ to be the field F.

As mentioned above, the deformations of the algebra depend on the second cohomology group in the Hochschild cohomology. Therefore, in order to have any hope of a deformation, we would have to consider posets with a non-zero 2-cocycle in the simplicial complex. This first arises as a two-point suspension of the circle, for which the minimal poset is

$$\begin{tikzcd} \cdot \arrow[d] \arrow[dr] & \cdot \arrow[d] \arrow[dl] \\ \cdot & \cdot \end{tikzcd}$$

Thus the minimal poset for which we could hope to have a nontrivial deformation is

$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] & \\ \cdot \arrow[d] \arrow[drr] & & \cdot \arrow[d] \arrow[dll] \\ \cdot \arrow[dr] & & \cdot \arrow[dl] \\ & \cdot & \end{tikzcd}$$

The corresponding poset algebra has six elements. There are four independent paths from top to bottom, and the deformation parameter is a form of cross ratio. (Unfortunately, if we take the poset algebra over $\mathbb{F}_2$, there is no nontrivial deformation because there is only one possible value of the parameter.) We can get a configuration of spheres connected by lines, by joining configurations with three point lines as in

$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] & & & & \cdot \arrow[dl] \arrow[dr] & \\ \cdot \arrow[d] \arrow[drr] & & \cdot \arrow[d] \arrow[dll] & \cdot \arrow[dl] \arrow[dr] & \cdot \arrow[d] \arrow[drr] & & \cdot \arrow[d] \arrow[dll] \\ \cdot \arrow[dr] & & \cdot \arrow[dl] & & \cdot \arrow[dr] & & \cdot \arrow[dl] \\ & \cdot & & & & \cdot & \end{tikzcd}$$

Again, however, the sparseness of the ground field $\mathbb{F}_2$ would preclude nontrivial deformation. Over larger ground fields, one does get nontrivial deformations, which should lead to nontrivial deformations of the representation algebra, but in that case the representation algebra will be so much more complicated that it will be difficult to make the same tight analysis of its radical

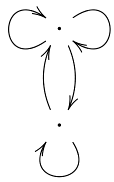
5. Degenerations of representation algebras

Another point of view is to consider not the deformations of the poset algebras or their representation algebras, but rather the degenerations.

We will replace the field k of characteristic not equal to 2 by a complete discrete valuation ring $\mathcal{O}$ with quotient field of characteristic zero and residue field of characteristic 2. We consider the representation algebra $A_{\mathcal{O}}$ of T_P. Over the quotient field K, we have the algebra determined as in §2. However, over the closed point, the basis graph degenerates to an algebra A_0. The primitive idempotents in A_0 are precisely the idempotents $\hat{E}(D)$ and the algebra is basic. The unweighted basis graph is obtained from the characteristic zero basis graph by coalescing all the idempotents from each J-class and adding extra loops to make up the lost dimensions. Because of all the loops, the quiver surely has oriented cycles. There are not only loops but also pairs of what were once matrix units and have become radical arrows in A_0.

Which of these arrows are in the quiver? The number of quiver arrows in $k(H(D)$ for k of characteristic 2 is the number of generators of the 2-group.

The following is the diagram of the quiver in Example 3 of $e_0A_0e_0$ with $\hat{E}(D_4) + \hat{E}(D_5) = e_0$.



Another difference from the case of characteristic not equal to 2 is that some of the compositions may be zero. Thus, for example, in Example 1, we have $W(B_1) \cdot W(B_2) = -2W(B_3)$ but in a field of characteristic 2, this composition is zero, so that $W(B_3)$ is no longer in the radical squared but only in the radical.

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Department of Mathematics, University of Pennsylvania, Philadelphia, PA, 19104-6395, USA

E-mail address: mgersten@upenn.edu

Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel

E-mail address: mschaps@macs.biu.ac.il