International Journal of Algebra and Computation
Vol. 17, No. 8 (2007) 1593–1610

THE QUIVER OF THE SEMIGROUP ALGEBRA OF A LEFT REGULAR BAND

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Received 3 August 2006
Revised 2 May 2007

Communicated by S. Margolis

Recently it has been noticed that many interesting combinatorial objects belong to a class of semigroups called *left regular bands*, and that random walks on these semigroups encode several well-known random walks. For example, the set of faces of a hyperplane arrangement is endowed with a left regular band structure. This paper studies the module structure of the semigroup algebra of an arbitrary left regular band, extending results for the semigroup algebra of the faces of a hyperplane arrangement. In particular, a description of the quiver of the semigroup algebra is given and the Cartan invariants are computed. These are used to compute the quiver of the face semigroup algebra of a hyperplane arrangement and to show that the semigroup algebra of the free left regular band is isomorphic to the path algebra of its quiver.

Keywords: Representation theory; quivers; semigroup algebras; semigroups; left regular bands; hyperplane arrangements.

MSC: 05E99, 16S99 (Primary), 52C35 (Secondary)

1. Introduction

A *left regular band* is a semigroup S satisfying $x^2 = x$ and $xyx = xy$ for all $x, y \in S$. Recent interest in left regular bands and their semigroup algebras arose due to the work of Brown [3], in which the representation theory of the semigroup algebra is used to study random walks on the semigroup. There are several interesting examples of such random walks, including the random walk on the chambers of a hyperplane arrangement. Several detailed examples are included in [3].

The starting point of this paper is the fact that the irreducible representations of the semigroup algebra of a left regular band are all one-dimensional. This implies that there is a canonical quiver (a directed graph) associated to the left regular band, and that the semigroup algebra is a quotient of the path algebra of the quiver. This paper determines a combinatorial description of this quiver and the Cartan invariants of the semigroup algebras and illustrates the theory through detailed examples.

The paper is structured as follows. Section 2 recalls the definition and collects some properties of left regular bands, and introduces the examples that will be used throughout the paper. Section 3 describes the irreducible representations of the semigroup algebra of a left regular band. In Sec. 4, a complete system of primitive orthogonal idempotents for the semigroup algebra is explicitly constructed. Section 5 describes the projective indecomposable modules of the semigroup algebra. Sections 6 through 9 deal with computing the quiver of the semigroup algebra. Sections 10 through 13 compute the Cartan invariants of the semigroup algebras. Finally, Sec. 14 discusses future directions for this project.

2. Left Regular Bands

See [3, Appendix B] for foundations of left regular bands and for proofs of the statements presented in this section.

A *left regular band* is a semigroup S satisfying the following two properties.

(LRB1) $x^2 = x$ for all $x \in S$.
(LRB2) $xyx = xy$ for all $x, y \in S$.

Define a relation on the elements of S by $y \leq x$ iff $yx = x$. This relation is a partial order (reflexive, transitive, and antisymmetric), so S is a poset.

Define another relation on the elements of S by $y \preceq x$ iff $xy = x$. This relation is reflexive and transitive, but not necessarily antisymmetric. Therefore, we get a poset L by identifying x and y if $x \preceq y$ and $y \preceq x$. Let $\operatorname{supp} : S \to L$ denote the quotient map. L is called the *support semilattice* of S and $\operatorname{supp} : S \to L$ is called the *support map.*

Proposition 2.1. *If S is a left regular band, then there is a semilattice L and a surjection* $\operatorname{supp} : S \to L$ *satisfying the following properties for all $x, y \in S$.*

(1) *If $y \leq x$, then* $\operatorname{supp}(y) \leq \operatorname{supp}(x)$;
(2) $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$;
(3) $xy = x$ *iff* $\operatorname{supp}(y) \leq \operatorname{supp}(x)$;
(4) *if S' is a subsemigroup of S, then the image of S' in L is the support semilattice of S'.*

Statement (1) says that supp is an order-preserving poset map, (2) says that supp is a semigroup map where we view L as a semigroup with product $\vee$, (3) follows from the construction of L, and (4) follows from the fact that (3) characterizes L up to isomorphism. If S has an identity element then L has a minimal element $\hat{0}$. If, in addition, L is finite, then L has a maximal element $\hat{1}$, and is therefore a lattice [6, Proposition 3.3.1]. In this case L is the *support lattice* of S.

Example 2.2 (The Free Left Regular Band). The *free left regular band $F(A)$ with identity* on a finite set A is the set of all (ordered) finite sequences of distinct elements from A with multiplication defined by

$$(a_1, \dots, a_l) \cdot (b_1, \dots, b_m) = (a_1, \dots, a_l, b_1, \dots, b_m)^{✄},$$

where ✄ means "delete any element that has occurred earlier". Equivalently, $F(A)$ is the set of all words on the alphabet A that do not contain any repeated letters.

The empty sequence is an element of $F(A)$, therefore $F(A)$ contains an identity element. The support lattice of $F(A)$ is the lattice L of subsets of A and the support map $\text{supp} : F(A) \to A$ sends a sequence $(a_1, \dots, a_l)$ to the set of elements in the sequence $\{a_1, \dots, a_l\}$. Figure 1 shows the Hasse diagrams of the poset $(F(A), \leq)$ and the support lattice of $F(A)$, where $A = \{a, b, c\}$.

Example 2.3 (Hyperplane Arrangements). A (*central*) *hyperplane arrangement* $\mathcal{A}$ is a finite collection of hyperplanes containing the origin in some real vector space $V = \mathbb{R}^d$, for some $d \in \mathbb{N}$. For each hyperplane $H \in \mathcal{A}$, let H^+ and H^- denote the two open half spaces of V determined by H. The choice of labels H^+ and H^- on the two open half spaces is arbitrary, but fixed throughout. For convenience, let H^0 denote H. A *face* of the arrangement $\mathcal{A}$ is a non-empty intersection of the form $\bigcap_{H \in \mathcal{A}} H^{\epsilon_H}$, where $\epsilon_H \in \{0, +, -\}$. Let $\mathcal{F}$ denote the set of all faces of $\mathcal{A}$. Define a relation on $\mathcal{F}$ by $x \leq y$ iff $x \subseteq \overline{y}$, where $\overline{y}$ denotes the closure of the set y. The relation is a partial order.

If $x = \bigcap_{H \in \mathcal{A}} H^{\epsilon_H}$ is a face, then let $\sigma_H(x) = \epsilon_H$ and let $\sigma(x) = (\sigma_H(x))_{H \in \mathcal{A}}$. The sequence $\sigma(x)$ is called the *sign sequence* of x. Define the product of two faces $x, y \in \mathcal{F}$ to be the face xy with sign sequence

$$\sigma_H(xy) = \begin{cases} \sigma_H(x), & \text{if } \sigma_H(x) \neq 0, \\ \sigma_H(y), & \text{if } \sigma_H(x) = 0. \end{cases}$$

This product has a geometric interpretation: the product xy of two faces x, y is the face entered by moving a small positive distance along a straight line from any point in x to a point in y. It is straightforward to verify that this product gives $\mathcal{F}$ the structure of an associative left regular band. Since all the hyperplanes in the arrangement contain the origin, $\mathcal{F}$ contains an identity element: $\cap_{H \in \mathcal{A}} H$. The left

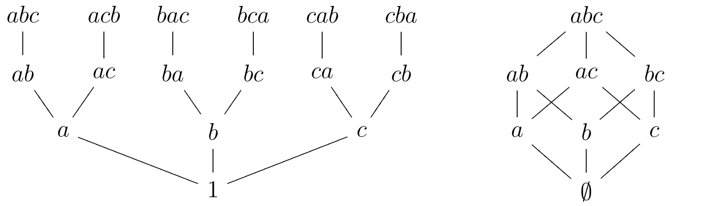

Fig. 1. The poset of the free left regular band $F(\{a, b, c\})$ on three generators and its support lattice.

regular band $\mathcal{F}$ is called the *face semigroup* of $\mathcal{A}$, and the semigroup algebra $k\mathcal{F}$ of $\mathcal{F}$ is called the face semigroup algebra of $\mathcal{A}$.

Let $\mathcal{L}$ denote the set of subspaces of V that can be obtained as the intersection of some hyperplanes in $\mathcal{A}$. Then $\mathcal{L}$ is a finite lattice, called the *intersection lattice* of $\mathcal{A}$, where the subspaces are ordered by inclusion and the meet operation is intersection. (Note that some authors order $\mathcal{L}$ by reverse inclusion rather than inclusion.) $\mathcal{L}$ is the support lattice of $\mathcal{F}$ and the support map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ maps a face $x \in \mathcal{F}$ to the intersection of all the hyperplanes of the arrangement that contain the face: $\operatorname{supp}(x) = \bigcap_{\{H\in\mathcal{A}:x\subseteq H\}} H$.

3. Representations of the Semigroup Algebra

Let k denote a field and S a left regular band. The semigroup algebra of S is denoted by kS and consists of all formal linear combinations $\sum_{s\in S} \lambda_s s$, with $\lambda_s \in k$ and multiplication induced by $\lambda_s s \cdot \lambda_t t = \lambda_s \lambda_t st$, where st is the product of s and t in the semigroup S. The following summarizes [3, Sec. 7.2].

Since S and L are semigroups and $\operatorname{supp} : S \to L$ is a semigroup morphism, the support map extends linearly to a surjection of semigroup algebras $\operatorname{supp} : kS \to kL$. The kernel of this map is nilpotent and the semigroup algebra kL is isomorphic to a product of copies of the field k, one copy for each element of L. Standard ring theory implies that ker(supp) is the Jacobson radical of kS and that the irreducible representations of kS are given by the components of the composition $kS \xrightarrow{\operatorname{supp}} kL \xrightarrow{\cong} \prod_{X\in L} k$. This last map sends $X \in L$ to the vector with 1 in the Y-component if $Y \geq X$ and 0 otherwise. The X-component of this surjection is the map $\chi_X : kS \to k$ defined on the elements $y \in S$ by

$$\chi_X(y) = \begin{cases} 1, & \text{if } \operatorname{supp}(y) \leq X, \\ 0, & \text{otherwise.} \end{cases}$$

The elements

$$E_X = \sum_{Y\geq X} \mu(X, Y)Y \tag{3.1}$$

in kL, one for each $X \in L$, correspond to the standard basis vectors of $\prod_{X\in L} k$ under the isomorphism $kL \cong \prod_{X\in L} k$. In the above μ denotes the Möbius function of the lattice L [6, Sec. 3.7]. The elements $\{E_X\}_{X\in L}$ form a basis of kL and a *complete system of primitive orthogonal idempotents* for kL (see the next section for the definition).

4. Primitive Idempotents of the Semigroup Algebra

Let A be a k-algebra. An element $e \in A$ is *idempotent* if $e^2 = e$. It is a *primitive idempotent* if e is idempotent and we cannot write $e = e_1 + e_2$, where e_1 and e_2 are nonzero idempotents in A with $e_1e_2 = 0 = e_2e_1$. Equivalently, e is primitive iff Ae is an indecomposable A-module. A set of elements $\{e_i\}_{i\in I} \subseteq A$ is a *complete* *system of primitive orthogonal idempotents* for A if e_i is a primitive idempotent for every i, if $e_ie_j = 0$ for $i \neq j$ and if $\sum_i e_i = 1$. If $\{e_i\}_{i\in I}$ is a complete system of primitive orthogonal idempotents for A, then $A \cong \bigoplus_{i\in I} Ae_i$ as left A-modules and $A \cong \bigoplus_{i,j\in I} e_iAe_j$ as k-vector spaces.

Let S denote a left regular band with identity. For each $X \in L$, fix an $x \in S$ with $\mathrm{supp}(x) = X$ and define elements in kS recursively by the formula

$$e_X = x - \sum_{Y>X} xe_Y. \tag{4.1}$$

Lemma 4.1. *Let $w \in S$ and $X \in L$. If* $\mathrm{supp}(w) \not\leq X$, *then* $we_X = 0$.

Proof. We proceed by induction on X. This is vacuously true if $X = \hat{1}$. Suppose the result holds for all $Y \in L$ with $Y > X$. Suppose $w \in S$ and $W = \mathrm{supp}(w) \not\leq X$. Using the definition of e_X and the identity $wxw = wx$ (LRB2),

$$we_X = wx - \sum_{Y>X} wxe_Y = wx - \sum_{Y>X} wx(we_Y).$$

By induction, $we_Y = 0$ if $W \not\leq Y$. Therefore, the summation runs over Y with $W \leq Y$. But $Y > X$ and $Y \geq W$ iff $Y \geq W \vee X$, so the summation runs over Y with $Y \geq W \vee X$.

$$we_X = wx - \sum_{Y>X} wx(we_Y) = wx - \sum_{Y\geq X\vee W} wxe_Y.$$

Now let z be the element of support $X \vee W$ chosen in defining $e_{X\vee W}$. So $e_{X\vee W} = z - \sum_{Y>X\vee W} ze_Y$. Note that $ze_{X\vee W} = e_{X\vee W}$ since $z = z^2$. Therefore, $z = \sum_{Y\geq X\vee W} ze_Y$. Since $\mathrm{supp}(wx) = W \vee X = \mathrm{supp}(z)$, it follows from Proposition 2.1(3) that $wx = wxz$. Combining the last two statements,

$$we_X = wx - \sum_{Y\geq X\vee W} wxe_Y = wx\left(z - \sum_{Y\geq X\vee W} ze_Y\right) = 0.$$

□

Theorem 4.2. *Let S denote a finite left regular band with identity and L its support lattice. Let k denote an arbitrary field. The elements $\{e_X\}_{X\in L}$ form a complete system of primitive orthogonal idempotents in the semigroup algebra kS.*

Proof. *Complete.* 1 is the only element of support $\hat{0}$. Hence, $e_{\hat{0}} = 1 - \sum_{Y>\hat{0}} e_Y$. Equivalently, $\sum_X e_X = 1$.

Idempotent. Since e_Y is a linear combination of elements of support at least Y, $e_Yz = e_Y$ for any z with $\mathrm{supp}(z) \leq Y$ (Proposition 2.1(3)). Using the definition of e_X, the facts $e_X = xe_X$ and $e_Y = e_Yy$, and Lemma 4.1,

$$e_X^2 = \left(x - \sum_{Y>X} xe_Y\right)e_X = xe_X - \sum_{Y>X} xe_Y(ye_X) = xe_X = e_X.$$

Orthogonal. We show that for every $X \in L$, $e_X e_Y = 0$ for $Y \neq X$. If $X = \hat{1}$, then $e_X e_Y = e_X x e_Y = 0$ for every $Y \neq X$ by Lemma 4.1 since $X = \hat{1}$ implies $X \not\leq Y$. Now suppose the result holds for $Z > X$. That is, $e_Z e_Y = 0$ for all $Y \neq Z$. If $X \not\leq Y$, then $e_X e_Y = 0$ by Lemma 4.1. If $X < Y$, then $e_X e_Y = x e_Y - \sum_{Z>X} x(e_Z e_Y) = x e_Y - x e_Y^2 = 0$.

Primitive. We will show that e_X lifts $E_X = \sum_{Y \geq X} \mu(X,Y)Y$ (see Eq. (3.1)) for all $X \in L$, a primitive idempotent in kL. (Then since e_X lifts a primitive idempotent, it is itself a primitive idempotent.) If $X = \hat{1}$, then $\operatorname{supp}(e_{\hat{1}}) = \hat{1} = E_{\hat{1}}$. Suppose the result holds for $Y > X$. Then $\operatorname{supp}(e_X) = \operatorname{supp}(x - \sum_{Y>X} x e_Y) = X - \sum_{Y>X}(X \vee E_Y)$. Since E_Y is a linear combination of elements $Z \geq Y$, it follows that $X \vee E_Y = E_Y$ if $Y > X$. Therefore, $\operatorname{supp}(e_X) = X - \sum_{Y>X} E_Y$. The Möbius inversion formula [6, Sec. 3] applied to $E_X = \sum_{Y \geq X} \mu(X,Y)Y$ gives $X = \sum_{Y \geq X} E_X$. Hence, $\operatorname{supp}(e_X) = X - \sum_{Y>X} E_Y = E_X$. □

Remark 4.3. We can replace $x \in S$ in Eq. (4.1) with any linear combination $\tilde{x} = \sum_{\operatorname{supp}(x)=X} \lambda_x x$ of elements of support X whose coefficients λ_x sum to 1. The proofs still hold since the element $\tilde{x}$ is idempotent and satisfies $\operatorname{supp}(\tilde{x}) = X$ and $\tilde{x}y = \tilde{x}$ for all y with $\operatorname{supp}(y) \leq X$. Unless explicitly stated we will use the idempotents constructed above.

Corollary 4.4. *The set* $\{x e_{\operatorname{supp}(x)} \mid x \in S\}$ *is a basis of* kS *of primitive idempotents* (*not necessarily orthogonal idempotents*).

Proof. Let $y \in S$. Then by Theorem 4.2 and Lemma 4.1,

$$y = y1 = y \sum_Z e_Z = \sum_{Z \geq \operatorname{supp}(y)} y e_Z = \sum_{Z \geq \operatorname{supp}(y)} (yz) e_Z,$$

where $z \in S$ was the element used to define e_Z. Since $\operatorname{supp}(yz) = \operatorname{supp}(y) \vee \operatorname{supp}(z) = Z$, every element $y \in S$ is a linear combination of elements of the form $x e_{\operatorname{supp}(x)}$. So the elements $x e_{\operatorname{supp}(x)}$, one for each x in S, span kS. Since the number of these elements is the cardinality of S, which is the dimension of kS, the set forms a basis of kS. The elements are idempotent since $(x e_X)^2 = (x e_X)(x e_X) = x e_X^2 = x e_X$ (since $xyx = xy$ for all $x, y \in S$). Since $x e_X$ lifts the primitive idempotent $E_X = \sum_{Y \geq X} \mu(X,Y)Y \in kL$, it is also a primitive idempotent (see the end of the proof of Theorem 4.2). □

5. Projective Indecomposable Modules of the Semigroup Algebra

For $X \in L$, let $S_X \subseteq S$ denote the set of elements of S of support X. For $y \in S$ and $x \in S_X$, define

$$y \cdot x = \begin{cases} yx, & \operatorname{supp}(y) \leq \operatorname{supp}(x), \\ 0, & \operatorname{supp}(y) \not\leq \operatorname{supp}(x). \end{cases}$$

Then $\cdot$ defines an action of kS on the k-vector space kS_X spanned by S_X.

Lemma 5.1. *Let $X \in L$. Then $\{xe_X \mid \mathrm{supp}(x) = X\}$ is a basis for $(kS)e_X$.*

Proof. Suppose $\sum_{w\in S} \lambda_w we_X \in kSe_X$. If $\mathrm{supp}(w) \not\leq X$, then $we_X = 0$. So, suppose $\mathrm{supp}(w) \leq X$. Then $\mathrm{supp}(wx) = \mathrm{supp}(w) \vee X = X$. Therefore,

$$\sum_{w\in S} \lambda_w we_X = \sum_{w\in S} \lambda_w (wx)e_X \in \mathrm{span}_k\{ye_X \mid \mathrm{supp}(y) = X\},$$

where x is the element chosen in the construction of e_X (recall that $e_X = xe_X$ since $x^2 = x$). So the elements span kSe_X. These elements are linearly independent being a subset of a basis of kS (Corollary 4.4). □

Proposition 5.2. *There is a kS-module isomorphism $kS_X \cong kSe_X$ given by right multiplication by e_X. Therefore, the kS-modules kS_X are all the projective indecomposable kS-modules. The radical of kS_X is $\mathrm{span}_k\{y - y' \mid y, y' \in S_X\}$.*

Proof. Define a map $\phi : kS_X \to kSe_X$ by $w \mapsto we_X$. Then ϕ is surjective since $\phi(y) = ye_X$ for $y \in S_X$ and since $\{ye_X \mid \mathrm{supp}(y) = X\}$ is basis for kSe_X (Lemma 5.1). Since $\dim kS_X = \#S_X = \dim kSe_X$, the map ϕ is an isomorphism of k-vector spaces.

ϕ is a kS-module map. Let $y \in S$ and let $x \in S_X$. If $\mathrm{supp}(y) \leq X$, then $\phi(y\cdot x) = \phi(yx) = yxe_X = y\phi(x)$. If $\mathrm{supp}(y) \not\leq X$, then $y \cdot x = 0$. Hence, $\phi(x \cdot y) = 0$. Also, since $\mathrm{supp}(y) \not\leq X$, it follows from Lemma 4.1 that $ye_X = 0$. Therefore, $y\phi(x) = yxe_X = yx(ye_X) = yx0 = 0$. So $\phi(y \cdot x) = y\phi(x)$. Hence ϕ is an isomorphism of kS-modules.

Since all the projective indecomposable kS-modules (up to isomorphism) are of the form kSe_X for a complete system of primitive orthogonal idempotents $\{e_X\}$, the kS-modules kS_X are all the indecomposable projective kS-modules. □

6. The Quiver of the Semigroup Algebra

Let A be a finite-dimensional k-algebra whose simple modules are all one-dimensional. The Ext-*quiver* or *quiver* of A is the directed graph Q with one vertex for each isomorphism class of simple modules and $\dim_k(\mathrm{Ext}^1_A(M_X, M_Y))$ arrows from X to Y, where M_X and M_Y are simple modules of the isomorphism classes corresponding to the vertices X and Y, respectively. The *path algebra* kQ of Q is the k-algebra spanned by paths of Q with multiplication induced by path composition: if two paths in Q compose to form another path, then that is the product; if the paths do not compose, then the product is 0. If Q is the quiver of A, then there exists a k-algebra surjection from kQ onto A. Although the quiver Q is canonical, this surjection is not.

Let S be a left regular band with identity and let L denote the support lattice of S. Let $X, Y \in L$ with $Y \leq X$ and fix $y \in S$ with $\mathrm{supp}(y) = Y$. Define a relation on the elements of S_X by $x \smile x'$ if there exists an element $w \in S$ satisfying $y < w$, $w < yx$, and $w < yx'$. (Equivalently, $yw = w$, $wx = yx$, $wx' = yx'$, and supp$(w) < X$.) Note that $x \smile x'$ iff $x \smile yx'$. Also note that for $X = \hat{1}$ and $Y = \hat{0}$, the relation becomes $x \smile x'$ iff there exists $w \neq 1$ such that $x > w$ and $x' > w$.

The relation $\smile$ is symmetric and reflexive, but not necessarily transitive. Let $\sim$ denote the transitive closure of $\smile$. Let $a_{XY} = {}^{\#}(S_X/{\sim}) - 1$, the number of equivalence classes of $\sim$ minus one. If $Y \not\leq X$, define $a_{XY} = 0$. In order to avoid confusion, we denote by a^S_{XY} the number a_{XY} computed in S. Since $u < v$ implies $yu < yv$ for all $u, v, y \in S$ (follows from (LRB2)), it follows that the relations $\smile$ and $\sim$ do not depend on the choice of y with $\text{supp}(y) = Y$.

Lemma 6.1. *Let S be a finite left regular band with identity and L its support lattice. Let M_X and M_Y denote the simple modules with irreducible characters χ_X and χ_Y, respectively. Then*

$$\dim(\text{Ext}^1_{kS}(M_X, M_Y)) = a_{XY}.$$

Proof. The proof is rather lengthy, so we postpone it until Appendix. □

Theorem 6.2. *Let S be a left regular band with identity and L the support lattice of S. Let k denote a field. The quiver of the semigroup algebra kS has L as the vertex set and a_{XY} arrows from the vertex X to the vertex Y.*

7. An Inductive Construction of the Quiver

In this section we describe how knowledge about the numbers $a^{S'}_{\hat{1}\hat{0}}$ for certain subsemigroups S' of S determine all the numbers a^S_{XY}. This allows for an inductive construction of the quiver of a left regular band.

Suppose S is a left regular band with identity. Let $X, Y \in L$ with $Y \leq X$ and let $y \in S$ be an element with $\text{supp}(y) = Y$. Then $yS = \{yw : w \in S\}$ and $S_{\leq X} = \{w \in S : \text{supp}(w) \leq X\}$ are subsemigroups of S.

Proposition 7.1. *Let S be a left regular band with identity, and let L denote the support lattice of S. Suppose $y \in S$ and $X \in L$. The quiver of the semigroup algebra $k(yS_{\leq X})$ of the left regular band $yS_{\leq X}$ is the full subquiver of the quiver of the semigroup algebra kS on the vertices in the interval* $[\text{supp}(y), X] \subseteq L$.

The Proposition follows from the following lemma that shows the number of arrows from X to Y in the quiver of kS is the number of arrows from $\hat{1}$ to $\hat{0}$ in the quiver of $k(yS_{\leq X})$, where $y \in S$ is any element of support Y. Recall that $a^{yS_{\leq X}}_{\hat{1}\hat{0}}$ denotes the number $a_{\hat{1}\hat{0}}$ computed in the left regular band $yS_{\leq X}$.

Lemma 7.2. *Let S be a left regular band with identity. Then $a^S_{XY} = a^{yS_{\leq X}}_{\hat{1}\hat{0}}$. That is, the number a_{XY} computed in S is the number $a_{\hat{1}\hat{0}}$ computed in $yS_{\leq X}$.*

Proof. If $\text{supp}(y) \not\leq X$, then $yS_{\leq X}$ is empty. So $a^S_{X,Y} = 0 = a^{yS_{\leq X}}_{\hat{1}\hat{0}}$. So suppose $\text{supp}(y) \leq X$.

Since $x \sim x'$ iff $x \sim yx'$ for any elements x, x' of support X, every equivalence class of $\sim$ (on S_X) contains an element of yS_X. Therefore, $a_{XY} + 1$ is the number of equivalence classes of $\sim$ restricted to yS_X.

Since $yS_{\leq X}$ is a subsemigroup of S, the support lattice of $yS_{\leq X}$ is the image of $yS_{\leq X}$ in L. Therefore, the support lattice of $yS_{\leq X}$ is the interval $[Y, X]$ in L. Since the top and bottom elements of $[Y, X]$ are X and Y, respectively, the number $a_{\hat{1}\hat{0}}^{yS_{\leq X}} + 1$ is the number of equivalence classes of $\sim$ restricted to yS_X. □

Therefore, if the numbers $a_{\hat{1}\hat{0}}^{yS_{\leq X}}$ are known for all the subsemigroups of S of the form $yS_{\leq X}$, then the quiver of kS is known. We illustrate this technique with two examples in the next two sections.

8. Example: The Free Left Regular Band

Let $S = F(A)$ denote the free left regular band on a finite set A (defined in Example 2.2). Recall that the support lattice L of S is the set of subsets of A.

Let $y \in S$ and $Y \subseteq A$ denote the set of elements occurring in the sequence y. Then yS is the set of all sequences of elements of A (without repetition) that begin with the sequence y. Therefore, yS is isomorphic to the free left regular band on $A \backslash Y$. If $X \subseteq A$ (so $X \in L$), then $S_{\leq X}$ is the set of all sequences containing only elements from X (without repetition). Therefore, $S_{\leq X}$ is also a free left regular band. It follows that $yS_{\leq X}$ is a free left regular band for any $y \in S$ and $X \subseteq A$. Therefore, the quiver of S is determined once the numbers $a_{\hat{0}\hat{1}} = a_{A\emptyset}$ are known for any free left regular band.

If two sequences $x, y \in S$ begin with the same element $a \in A$, then $ax = x$ and $ay = y$. Therefore, $x \sim y$. Conversely, if $x \smile y$, then there is a nonempty sequence w such that $wx = x$ and $wy = y$. Then x and y both begin with the first element of w. Therefore, $x \sim y$ iff x and y are sequences beginning with the same element. So, the equivalence classes of $\sim$ are determined by the first elements of the sequences in S. Hence, $a_{\hat{1}\hat{0}} = {}^{\#}(A) - 1$. This argument applies to any free left regular band with identity, so $a_{XY} = {}^{\#}(X \backslash Y) - 1$ since $yS_{\leq X}$ is isomorphic to the free left regular band on the elements $X \backslash Y$.

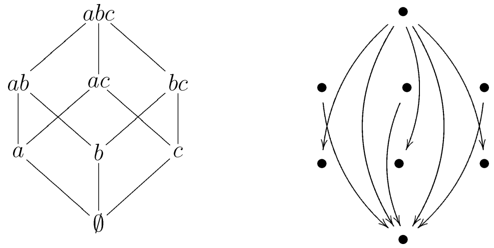

Fig. 2. The support lattice and the quiver of the semigroup algebra of the free left regular band on three generators. See also Fig. 1.

Theorem 8.1 (Brown, private communication). *Let* $S = F(A)$ *be the free left regular band on a finite set* A *and let* k *denote a field. Then the quiver of the semigroup algebra* kS *has one vertex* X *for each subset* X *of* A *and* ${}^{\#}(X \backslash Y) - 1$ *arrows from* X *to* Y *if* $Y \subseteq X$ *(and no other arrows or vertices).*

9. Example: The Face Semigroup of a Hyperplane Arrangement

Let $\mathcal{F}$ be the face semigroup of a central hyperplane arrangement $\mathcal{A}$ and let $\mathcal{L}$ be the intersection lattice of $\mathcal{A}$ (see Example 2.3). Let $X, Y \in \mathcal{L}$ and y be a face of support Y. Then the subsemigroup $y\mathcal{F}_{\leq X}$ is the semigroup of faces of a hyperplane arrangement with intersection lattice $[Y, X] \subseteq \mathcal{L}$. (Explicitly, this hyperplane arrangement is given by $\{X \cap H : H \in \mathcal{A}, Y \subseteq H, X \not\subseteq H\}$.) Therefore, we know all the numbers a_{XY} for $\mathcal{F}$ if we know the number $a_{\hat{1}\hat{0}}$ for the face semigroup of an arbitrary arrangement.

If $\mathcal{L}$ contains only one element, then $\hat{0} = \hat{1}$ and $a_{\hat{1}\hat{0}} = 0$. Suppose that $\mathcal{L}$ contains at least two elements. It is well known that for any two distinct chambers c and d, there exists a sequence of chambers $c_0 = c, c_1, \ldots, c_i = d$ such that c_{j-1} and c_j share a common codimension one face w_j for each $1 \leq j \leq i$ [2, Sec. I.4E Proposition 3]. Therefore, $c_{j-1} \smile c_j$ unless w_j is of support $\hat{0}$, in which case $\mathcal{L}$ has two elements. Equivalently, $c \sim d$ iff the arrangement is of rank greater than 2. So if $\mathcal{L}$ has exactly two elements, then $a_{\hat{1}\hat{0}} = 1$ and if $\mathcal{L}$ has more than two elements then $a_{\hat{1}\hat{0}} = 0$.

Theorem 9.1 ([5, Corollary 8.4]). *The quiver* Q *of the semigroup algebra* $k\mathcal{F}$ *coincides with the Hasse diagram of* $\mathcal{L}$. *That is, there is exactly one arrow* $X \to Y$ *iff* $Y \lessdot X$.

In [5], the relations of the quiver are also determined. Let I be the ideal generated by the following elements, one for each interval $[Z, X]$ of length two in $\mathcal{L}$,

$$\sum_{Y : Z \lessdot Y \lessdot X} X \to Y \to Z.$$

Then $k\mathcal{F} \cong kQ/I$ as k-algebras, where kQ is the path algebra of Q.

10. Idempotents in the Subalgebras $k(yS)$ and $kS_{\geq X}$

This section describes the subalgebras of kS generated by the subsemigroups yS and $S_{\leq Y}$ of S.

Let S be a left regular band. Recall that for $y \in S$, the set $yS = \{yw : w \in S\} = \{w \in S : w \geq y\}$ is a subsemigroup of S (and hence a left regular band). Note that if $\operatorname{supp}(y') = \operatorname{supp}(y)$ then the left regular bands yS and $y'S$ are isomorphic with isomorphism given by multiplication by y (the inverse is multiplication by y'). Since yS is a subsemigroup of S, the support lattice of yS is the image of yS in L by Proposition 2.1, which is the interval $[Y, \hat{1}]$.

Proposition 10.1. *Let* S *be a left regular band, let* $y \in S$ *and let* $Y = \operatorname{supp}(y)$. *There exists a complete system of primitive orthogonal idempotents* $\{e_X : X \in L\}$ *in*

kS such that $\{e_X : X \geq Y\}$ is a complete system of primitive orthogonal idempotents in the semigroup algebra $k(yS)$. Moreover, $k(yS) = (\sum_{X \geq Y} e_X)kS$.

Proof. For each $X \in L$, fix $x \in S$ with $\mathrm{supp}(x) = X$. If $X \geq Y$, then replace x with yx. Note that $\mathrm{supp}(yx) = \mathrm{supp}(x)$ since $X \geq Y$. Therefore, $x \geq y$ if $X \geq Y$. The formula $e_X = x - \sum_{W>X} xe_W$ for $X \in L$ defines a complete system of primitive orthogonal idempotents for kS (Theorem 4.2). And since the support lattice of yS is $[Y, \hat{1}] \subseteq L$, the elements $e_X = x - \sum_{W>X} xe_W$ for $X \geq Y$ define a complete system of primitive orthogonal idempotents in $k(yS)$. Since y is the identity of yS, we have $y = \sum_{X \geq Y} e_X$. Therefore, $k(yS) = y(kS) = (\sum_{X \geq Y} e_X)kS$. □

If $Y \in L$, then $S_{\leq Y} = \{w \in S : \mathrm{supp}(w) \leq Y\}$ is a subsemigroup of S. The support lattice of $S_{\leq Y}$ is the interval $[\hat{0}, Y]$ of L. Let $\mathrm{proj}_{kS_{\leq Y}} : kS \to kS_{\leq X}$ denote the projection onto the subspace $kS_{\leq X}$ of kS.

Proposition 10.2. *Let S be a left regular band and $Y \in L$. Let $\{e_X : X \in L\}$ denote a complete system of primitive orthogonal idempotents of kS. Then $\{\mathrm{proj}_{kS_{\leq Y}}(e_X) : X \leq Y\}$ is a complete system of primitive orthogonal idempotents of $kS_{\leq Y}$. Moreover, the semigroup algebra $k(S_{\leq Y})$ is isomorphic to $kS(\sum_{X \leq Y} e_X)$.*

Proof. The map $\mathrm{proj}_{kS_{\leq Y}}$ is an algebra morphism $kS \to kS_{\leq Y}$. This follows from the fact that $\mathrm{supp}(wx) = \mathrm{supp}(w) \vee \mathrm{supp}(x)$ for any $x, w \in S$. So if $X \leq Y$, then $\mathrm{proj}_{kS_{\leq Y}}(e_X) = x - \sum_{W>X} x\,\mathrm{proj}_{kS_{\leq Y}}(e_W)$ since $e_X = x - \sum_{W>X} xe_W$. Therefore, the elements $\mathrm{proj}_{kS_{\leq Y}}(e_X)$ for $X \leq Y$ form a complete system of primitive orthogonal idempotents for the semigroup algebra of the left regular band $S_{\leq Y}$ (Theorem 4.2). Since $\mathrm{proj}_{kS_{\leq Y}}$ is an algebra morphism, it restricts to a surjective morphism of algebras $\mathrm{proj}_{kS_{\leq Y}} : kS(\sum_{X \leq Y} e_X) \to k(S_{\leq Y})$. Since $kS_X \cong (kS)e_X$ for all $X \in L$ as kS-modules (Proposition 5.2), $\dim(kS_{\leq Y}) = \dim(\sum_{X \leq Y}(kS)e_X)$. So $\mathrm{proj}_{kS_{\leq Y}}$ is an isomorphism. Its inverse is right multiplication by $\sum_{X \leq Y} e_X$. □

11. Cartan Invariants of the Semigroup Algebra

The *Cartan invariants* of a finite-dimensional k-algebra A are the numbers $\dim_k(\mathrm{Hom}_A(Ae_X, Ae_Y))$, where $\{e_X\}_{X \in I}$ is a complete system of primitive orthogonal idempotents for A. They are independent of the choice of $\{e_X\}_{X \in I}$.

Let S be a left regular band with identity and let L denote the support lattice of S. For $X, Y \in L$, define numbers $m(Y, X)$ follows. If $Y \not\leq X$, then $m(Y, X) = 0$. If $Y \leq X$, then define $m(Y, X)$ by the formulas

$$\sum_{W \leq Y \leq X} m(Y, X) = {}^{\#}(wS_X), \tag{11.1}$$

one for each $W \in L$, where w is an element of support W. (Recall that the number ${}^{\#}(wS_X)$ does not depend on the choice of w with $\mathrm{supp}(w) = W$.)

Equivalently,

$$m(Y,X) = \sum_{Y \le W \le X} \mu(Y,W)\ {}^{\#}(wS_X),$$

where μ is the Möbius function of L [6, Sec. 3.7].

Proposition 11.1. *Let S be a left regular band with identity. Let $\{e_X\}_{X \in L}$ denote a complete system of primitive orthogonal idempotents for kS. Then for any X, Y,*

$$\dim(e_Y kS e_X) = \dim \operatorname{Hom}_{kS}(kSe_Y, kSe_X) = m(Y,X).$$

Therefore, the numbers $m(Y,X)$ are the Cartan invariants of kS.

Proof. The first equality follows from the identity $\operatorname{Hom}_A(Ae, Af) \cong eAf$ for idempotents e, f of a k-algebra A. If $Y \not\le X$, then it follows from (LRB2) and Lemma 4.1 that $e_Y kS e_X = 0$. Suppose that $Y \le X$. From the previous section, $k(yS) = \sum_{W \ge Y} e_W kS$ for some complete system of primitive orthogonal idempotents. Combined with the isomorphism $kS_X \cong kSe_X$ we get $k(yS_X) \cong \bigoplus_{Y \le W \le X} e_W kS e_X$. Therefore,

$$\sum_{Y \le W \le X} m(W,X) = \dim(k(yS_X)) = \sum_{Y \le W \le X} \dim(e_W kS e_X).$$

The result now follows by induction. If $X = Y$, then $\dim e_X kS e_X = m(X,X)$. Suppose the result holds for all W with $Y < W \le X$. Then

$$\begin{aligned}
\dim e_Y kS e_X &= \sum_{Y \le W \le X} m(W,X) - \sum_{Y < W \le X} \dim e_W kS e_X \\
&= \sum_{Y \le W \le X} m(W,X) - \sum_{Y < W \le X} m(W,X) \\
&= m(Y,X).
\end{aligned}$$

□

12. Example: The Face Semigroup of a Hyperplane Arrangement

Let $\mathcal{F}$ denote the semigroup of faces of a hyperplane arrangement $\mathcal{A}$. Then ${}^{\#}(w\mathcal{F}_X)$ is the number of faces of support X containing w as a face. Zaslavsky's theorem [7] gives that this is $\sum_{W \le Y \le X} |\mu(Y,X)|$, where μ is the Möbius function of the intersection lattice of $\mathcal{A}$. Comparing this with Eq. (11.1), we conclude that the Cartan invariants of $k\mathcal{F}$ are $m(Y,X) = |\mu(Y,X)|$. These were also computed in [5, Proposition 6.4].

13. Example: The Free Left Regular Band

Let S be a free left regular band on a finite set A. The support lattice of S is the lattice of subsets of A. Therefore, $\mu(Y,W) = (-1)^{\#(W \backslash Y)}$ [6, Example 3.8.3] for any $Y, W \in L$. And ${}^{\#}(wS_X) = {}^{\#}(X \backslash W)!$ since the number of elements of maximal support in the free left regular band on A is precisely $({}^{\#}A)!$. If $n = {}^{\#}X$ and $j = {}^{\#}Y$, and $Y \subseteq X$, then

$$\begin{aligned} m(Y,X) &= \sum_{Y \le W \le X} \mu(Y,W)\ {}^{\#}(wS_X) \\ &= \sum_{Y \le W \le X} (-1)^{{}^{\#}W - j}\ (n - {}^{\#}W)! \\ &= \sum_{i=j}^{n} \sum_{\substack{Y \subseteq W \subseteq X \\ {}^{\#}W = i}} (-1)^{i-j} (n-i)! \\ &= \sum_{i=j}^{n} (-1)^{i-j} (n-i)! \binom{n-j}{i-j} \\ &= (n-j)! \sum_{i=j}^{n} \frac{(-1)^{i-j}}{(i-j)!} \\ &= (n-j)! \sum_{i=0}^{n-j} \frac{(-1)^{i}}{i!}. \end{aligned}$$

Therefore, the number $m(Y,X)$ depends only on the cardinality of $X \backslash Y$ and we denote it by m_i, where $i = {}^{\#}(X \backslash Y)$.

We will now prove that these numbers count paths in the quiver of kS. For a set A of cardinality n, let Q_n be the directed graph with one vertex for each subset of A and ${}^{\#}(X \backslash Y) - 1$ arrows from X to Y if $Y \subseteq X$. Let p_n denote the number of paths in Q_n beginning at A and ending at $\emptyset$. Note that if $Y \subseteq X \subseteq A$, then the number of paths beginning at X and ending at Y in Q_n is p_m, where $m = {}^{\#}(X \backslash Y)$.

For each $0 \le i \le n-1$, there are $n-i-1$ arrows from A to sets of size i, and there are $\binom{n}{i}$ such sets, so $p_n = \sum_{0 \le i \le n-1} \binom{n}{i} (n-i-1) p_i$ for $n \ge 1$. Equivalently,

$$\sum_{0 \le i \le n} \binom{n}{i} p_i = \sum_{0 \le i \le n-1} \binom{n}{i} (n-i) p_i.$$

If m_i satisfy the above recurrence, then $m_i = p_i$ for all i since $m_0 = 1 = p_0$. Well,

$$\begin{aligned} &\sum_{0 \le i \le n-1} \binom{n}{i} (n-i) m_i \\ &\quad = \sum_{0 \le i \le n-1} \frac{n!}{(n-i-1)!\ i!} m_i \\ &\quad = \sum_{0 \le i \le n-1} \frac{n!}{(n-i-1)!\ i!} \left(i! \sum_{0 \le j \le i} \frac{(-1)^j}{j!} \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{0\le i\le n-1} \frac{n!}{(n-i-1)!}\left(\sum_{0\le j\le i}\frac{(-1)^j}{j!}\right)\\
&= \sum_{1\le k\le n} \frac{n!}{(n-k)!}\left(\sum_{0\le j\le k-1}\frac{(-1)^j}{j!}\right)\\
&= \sum_{1\le k\le n} \frac{n!}{(n-k)!}\left(\sum_{0\le j\le k}\frac{(-1)^j}{j!} - \frac{(-1)^k}{k!}\right)\\
&= \sum_{1\le k\le n} \frac{n!}{(n-k)!}\left(\sum_{0\le j\le k}\frac{(-1)^j}{j!}\right) - \sum_{1\le k\le n}\frac{n!}{(n-k)!}\left(\frac{(-1)^k}{k!}\right)\\
&= \sum_{1\le k\le n} \binom{n}{k}\left(k!\sum_{0\le j\le k}\frac{(-1)^j}{j!}\right) - \sum_{1\le k\le n}\binom{n}{k}(-1)^k\\
&= \sum_{1\le k\le n} \binom{n}{k} m_k + 1\\
&= \sum_{1\le k\le n} \binom{n}{k} m_k + \binom{n}{0} m_0\\
&= \sum_{0\le k\le n} \binom{n}{k} m_k.
\end{aligned}
$$

Theorem 13.1 (Brown, private communication). *Let $S = F(A)$ be the free left regular band on a finite set A. Then $kS \cong kQ$, where kQ is the path algebra of the quiver Q of kS.*

Proof. Since Q is the quiver of kS, there is an algebra surjection $kQ \to kS$, where kQ is the path algebra of Q. The canonical basis for kQ is the set of paths in Q, so using the fact that $m(Y, X) = \dim(e_Y kSe_X)$ counts the number of paths in Q from X to Y (see the preceding two paragraphs), we have $\dim(kQ) = \sum_{Y,X} m(Y, X) = \sum_{Y,X} \dim(e_Y kSe_X) = \dim(kS)$. □

14. Future Directions

We conclude this paper by providing a few problems for future exploration.

Although this paper successfully determines the quiver of the semigroup algebra of a left regular band, it says nothing about the quiver relations. *Describe the quiver relations of the semigroup algebra of a left regular band with identity.*

The face semigroup algebra of a hyperplane arrangement is a Koszul algebra [5, Proposition 9.4] and its Koszul dual is the incidence algebra of the opposite lattice of the support lattice of the semigroup. Since this algebra is the semigroup algebra of a left regular band, it is natural to ask this question for all left regular bands. *Determine which class of left regular bands give Koszul semigroup algebras and identify their Koszul duals.* One source of examples of left regular bands giving Koszul algebras comes from *interval greedoids* (see [1] for an introduction to interval greedoids). This will be explored in an upcoming paper.

Another nice property of the face semigroup algebra of a hyperplane arrangement is that the quiver of the semigroup algebra coincides with the support lattice of the semigroup. In fact, the support lattice completely determines the semigroup algebra. *Determine the left regular bands S for which the quiver of kS coincides with the support lattice of L.* (From our description of the quiver of kS, we have a description of these left regular bands in terms of the equivalence classes of $\sim$.) *Determine those S for which the support lattice L completely determines kS.*

A *band* is a semigroup B satisfying $b^2 = b$ for all $b \in B$. Since left regular bands are bands it is natural to try to generalize these results to arbitrary bands. *Describe the quiver of the semigroup algebra kB of a band B with identity. Construct a complete system of primitive orthogonal idempotents for kB. Determine the bands B for which kB is a Koszul algebra.*

Appendix: Proof of Lemma 6.1

Lemma 6.1. *Let S be a finite left regular band with identity and L its support lattice. Let M_X and M_Y denote the simple modules with irreducible characters χ_X and χ_Y, respectively, where $X, Y \in L$. Then*

$$\dim(\mathrm{Ext}^1_{kS}(M_X, M_Y)) = a_{XY}. \tag{A.1}$$

Proof. Here is an outline of the proof. Let $X \in L$. Recall that $M_X = k$ is a vector space and the action of kS on M_X is given by χ_X: if $y \in S$ and $\lambda \in k$, then $y \cdot \lambda = \chi_X(y)\lambda$. Let $K = \ker(\chi_X|_{kS})$.

(1) Using basic homological algebra, the computation of the dimension of $\mathrm{Ext}^1_{kS}(M_X, M_Y)$ is reduced to computing the dimensions of $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$ and $\mathrm{Hom}_{kS}(kS_X, M_Y)$.
(2) For $Y \not< X$ we have $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y) = 0$, which implies $\mathrm{Ext}^1_{kS}(M_X, M_Y) = 0$ for $Y \not< X$. This agrees with $a_{XY} = 0$ for $Y \not< X$.
(3) If $Y < X$, then $\mathrm{Ext}^1_{kS}(M_X, M_Y) \cong \mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$. So we need only show that the dimension of the latter space is a_{XY}.
(4) Let $f \in \mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$. We show: if $x \sim x'$, then $f(x - x') = 0$; and if $u \sim x$ and $u' \sim x'$, then $f(u - u') = f(x - x')$. This implies $\dim(\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)) \le a_{XY}$.
(5) We construct a_{XY} linearly independent kS-module maps in the k-vector space $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$, establishing the inequality

$$\dim(\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)) \ge a_{XY}.$$

At this point the proof of the lemma is complete: step (2) shows Eq. (A.1) holds for $Y \not< X$; steps (3)–(5) show Eq. (A.1) holds for $Y < X$.

Step 1. Computing $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$ and $\mathrm{Hom}_{kS}(kS_X, M_Y)$ is sufficient to determine $\mathrm{Ext}^1_{kS}(M_X, M_Y)$.

Since the following is a short exact sequence of kS-modules and kS_X is projective,

$$0 \to \ker(\chi_X|_{kS}) \to kS_X \xrightarrow{\chi_X} M_X \to 0,$$

by [4, Proposition 7.2, Chap. V], the following is an exact sequence,

$$\mathrm{Hom}_{kS}(kS_X, M_Y) \to \mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y) \\ \to \mathrm{Ext}^1_{kS}(M_X, M_Y) \to 0.$$

Therefore, $\mathrm{Ext}^1_{kS}(M_X, M_Y)$ will be determined once $\mathrm{Hom}_{kS}(kS_X, M_Y)$ and $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$ are determined.

Step 2. If $Y \not< X$, then $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y) = 0$. This implies Eq. $(A.1)$ holds for $Y \not< X$ since $a_{XY} = 0$ for $Y \not< X$.

Let K denote the kernel of $\chi_X|_{kS_X}$. Then K is spanned by the differences of elements of support X. If $f \in \mathrm{Hom}_{kS}(K,\ M_Y)$ and x, x' are elements of support X, then $f(x - x') = 1f(x - x') = \chi_Y(y)f(x - x') = y \cdot f(x - x') = f(y \cdot (x - x'))$, for any element y of support Y. So if $Y \not\leq X$ or if $Y = X$, then $f = 0$. Therefore, $\mathrm{Hom}_{kS}(K, M_Y) = 0$ if $Y \not< X$. It follows from the exact sequence above that

$$\mathrm{Ext}^1_{kS}(M_X, M_Y) = 0 = a_{XY} \quad \text{for } Y \not< X.$$

Step 3. If $Y < X$, then $\mathrm{Ext}^1_{kS}(M_X, M_Y) \cong \mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$.

Suppose $Y < X$. If $f \in \mathrm{Hom}_{kS}(kS_X, M_Y)$, then for all $x \in S_X$, $f(x) = f(x^2) = f(x \cdot x) = x \cdot f(x) = \chi_Y(x)f(x) = 0f(x) = 0$ for all $x \in S$ with $\mathrm{supp}(x) = X$. Therefore, $\mathrm{Hom}_{kS}(kS_X, M_Y) = 0$. Hence, by the exact sequence in Step (1) above,

$$\mathrm{Ext}^1_{kS}(M_X, M_Y) \cong \mathrm{Hom}_{kS}(K, M_Y) \quad \text{for } Y < X.$$

Step 4. Let $f \in \mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$. If $x \sim x'$, then $f(x - x') = 0$; and if $u \sim x$ and $u' \sim x'$, then $f(u - u') = f(x - x')$. This implies $\dim(\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)) \leq a_{XY}$.

Suppose $x \smile x'$. Then there exists a $w \in S$ with $y < w$, $\mathrm{supp}(w) < X$, $wx = yx$, and $wx = yx'$. Then $x - x' \in K$, and for any $f \in \mathrm{Hom}_{kS}(K,\ M_Y)$, we have $f(x - x') = \chi_Y(y)f(x - x') = f(yx - yx') = f(wx - wx') = f(w \cdot (x - x')) = w \cdot f(x - x') = \chi_Y(w)f(x - x') = 0f(x - x') = 0$. Therefore, $f(x - x') = 0$ if $x \smile x'$. If $x \sim x'$, then there exist $x_0 = x, x_1, \ldots, x_i = x'$ such that $x_{j-1} \smile x_j$ for $1 \leq j \leq i$, and $f(x - x') = f(x_0 - x_1) + f(x_1 - x_2) + \cdots + f(x_{i-1} + x_i) = 0$. Therefore, $f(x - x') = 0$ if $x \sim x'$. So f can only be nonzero on differences of elements in different equivalence classes of $\sim$. Moreover, the equivalence classes determine f: if

$u \sim x$ and $u' \sim x'$, then $f(u-u') = f(u-x) + f(x-x') + f(x'-u') = f(x-x')$. Therefore,

$$\dim(\mathrm{Ext}^1_{kS}(M_X, M_Y)) = \dim(\mathrm{Hom}_{kS}(K, M_Y)) \leq a_{XY}.$$

Step 5. There are a_{XY} linearly independent kS-module maps in the k-vector space $\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)$. This implies the inequality,

$$\dim(\mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y)) \geq a_{XY}.$$

Fix y with $\mathrm{supp}(y) = Y$ and let $x, x' \in S_X$ with $x \not\sim x'$. Since $\{u - x : u \neq x, \mathrm{supp}(u) = X\}$ is a basis for K, we get a well-defined linear function $f : K \to k$ by defining

$$f(u-x) = \begin{cases} 1, & \text{if } u \sim x', \\ 0, & \text{otherwise.} \end{cases}$$

We now show that $f : K \to M_Y$ is a kS-module map. That is, $f(w \cdot (u-x)) = \chi_Y(w) \cdot f(u-x)$ for all $w \in S$ and for all $u \in S_X$.

Suppose $\mathrm{supp}(w) \not\leq Y$. Then $w \cdot f(u-x) = 0$ since w acts trivially on M_Y. If $\mathrm{supp}(w) \not< X$, then w acts trivially on K and so $w \cdot f(u-x) = 0 = f(w \cdot (u-x))$. So suppose $\mathrm{supp}(w) < X$. Then $f(w \cdot (u-x)) = f(wu - wx) = f(wu-x) - f(wx-x)$. Since $v \sim x'$ iff $yv \sim x'$ for any $v \in S_X$, it follows that $f(wu-x) = f(ywu-x)$ and $f(wx-x) = f(ywx-x)$. If $\mathrm{supp}(yw) = X$, then $ywu = yw = ywx$ (LRB2), so $f(w \cdot (u-x)) = 0$. If $\mathrm{supp}(yw) < X$, then we have an element $v = yw$ satisfying $v > y$, $\mathrm{supp}(v) < X$, $v(wu) = y(wu)$, and $v(wx) = y(wx)$. That is, $wu \sim wx$ and it follows that $f(wu-x) = f(wx-x)$. So $f(w \cdot (u-x)) = 0$.

Suppose $\mathrm{supp}(w) \leq Y$. Then w acts as the identity on M_Y. Hence, $w \cdot f(u-x) = f(u-x)$. Since $\mathrm{supp}(w) \leq Y$ and $Y \leq X$, we have that $\mathrm{supp}(w) \leq X$. Therefore, $f(w \cdot (u-x)) = f(wu - wx) = f(wu-x) - f(wx-x)$. Since $v \sim x'$ iff $yv \sim x'$, we have $f(wu-x) = f(y(wu)-x) = f(yu-x) = f(u-x)$ since $\mathrm{supp}(w) \leq Y$. Similarly, $f(wx-x) = f(x-x) = 0$. Therefore, $f(w \cdot (u-x)) = f(u-x)$.

This establishes that $f : K \to M_Y$ is a kS-module map. And since f is nonzero only on differences of the form $u - u'$ with $u \sim x$ and $u' \sim x'$, there are exactly a_{XY} such kS-module maps. These maps are linearly independent, therefore

$$\dim(\mathrm{Ext}^1_{kS}(M_X, M_Y)) = \dim(\mathrm{Hom}_{kS}(K, M_Y)) \geq a_{XY}.$$

The proof of the lemma is complete. □

Acknowledgments

The author is indebted to Ken Brown for introducing him to the study of left regular bands and for sharing his thoughts and results. The referee made very useful suggestions to improve the exposition of the proof of Lemma 6.1 in Appendix. This research was funded in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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