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0-HECKE ALGEBRAS

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Abstract

The structure of a 0-Hecke algebra H of type (W, R) over a field is examined. H has 2^n distinct irreducible representations, where $n = |R|$, all of which are one-dimensional, and correspond in a natural way with subsets of R. H can be written as a direct sum of 2^n indecomposable left ideals, in a similar way to Solomon's (1968) decomposition of the underlying Coxeter group W.

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1. Introduction

NOTATION. $\{i_1, \ldots, \hat{i}_s, \ldots, i_n\}$ denotes the set $\{i_1, \ldots, i_n\} - \{i_s\}$, $\cup$ denotes set union and $\cap$ denotes set intersection. $(xyx\ldots)_n$ denotes the product of the first n terms of the sequence $x, y, x, y, x, \ldots$ ACC denotes the ascending chain condition and DCC denotes the descending chain condition. Let S be a set and A a subset of S. Then $|A|$ denotes the number of elements in A, and $\hat{A}$ denotes the complement of A in S.

Let K be any field, and let (W, R) be a finite Coxeter system, with root system Φ, positive system Φ^+ and simple system Π. For each $J \subseteq R$, let Φ_J, Φ_J^+ and Π_J be the corresponding root system, positive system and simple system. $w_i \in R$ is the reflection in the hyperplane perpendicular to $r_i \in \Pi$. For each $J \subseteq R$, let

$$X_J = \{w \in W\colon w(\Pi_J) \subseteq \Phi^+\} \quad \text{and} \quad Y_J = \{w \in W\colon w(\Pi_J) \subseteq \Phi^+, w(\Pi_{\hat{J}}) \subseteq \Phi^-\},$$

where $\hat{J} = R - J$. We shall assume all the standard results on finite Coxeter systems, as found in Bourbaki (1968), Carter (1972) and Steinberg (1967).

1.1 DEFINITION. The *0-Hecke algebra* H over K of type (W, R) is the associative algebra over K with identity 1 generated by $\{a_i: w_i \in R\}$ subject to the relations:

(i) $a_i^2 = -a_i$ for all $w_i \in R$,

(ii) $(a_i a_j a_i \dots)_{n_{ij}} = (a_j a_i a_j \dots)_{n_{ij}}$ for all $w_i, w_j \in R$, $w_i \neq w_j$, where n_{ij} = the order of $w_i w_j$ in W.

For all $w \in W$, define $a_w = a_{i_1} \dots a_{i_s}$, where $w = w_{i_1} \dots w_{i_s}$ is a reduced expression for $w \in W$ in terms of the elements of R. Note that $a_{1_W} = 1$, where 1_W denotes the identity element of W. It is easy to show that a_w is independent of the reduced expression for w, and that every element of H is a K-linear combination of elements a_w, for $w \in W$.

By Bourbaki (1968) (Exercise 23, p. 55), $\{a_w: w \in W\}$ are linearly independent over K and so form a K-basis of H.

1.2 SOME EXAMPLES. (i) Let $G = G(q)$ be a Chevalley group over the finite field $F = GF(q)$ of q elements, where $q = p^m$ for some prime p and positive integer m. Then G has a (B, N) pair (G, B, N, R) and Weyl group W such that for each $w_i \in R$ there is a positive integer c_i such that $|B: B \cap B^{w_i}| = q^{c_i}$. If K is a field of characteristic p, then the Hecke algebra $H_K(G, B)$ is a 0-Hecke algebra.
(ii) Let G be a finite group with a split (B, N) pair (G, B, N, R, U) of rank n and characteristic p with Weyl group W, and let K be a field of characteristic p. Then the Hecke algebra $H_K(G, B)$ is a 0-Hecke algebra of type (W, R) over K.

1.3 LEMMA. *For all $w_i \in R$ and all $w \in W$,*

$$a_i a_w = \begin{cases} a_{w_i w} & \text{if } l(w_i w) = l(w) + 1, \\ -a_w & \text{if } l(w_i w) = l(w) - 1; \end{cases}$$

$$a_w a_i = \begin{cases} a_{w w_i} & \text{if } l(w w_i) = l(w) + 1, \\ -a_w & \text{if } l(w w_i) = l(w) - 1. \end{cases}$$

PROOF. If $l(w_i w) = l(w) + 1$, then $a_{w_i w} = a_i a_w$ by the definition of $a_{w_i w}$. Suppose $l(w_i w) = l(w) - 1$; then there is a reduced expression for w beginning with w_i: say $w = w_i w'$ where $l(w) = l(w') + 1$. Then $a_w = a_i a_{w'}$, and so

$$a_i a_w = a_i a_i a_{w'} = -a_i a_{w'} = -a_w.$$

Similarly for $a_w a_i$.

1.4 COROLLARY. (1) *For all w, $w' \in W$,*

(a) *$a_w a_{w'} = \pm a_{w''}$ for some $w'' \in W$, with $l(w'') \geq \max(l(w), l(w'))$;*

(b) *$a_w a_{w'} = a_{ww'}$ if and only if $l(ww') = l(w) + l(w')$;*

(c) $a_w a_{w'} = (-1)^{l(w')} a_w$ *if and only if* $w(r_i) \in \Phi^-$ *for each* $r_i \in \Pi_J$, *where* $J = \{w_i \in R$: w_i *occurs in some reduced expression for* $w'\}$.

(d) $a_w a_{w'} = (-1)^{l(w)} a_{w'}$ *if and only if* $(w')^{-1}(r_i) \in \Phi^-$ *for each* $r_i \in \Pi_J$, *where* $J = \{w_i \in R$: w_i *occurs in some reduced expression for* $w\}$;

(e) $a_w a_{w'} = \pm a_{w''}$ *with* $l(w'') > l(w)$, *where* $l(w) \geqslant l(w')$, *if and only if there exists* $r_i \in \Pi_J$ *such that* $w(r_i) \in \Phi^+$, *where* $J = \{w_j \in R$: w_j *occurs in some reduced expression for* $w'\}$.

(2) *Let* w_0 *be the unique element of maximal length in* W. *Then for all* $w \in W$,

$$a_w a_{w_0} = (-1)^{l(w)} a_{w_0} \quad \text{and} \quad a_{w_0} a_w = (-1)^{l(w)} a_{w_0}.$$

2. The nilpotent radical of H

Let N be the nilpotent radical of H. Since H is a finite-dimensional algebra over K, H has the DCC and ACC and so N is also the Jacobson radical of H, and is the unique maximal nilpotent ideal of H.

There is a natural composition series for H, consisting of (two-sided) ideals of H such that every factor is a one-dimensional H-module. This series arises as follows: list the basis elements $\{a_w\colon w \in W\}$ in order of increasing length of w, and if w, $w' \in W$ have the same length it does not matter in which order a_w and $a_{w'}$ occur on the list. Rename these elements $h_1, h_2, \ldots, h_{|W|}$ respectively. Note that $h_1 = 1$ and $h_{|W|} = a_{w_0}$. Let H_j be the ideal of H generated by $\{h_m\colon m \geqslant j\}$. H_j has K-basis $\{h_m\colon m \geqslant j\}$ and dimension $|W| - j + 1$. Then

2.1 $$H = H_1 > H_2 > \ldots > H_{|W|} = a_{w_0} H > 0$$

is the natural composition series of H described above. H_i/H_{i+1} is a one-dimensional H-module, $1 \leqslant i \leqslant |W|$, where $H_{|W|+1} = 0$, with basis $h_i + H_{i+1}$, where $h_i = a_w$ for some $w \in W$. Either $a_w^2 = (-1)^{l(w)} a_w$ or $a_w^2 \in H_{i+1}$; in the first case, the factor ring H_i/H_{i+1} is generated by an idempotent, and in the second case it is nilpotent.

2.2 Lemma. *The number of factors which are generated by an idempotent is equal to* 2^n, *where* $n = |R|$.

Proof. The factors generated by idempotents correspond to elements $w \in W$ such that $a_w^2 = (-1)^{l(w)} a_w$. Let $w \in W$ be such an element. Write $w = w_{i_1} \ldots w_{i_s}$, where $l(w) = s$, and let $J = \{w_{i_j}\colon 1 \leqslant j \leqslant s\}$. Then $w \in W_J$, and by 1.4(1c), $w(\Pi_J) \subseteq \Phi^-$. Hence $w = w_{0J}$, the unique element of maximal length in W_J. Conversely, for each subset J of R, $a_{w_{0J}}^2 = (-1)^{l(w_{0J})} a_{w_{0J}}$. Hence the number of factors which are generated by an idempotent is equal to the number of subsets of R, that is, 2^n, where $n = |R|$.

By Schreier's theorem, any series of ideals of H can be refined to a composition series, and all so obtained have the same number of terms in them as the natural series, and with the factors in one–one correspondence with those of the natural series. In particular, consider $H > N \geqslant 0$. This can be refined to a composition series $H = H'_1 > \dots > H'_{|W|} > H'_{|W|+1} = 0$, where $N = H'_r$, $2 < r \leqslant |W|+1$. Now each factor H'_i/H'_{i+1}, $i \geqslant r$, is nilpotent as $H'_i \leqslant N$, and each factor H'_i/H'_{i+1}, $i+1 \leqslant r$, must be generated by an idempotent as $H'_i/N \leqslant H/N$, a semi-simple ring. Hence the number of factors which are nilpotent is equal to the dimension of N. Thus, $\dim N = |W| - 2^n$, where $n = |R|$.

We can, however, give a precise basis of N.

2.3 THEOREM. *Let $w \in W$, and suppose $w \neq w_{0J}$ for any $J \subseteq R$. Write $w = w_{i_1} \dots w_{i_s}$, $l(w) = s$, and let $J(w) = \{w_{i_j}\colon 1 \leqslant j \leqslant s\}$. Then $E(w) = a_w + (-1)^{l(w_{0J(w)})+l(w)+1} a_{w_{0J(w)}}$ is nilpotent, and $\{E(w)\colon w \in W,\ w \neq w_{0J}$ for any $J \subseteq R\}$ is a basis of N.*

PROOF. Show $E(w)$ is nilpotent by induction on $l(w_{0J(w)}) - l(w)$. Note that if $w = w_{0J}$ for some $J \subseteq R$ then $E(w) = 0$. Suppose $l(w_{0J(w)}) - l(w) = 1$. Then since a reduced expression for w involves all $w_i \in J(w)$, $w \neq w_{0J(w)}$, there exists $r_j \in \Pi_{J(w)}$ such that $w(r_j) \in \Phi^+$. So $a_w^2 = (-1)^{l(w)-1} a_{w_{0J(w)}}$. Thus

$$\begin{aligned} E(w)^2 &= a_w^2 + a_w a_{w_{0J(w)}} + a_{w_{0J(w)}} a_w + a_{w_{0J(w)}}^2 \\ &= a^b_{w_{0J(w)}} \quad \text{where } b = (-1)^{l(w)-1} + 2(-1)^{l(w)} + (-1)^{l(w_{0J(w)})} \\ &= 0 \text{ as } l(w_{0J(w)}) = l(w) + 1. \end{aligned}$$

Now suppose $l(w_{0J(w)}) - l(w) > 1$. Consider the product $a_w a_w$. Since $w \neq w_{0J(w)}$, there exists $r_j \in \Pi_{J(w)}$ such that $w(r_j) \in \Phi^+$. As any reduced expression for w involves all $w_i \in J(w)$, we have $a_w a_w = (-1)^{2l(w)-l(w')} a_{w'}$, with $w' \in W_{J(w)}$ and $l(w') > l(w)$. Further, $J(w') = J(w)$. Then

$$\begin{aligned} E(w)^2 &= a_w^2 + 2(-1)^{l(w_{0J(w)})+1} a_{w_{0J(w)}} + (-1)^{l(w_{0J(w)})} a_{w_{0J(w)}} \\ &= (-1)^{l(w')} a_{w'} + (-1)^{l(w_{0J(w)})+1} a_{w_{0J(w)}} \\ &= (-1)^{l(w')} (a_{w'} + (-1)^{l(w_{0J(w')})+l(w')+1} a_{w_{0J(w')}}) \\ &= (-1)^{l(w')} E(w'). \end{aligned}$$

As $l(w') > l(w)$, either $w' = w_{0J(w)}$ and thus $E(w)^2 = 0$ or $w' \neq w_{0J(w)}$ and then by induction $E(w')$ is nilpotent. Thus $E(w)$ is nilpotent.

Finally, note that we get a nilpotent element for each $w \in W$, $w \neq w_{0J}$ for any $J \subseteq R$. The set of all $E(w)$, $w \neq w_{0J}$ for any $J \subseteq R$, is obviously linearly independent, and there are $|W| - 2^n$ elements in all, where $n = |R|$. Hence they are a K-basis for N.

2.4 COROLLARY. *H/N is commutative.*

PROOF. We show that $a_i a_j - a_j a_i \in N$ for all $w_i, w_j \in R$. If $a_i a_j = a_j a_i$, the result is obvious. So suppose $a_i a_j \neq a_j a_i$. Then we can form $E(w_i w_j)$ and $E(w_j w_i)$ and $E(w_i w_j) - E(w_j w_i) = a_i a_j - a_j a_i \in N$ as each of $E(w_i w_j)$ and $E(w_j w_i)$ is in N.

3. The irreducible representations of H

Consider the one-dimensional H-modules which arise from the natural composition series of H. Let the factor H_i/H_{i+1} be generated as left H-module by $a_w + H_{i+1}$. The action of H on this element is determined as follows: for each $w_i \in R$,

$$a_i(a_w + H_{i+1}) = \begin{cases} -(a_w + H_{i+1}) & \text{if } w^{-1}(r_i) \in \Phi^-, \\ 0 & \text{if } w^{-1}(r_i) \in \Phi^+. \end{cases}$$

For any $w \in W$, let $J(w) = \{w_{i_j} : 1 \leqslant j \leqslant s\}$ where $w = w_{i_1} \dots w_{i_s}$ is a reduced expression for w. Then for $w' \in W$,

$$a_{w'}(a_w + H_{i+1}) = \begin{cases} (-1)^{l(w')}(a_w + H_{i+1}) & \text{if } w^{-1}(\Pi_{J(w')}) \subseteq \Phi^-, \\ 0 & \text{if there exists } r_i \in \Pi_{J(w')} \text{ such that } w^{-1}(r_i) \in \Phi^+. \end{cases}$$

Hence the action of H on $a_w + H_{i+1}$ depends on w^{-1}.

3.1 DEFINITION. For each $J \subseteq R$, let λ_J be the one-dimensional representation of H defined by

$$\lambda_J(a_i) = \begin{cases} 0 & \text{if } w_i \in J, \\ -1 & \text{if } w_i \in \hat{J}. \end{cases}$$

For all $w \in W$, let $w = w_{i_1} \dots w_{i_s}$ with $l(w) = s$. Then $\lambda_J(a_w) = \lambda_J(a_{i_1}) \dots \lambda_J(a_{i_s})$. Extend λ_J to H by linearity.

For each $J \subseteq R$, let $H_{i(J)}/H_{i(J)+1}$ be the factor of the natural series which is generated by $a_{w_0 \hat{J}} + H_{i(J)+1}$. Then the left H-module $H_{i(J)}/H_{i(J)+1}$ affords the representation λ_J of H.

Since each composition factor of H is one-dimensional, it follows that all irreducible representations of H are one-dimensional. Let μ be an irreducible representation of H. Then μ is completely determined by the values $\mu(a_i)$ for all $w_i \in R$. Since μ is an algebra homomorphism, $\mu(a_i)^2 = -\mu(a_i)$ for all $w_i \in R$. Let $\mu(a_i) = u_i \in K$ for all $w_i \in R$. Then $u_i^2 = -u_i$ in K implies that $u_i = 0$ or $u_i = -1$.

Thus each irreducible representation of H can be described by an n-tuple $(u_1, \ldots, u_n)$, where $n = |R|$, with $u_i = 0$ or -1 for all i. In particular, λ_J corresponds to the n-tuple $(u_1, \ldots, u_n)$ where $u_i = 0$ if $w_i \in J$ and $u_i = -1$ if $w_i \in \hat{J}$. There are 2^n such irreducible representations, and they all occur in the natural series of H.

2^n maximal ideals of H are determined as follows: for each $J \subseteq R$, form the n-tuple $(u_1, \ldots, u_n)$, where $u_i = 0$ if $w_i \in J$ and $u_i = -1$ otherwise. Let M_J be the left ideal of H generated by $\{a_i - u_i 1 : w_i \in R\}$. Then $M_J = \ker \lambda_J$, and as each λ_J is irreducible, M_J is a maximal left ideal of H.

Now H/N is semi-simple Artinian. So by extending K to its algebraic closure $\bar{K}$ and considering H as an algebra over $\bar{K}$, we deduce that

$$H/N \cong \bar{K} \oplus \bar{K} \oplus \ldots \oplus \bar{K}, \quad \text{a direct sum of } 2^n \text{ fields.}$$

(Actually, we will show that

$$H/N \cong K \oplus K \oplus \ldots \oplus K, \quad 2^n \text{ copies of } K,$$

regardless of which field K is.)

4. Some decompositions of H

For each $J \subseteq R$, let H_J be the subalgebra of H generated by $\{a_i : w_i \in J\}$.

4.1 Definition. For each $J \subseteq R$, let

$$e_J = \sum_{w \in W_J} a_w, \quad o_J = (-1)^{l(w_{0J})} a_{w_{0J}}.$$

4.2 Lemma. *For all $w_i \in J$,*

$$a_i e_J = 0 = e_J a_i \quad \text{and} \quad a_i o_J = -o_J = o_J a_i.$$

Proof. Use 1.3.

4.3 Lemma. *Let $w_{0J} = w_{i_1} \ldots w_{i_s}$, $l(w_{0J}) = s$. Then*

$$e_J = (1 + a_{i_1}) \ldots (1 + a_{i_s})$$

and is independent of the reduced expression for w_{0J}.

Notation. For all $w \in W$, if $w = w_{i_1} \ldots w_{i_t}$ with $l(w) = t$, write

$$[1 + a_w] = (1 + a_{i_1}) \ldots (1 + a_{i_t}).$$

By the following proof it follows that $[1 + a_w]$ is independent of the reduced expression for w.

PROOF. Firstly, we show that $[1+a_{w_{0J}}]$ is independent of the reduced expression for w_{0J}. Since we can pass from one reduced expression for w_{0J} to another by substitutions of the form $(w_i w_j w_i \dots)_{n_{ij}} = (w_j w_i w_j \dots)_{n_{ij}}$, $i \neq j$, where n_{ij} is the order of $w_i w_j$ in W, we need to show that

$$[1+a_{(w_i w_j w_i \dots)_{n_{ij}}}] = [1+a_{(w_j w_i w_j \dots)_{n_{ij}}}].$$

To do this, we use induction on n, $n \leqslant n_{ij}$, to show that

$$[1+a_{(w_i w_j w_i \dots)_n}] = 1 + \sum_{m=1}^{n} a_{(w_i w_j w_i \dots)_m} + \sum_{m=1}^{n-1} a_{(w_j w_i w_j \dots)m}.$$

This is clearly true for $n=1$. Suppose it is true for all integers $\leqslant k$, and suppose that k is odd. Then

$$\begin{aligned}
[1+a_{(w_i w w_i \dots)_{k+1}}] &= [1+a_{(w_i w_j w_i \dots)_k}](1+a_j) \\
&= \left(1 + \sum_{m=1}^{k} a_{(w_i w_j w_i \dots)_m} + \sum_{m=1}^{k-1} a_{(w_j w_i w_j \dots)_m}\right)(1+a_j) \\
&= \left(1 + \sum_{m=1}^{k} a_{(w_i w_j w_i \dots)_m} + \sum_{m=1}^{k-1} a_{(w_j w_i w_j \dots)_m}\right) + a_j \\
&\quad + \sum_{m=0}^{\frac{1}{2}(k-1)} a_{(w_i w_j w_i \dots)_{2m+1}} a_j + \sum_{m=1}^{\frac{1}{2}(k-1)} a_{(w_i w_j w_i \dots)_{2m}} a_j \\
&\quad + \sum_{m=1}^{\frac{1}{2}(k-1)} a_{(w_j w_i w_j \dots)_{2m-1}} a_j + \sum_{m=1}^{\frac{1}{2}(k-1)} a_{(w_j w_i w_j \dots)_{2m}} a_j.
\end{aligned}$$

Now,

$$a_{(w_i w_j w_i \dots)_{2m-1}} a_j = -a_{(w_i w_j w_i \dots)_{2m}} a_j, \quad 1 \leqslant m \leqslant \tfrac{1}{2}(k-1),$$

and

$$a_{(w_j w_i w_j \dots)_{2m-1}} a_j = -a_{(w_j w_i w_j \dots)_{2m-2}} a_j, \quad 1 \leqslant m \leqslant \tfrac{1}{2}(k-1),$$

where $a_{(w\, w_i w_j \dots)_0} = 1$. Then

$$\begin{aligned}
[1+a_{(w_i w_j w_i \dots)_{k+1}}] &= 1 + \sum_{m=1}^{k} a_{(w_i w_j w_i \dots)_m} + \sum_{m=1}^{k-1} a_{(w_j w_i w_j \dots)_m} \\
&\quad + a_{(w_i w_j w_i \dots)_k} a_j + a_{(w_j w_i w_j \dots)_{k-1}} a_j \\
&= 1 + \sum_{m=1}^{k+1} a_{(w_i w_j w_i \dots)_m} + \sum_{m=1}^{k} a_{(w_j w_i w_j \dots)_m}.
\end{aligned}$$

Similarly, we get the above result if we assume k is even.

Similarly, for all $n \leqslant n_{ij}$,

$$[1+a_{(w_j w_i w_j \dots)_n}] = 1 + \sum_{m=1}^{n} a_{(w_j w_i w_j \dots)_m} + \sum_{m=1}^{n-1} a_{(w_i w_j w_i \dots)_m}.$$

Then, for all $n \leqslant n_{ij}$,

$$[1+a_{(w_i w_j w_i \dots)_n}] - [1+a_{(w_j w_i w_j \dots)_n}] = a_{(w_i w_j w_i \dots)_n} - a_{(w_j w_i w_j \dots)_n}.$$

When $n = n_{ij}$, this difference is zero, and so

$$[1+a_{(w_i w_j w_i \dots)_{n_{ij}}}] = [1+a_{(w_j w_i w_j \dots)_{n_{ij}}}]$$

and thus $[1+a_{w_{0J}}]$ is independent of the reduced expression for w_{0J} chosen.

Finally, $[1+a_{w_{0J}}]$ is a linear combination of certain a_w with $w \in W_J$. We show by induction on $l(w)$ for all $w \in W_J$ that a_w occurs in the expansion of $[1+a_{w_{0J}}]$ with coefficient 1. If $l(w) = 0$, then $w = 1$ and obviously 1 occurs with coefficient 1. Suppose $l(w) > 0$. Let $w = w'w_j$, $w' \in W_J$, $w_j \in J$, where $l(w) = l(w')+1$. By induction $a_{w'}$ occurs in $[1+a_{w_{0J}}]$ with coefficient 1. Choose an expression for w_{0J} ending in w_j, and then $[1+a_{w_{0J}}] = [1+a_{w_{0J}w_j}](1+a_j)$. Since $l(w'w_j) > l(w')$, the only contribution to $a_{w'}$ from the last bracket is from the 1. If instead we take a_j from the last bracket, we get a_w, with coefficient 1. Now suppose a_w occurs in $[1+a_{w_{0J}w_j}]$ with coefficient m. Then

$$ma_w(1+a_j) = ma_w + ma_w a_j = ma_w - ma_w = 0 \quad \text{as } w(r_j) \in \Phi^-.$$

Thus a_w occurs in the expansion of $[1+a_{w_{0J}}]$ with coefficient 1, and hence $e_J = [1+a_{w_{0J}}]$.

4.4 Corollary. (1) *If* $J, L \subseteq R$, $J \cap L \neq \varnothing$, *then* $o_J e_L = 0$ *and* $e_J o_L = 0$.
(2) *If* $L \subseteq J \subseteq R$, *then* $e_L e_J = e_J = e_J e_L$ *and* $o_L o_J = o_J = o_J o_L$.

Proof. Use 4.2 and 4.3.

4.5 Lemma. *Let* $y \in Y_J$ *for some* $J \subseteq R$. *Then* $a_y o_{\hat{J}} = a_y$ *and* $a_y o_{\hat{J}} e_J = \sum_{w \in W_J} a_{yw}$, *with* $l(yw) = l(y) + l(w)$ *for all* $w \in W_J$, *that is,* $a_y o_{\hat{J}} e_J$ is equal to a_y plus a sum of certain a_w with $l(w) > l(y)$.

Proof. If $y \in Y_J$, then $y = ww_{0\hat{J}}$ for some $w \in W$ with $l(y) = l(w) + l(w_{0\hat{J}})$. Hence $a_y o_{\hat{J}} = (-1)^{l(w_{0\hat{J}})} a_w a_{w_{0\hat{J}}} a_{w_{0\hat{J}}}$, and so $a_y o_{\hat{J}} = a_y$. Now for all $w \in W_J$, as $y \in Y_J \subseteq X_J$, we have $l(yw) = l(y) + l(w)$. So for all $w \in W_J$, $a_y a_w = a_{yw}$. Thus

$$a_y o_{\hat{J}} e_J = a_y e_J = \sum_{w \in W_J} a_y a_w = \sum_{w \in W_J} a_{yw} = a_y + \sum_{w \in W_J, w \neq 1} a_{yw},$$

and $l(yw) > l(y)$ if $w \neq 1$, $w \in W_J$.

4.6 LEMMA. *For $y \in Y_J$, a_y occurs in the expansion of $a_y e_J o_{\hat{J}}$ with coefficient 1, and if, for any $w \in W$, a_w occurs in the expansion of $a_y e_J o_{\hat{J}}$ with non-zero coefficient, then $w = y$ or $l(w) > l(y)$.*

PROOF. By 4.5, $a_y e_J = \sum_{w \in W_J} a_{yw}$, with $l(yw) = l(y) + l(w)$ for all $w \in W_J$. So

$$a_y e_J o_{\hat{J}} = \sum_{w \in W_J} a_{yw} o_{\hat{J}} = a_y o_{\hat{J}} + \sum_{w \in W_J, w \neq 1} a_{yw} o_{\hat{J}}.$$

From the proof of 4.5, $a_y o_{\hat{J}} = a_y$, and for all $w \in W_J$, $w \neq 1$,

$$a_{yw} o_{\hat{J}} = a_{yw}(-1)^{l(w_0 \hat{J})} a_{w_{0\hat{J}}} = \pm a_{w'}$$

for some $w' \in W$ with $l(w') \geqslant l(yw) > l(y)$.

4.7 THEOREM. (i) *The elements $\{a_y o_{\hat{J}} e_J = a_y e_J \colon y \in Y_J, J \subseteq R\}$ are linearly independent and form a basis of H.*

(ii) *The elements $\{a_y e_J o_{\hat{J}} \colon y \in Y_J, J \subseteq R\}$ are linearly independent and form a basis of H.*

PROOF. (i) Suppose that for each $y \in Y_J$ and each $J \subseteq R$ there is an element $k_y \in K$ such that $\sum_{J \subseteq R} \sum_{y \in Y_J} k_y a_y e_J = 0$. Let

$$S_n = \sum_{J \subseteq R} \sum_{y \in Y_J, l(y) \geqslant n} k_y a_y e_J.$$

We show that if $S_n = 0$, then $k_y = 0$ whenever $l(y) = n$ and hence $S_{n+1} = 0$.

Let $y_1, \ldots, y_t$ be those elements of W for which $l(y_i) = n$. Then by 4.5, if $y_i \in Y_{J(i)}$ for some $J(i) \subseteq R$,

$$a_{y_i} e_{J(i)} = a_{y_i} + (\text{a linear combination of certain } a_w \text{ where } l(w) > l(y_i)).$$

Hence,

$$S_n = \sum_{i=1}^{t} k_{y_i} a_{y_i} + (\text{a linear combination of certain } a_w \text{ with } l(w) > n).$$

If $S_n = 0$, then as $\{a_w \colon w \in W\}$ are a basis of H, we must have $k_{y_i} = 0$ for all i, $1 \leqslant i \leqslant t$. Then $S_{n+1} = 0$.

Since $S_0 = 0$, $k_y = 0$ for all y whenever $l(y) = 0$, and then $S_1 = 0$. By induction, all k_y are zero, and so $\{a_y e_J \colon y \in Y_J, J \subseteq R\}$ is a set of linearly independent elements. As there are $|W|$ of them, they must form a basis of H.

(ii) This is proved using similar arguments.

4.8 Corollary. (i) *For any $L \subseteq R$, the elements of the set*

$$\{a_y o_{\hat{J}} e_J o_{\hat{L}} = a_y e_J o_{\hat{L}} \colon y \in Y_J, J \subseteq L\}$$

are linearly independent.

(ii) *For any $L \subseteq R$, the elements of the set $\{a_y e_J o_{\hat{J}} e_L \colon y \in Y_J, J \supseteq L\}$ are linearly independent.*

Proof. (i) $a_y e_J o_{\hat{L}} = \sum_{w \in W_J} a_{yw} o_{\hat{L}}$. As $J \subseteq L$, $\hat{L} \subseteq \hat{J}$ and so $a_{w_0 J} o_{\hat{L}} = a_{w_0 \hat{J}}$. Then

$$\begin{aligned} a_y e_J o_{\hat{L}} &= a_y o_{\hat{L}} + \sum_{w \in W_J, w \neq 1} a_{yw} o_{\hat{L}} \\ &= a_y + \sum_{w \in W_J, w \neq 1} a_{yw} o_{\hat{L}} \quad \text{as } y \in Y_J \\ &= a_y + (\text{a linear combination of certain } a_w \text{ with } l(w) > l(y)). \end{aligned}$$

The result now follows by using an argument similar to that used in the proof of 4.7.

(ii) For any $y \in Y_J$, $a_y e_J o_{\hat{J}} = a_y + (\sum_{w \in W} k_w a_w)$, where $k_w \in K$ and $k_w = 0$ if $l(w) \leqslant l(y)$. Then

$$\begin{aligned} a_y e_J o_{\hat{J}} e_L &= a_y e_L + \Big(\sum_{w \in W} k_w a_w\Big) e_L, \quad k_w \in K \text{ given as above,} \\ &= a_y + \Big(\sum_{w \in W} k'_w a_w\Big) \quad \text{for certain } k'_w \in K, \text{ with } k'_w = 0 \text{ if } l(w) \leqslant l(y). \end{aligned}$$

Once again the result is given using an argument similar to that given in the proof of 4.7.

4.9 Theorem. (i) *For each $a \in H$ and for any $J \subseteq R$, there exist elements $k_y \in K$ such that*

$$a o_{\hat{J}} e_J = \sum_{y \in Y_J} k_y a_y e_J = \Big(\sum_{y \in Y_J} k_y a_y o_{\hat{J}} e_J\Big).$$

(ii) *For each $a \in H$ and for any $J \subseteq R$, there exist elements $k_y \in K$ such that*

$$a e_J o_{\hat{J}} = \sum_{y \in Y_J} k_y a_y e_J o_{\hat{J}}.$$

Proof. (i) As $\{a_w \colon w \in W\}$ is a basis of H, we may write $a = \sum_{w \in W} u_w a_w$ with $u_w \in K$ for all $w \in W$. It is thus sufficient to express $a_w o_{\hat{J}} e_J$ as a linear combination of the elements $\{a_y e_J \colon y \in Y_J\}$ for all $w \in W$. Use induction on $l(w)$ to prove this.

If $l(w) = 0$, then $w = 1$ and $1 o_{\hat{J}} e_J = (-1)^{l(w_0 \hat{J})} a_{w_0 \hat{J}} e_J$. The result is true for $w = 1$ as $w_{0\hat{J}} \in Y_J$.

Suppose $l(w)>0$. Let $w = w_i w'$ for some $w_i \in R$, $w' \in W$, $l(w) = l(w')+1$. By induction,

$$a_{w'} o_{\hat{J}} e_J = \sum_{y \in Y_J} u_y a_y e_J \quad \text{for some } u_y \in K.$$

Then

$$a_w o_{\hat{J}} e_J = a_i a_{w'} o_{\hat{J}} e_J = \sum_{y \in Y_J} u_y a_i a_y e_J.$$

Hence for each $y \in Y_J$ we have to express $a_i a_y e_J$ as a combination of $\{a_v e_J : v \in Y_J\}$. Now for any $y \in Y_J$,

$$(4.10) \qquad a_i a_y e_J = \begin{cases} -a_y e_J, & \text{if } y^{-1}(r_i) \in \Phi^-, \\ 0, & \text{if } y^{-1}(r_i) = r_j \text{ for some } r_j \in \Pi_J, \\ & \quad \text{as then } a_i a_y = a_y a_j, \\ a_{w_i y} e_J, & \text{where } w_i y \in Y_J \text{ if } y^{-1}(r_i) \in \Phi^+, \\ & \quad y^{-1}(r_i) \neq r_j \text{ for any } r_j \in \Pi_J. \end{cases}$$

The result follows.

(ii) Since $\{a_y e_L o_{\hat{L}} : y \in Y_L, L \subseteq R\}$ is a basis of H, there exist elements $u_y \in K$ such that

$$a e_J o_{\hat{J}} = \sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L o_{\hat{L}}.$$

Choose any $M \subseteq R$ with $M \cap \hat{J} \neq \emptyset$. Then $a e_J o_{\hat{J}} e_M = 0$; so

$$\sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L o_{\hat{L}} e_M = 0.$$

But $o_{\hat{L}} e_M = 0$ if $\hat{L} \cap M \neq \emptyset$. So the only non-zero terms in the above equation involve those $L \subseteq R$ for which $\hat{L} \cap M = \emptyset$. Thus

$$\sum_{L, M \subseteq L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L o_{\hat{L}} e_M = 0.$$

By 4.8(ii), $u_y = 0$ for all $y \in Y_L$, $M \subseteq L \subseteq R$. Hence we have that $u_y = 0$ for all $y \in Y_L$, with $L \cap \hat{J} \neq \emptyset$. Thus

$$a e_J o_{\hat{J}} = \sum_{L \subseteq J} \sum_{y \in Y_L} u_y a_y e_L o_{\hat{L}}.$$

Let $S_J = \{w \in W : u_w \neq 0,\ w \in Y_L \text{ for some } L \subset J\}$. Suppose $S_J \neq \emptyset$. Choose an element $y_0 \in S_J$ of minimal length, and suppose $y_0 \in Y_{J_0}$ for some $J_0 \subset J$. Consider

$$a e_J o_{\hat{J}} o_{\hat{J}_0} = \sum_{L \subseteq J} \sum_{y \subseteq Y_L} u_y a_y e_L o_{\hat{L}} o_{\hat{J}_0}.$$

As $J_0 \subset J$, $e_J o_{\hat{J}} o_{\hat{J}_0} = e_J o_{\hat{J}_0} = 0$. Then

$$(*) \qquad \sum_{L \subset J} \sum_{y \in Y_L} u_y a_y e_L o_{\hat{L}} o_{\hat{J}_0} = 0.$$

Now if $L \subset J$ and $y \in Y_L$,

$$a_y e_L o_{\hat{L}} o_{\hat{J}_0} = a_y o_{\hat{J}_0} + \sum_{w \in W, l(w) > l(y)} k_w a_w$$

where $k_w \in K$, and $a_y o_{\hat{J}_0} = \pm a_w$, for some $w \in W$ with $l(w) \geqslant l(y)$.

Since y_0 is of minimal length in S_J, the coefficient of a_{y_0} on the left side of (*) is u_{y_0}. As $\{a_w: w \in W\}$ is a basis of H, so $u_{y_0} = 0$, which is a contradiction. Hence $S_J = \varnothing$ and $ae_J o_{\hat{J}} = \sum_{y \in Y_J} u_y a_y e_J o_{\hat{J}}$.

REMARK. Let $z \in Z$. Then z can be regarded as an element of K in a natural way —it is the element $z1_K = 1_K + \ldots + 1_K$ (z times), where 1_K is the identity of K.

4.11 COROLLARY. (1) *For each $w \in W$, there exist rational integers $u_y = u_y(w)$ such that $a_w o_{\hat{J}} e_J = \sum_{y \in Y_J} u_y a_y o_{\hat{J}} e_J$.*

(2) *For each $w \in W$, there exist rational integers $u_y = u_y(w)$ such that*

$$a_w e_J o_{\hat{J}} = \sum_{y \in Y_J} u_y a_y e_J o_{\hat{J}}.$$

PROOF. (1) Follows from the proof of 4.9(i).

(2) List the elements $y_1, \ldots, y_m$ of Y_J in order of increasing length; if $i < j$ then $l(y_i) \leqslant l(y_j)$. Let c_{ij} be the coefficient of a_{y_i} in $a_{y_j} e_J o_{\hat{J}}$. Clearly c_{ij} is an integer as $a_{y_j} e_J o_{\hat{J}}$ is an integral combination of certain elements $a_{w'}$, $w' \in W$. Also, $c_{ii} = 1$ for all i, $1 \leqslant i \leqslant m$, and $c_{ij} = 0$ if $i < j$ by 4.6. Let h_i be the coefficient of a_{y_i} in $a_w e_J o_{\hat{J}}$. Clearly h_i is an integer, and

$$h_i = \sum_{j=1}^{m} k_j c_{ij} \quad \text{where} \quad a_w e_J o_{\hat{J}} = \sum_{i=1}^{m} k_i a_{y_i} e_J o_{\hat{J}}$$

for some $k_i \in K$. Hence, $h_i = \sum_{j=1}^{i-1} k_j c_{ij} + k_i$. Let $i = 1$. Then $h_1 = k_1$, an integer. Now use increasing induction on i to show k_i is an integer for all i, $1 \leqslant i \leqslant m$.

4.12 THEOREM. (1) *$Ho_{\hat{J}} e_J$ is a left ideal of H with K-basis $\{a_y o_{\hat{J}} e_J = a_y e_J : y \in Y_J\}$. Hence $\dim Ho_{\hat{J}} e_J = |Y_J|$. Let $Y_J = \{y_1, \ldots, y_s\}$, with $l(y_i) \leqslant l(y_j)$ if $i < j$, and let $H_{J,i} = \{\sum_{j=i}^{s} k_j a_{y_j} o_{\hat{J}} e_J : k_j \in K\}$; then*

$$Ho_{\hat{J}} e_J = H_{J,1} > H_{J,2} > \ldots > H_{J,s} > 0$$

is a composition series of $Ho_{\hat{J}} e_J$ of left H-modules, and $H_{J,i}/H_{J,i+1}$ affords the representation λ_M of H, where $y_i^{-1} \in Y_M$, and $H_{J,s+1} = 0$. Finally, $H = \sum_{J \subseteq R}^{\oplus} Ho_{\hat{J}} e_J$, a direct sum of 2^n left ideals, where $n = |R|$.

(2) *$He_J o_{\hat{J}}$ is a left ideal of H with K-basis $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$. Hence $\dim He_J o_{\hat{J}} = |Y_J|$. Let $Y_J = \{y_1, \ldots, y_s\}$, with $l(y_i) \leqslant l(y_j)$ if $i < j$, and let*

$$H_{J,i} = \left\{ \sum_{j=i}^{s} k_j a_{y_j} e_J o_{\hat{J}} : k_j \in K \right\};$$

then

$$He_J o_{\hat{J}} = H_{J,1} > H_{J,2} > \dots > H_{J,s} > 0$$

is a composition series of $He_J o_{\hat{J}}$ *of left H-modules, and* $H_{J,i}/H_{J,i+1}$ *affords the representation* λ_M *of H, where* $y_i^{-1} \in Y_M$, *and* $H_{J,s+1} = 0$. *Finally,* $H = \sum_{J \subseteq R}^{\oplus} He_J o_{\hat{J}}$, *a direct sum of* 2^n *left ideals, where* $n = |R|$.

PROOF. The results follow by 4.7, 4.8, 4.10 and the fact that

$$\dim H = |W| = \sum_{J \subseteq R} |Y_J|.$$

4.13 COROLLARY. $Ho_{\hat{J}} e_J$ *and* $He_J o_{\hat{J}}$ *are indecomposable left ideals of H, for all* $J \subseteq R$, *and they are isomorphic as left ideals of H.*

PROOF. From the theory of Artinian rings and the fact that H/N is a direct sum of 2^n irreducible components (see remarks at the end of Section 3), it follows that H can be expressed as the direct sum of 2^n indecomposable left ideals. Hence $Ho_{\hat{J}} e_J$ and $He_J o_{\hat{J}}$ must be indecomposable left ideals of H for all $J \subseteq R$.

To show they are isomorphic, first note that $He_J o_{\hat{J}} = Ho_{\hat{J}} e_J o_{\hat{J}}$. Then define the homomorphism $f_J: Ho_{\hat{J}} e_J \to He_J o_{\hat{J}}$ by $f_J(ao_{\hat{J}} e_J) = ao_{\hat{J}} e_J o_{\hat{J}}$, for all $ao_{\hat{J}} e_J \in Ho_{\hat{J}} e_J$. As f_J is given by right multiplication by $o_{\hat{J}}$, it is well defined and is a homomorphism of left ideals of H. f_J is onto, since $He_J o_{\hat{J}} = Ho_{\hat{J}} e_J o_{\hat{J}}$ and an element $ao_{\hat{J}} e_J o_{\hat{J}} \in He_J o_{\hat{J}}$ is the image under f_J of $ao_{\hat{J}} e_J$. f_J is one–one as $\dim Ho_{\hat{J}} e_J = \dim He_J o_{\hat{J}}$. Hence f_J is an isomorphism of left ideals of H.

4.14 COROLLARY. (1) *For any* $L \subseteq R$,

$$Ho_{\hat{L}} = \sum_{J \subseteq L}^{\oplus} Ho_{\hat{J}} e_J o_{\hat{L}}, \quad \text{and } \dim Ho_{\hat{L}} = \sum_{J \subseteq L} |Y_J| = |X_{\hat{L}}|.$$

(2) *For any* $L \subseteq R$,

$$He_L = \sum_{J \supseteq L}^{\oplus} He_J o_{\hat{J}} e_L, \quad \text{and } \dim He_L = \sum_{J \supseteq L} |Y_J| = |X_L|.$$

PROOF. Use 4.12 and 4.8.

4.15 THEOREM. *For any* $J \subseteq R$,

$$He_J = \{a \in H: aa_i = 0 \text{ for all } w_i \in J\}$$

$$= \{a \in H: a(1 + a_i) = a \text{ for all } w_i \in J\}.$$

Further, $He_J = \sum^{\oplus}_{J \subseteq L} Ho_{\hat{L}} e_L$, *and* He_J *has basis* $\{a_w e_J : w \in X_J\}$ *and dimension* $|X_J|$. *Finally,*

$$Ho_{\hat{J}} e_J = \{a \in H : aa_i = 0 \text{ for all } w_i \in J,\ ae_L = 0 \text{ for all } L \supset J\}$$

$$= He_J \cap \Big(\bigcap_{J \supset L} \ker e_L\Big),$$

where ker $e_L = \{a \in H : ae_L = 0\}$.

PROOF. Clearly, $He_J \leqslant \{a \in H : aa_i = 0$ for all $w_i \in J\}$. Conversely, take $a \in H$ and suppose $aa_i = 0$ for all $w_i \in J$. Then $a(1 + a_i) = a$ for all $w_i \in J$, and so $ae_J = a$, and so $a \in He_J$. Thus the first part is proved.

Now $Ho_{\hat{L}} e_L \leqslant He_J$ for all $L \supseteq J$, and so $\sum^{\oplus}_{L \supseteq J} Ho_{\hat{L}} e_L \leqslant He_J$. By 4.14, $\dim He_J = |X_J|$, and as $\dim Ho_{\hat{L}} e_L = |Y_L|$, we have $He_J = \sum^{\oplus}_{L \supseteq J} Ho_{\hat{L}} e_L$.

Let $a = \sum_{w \in W} u_w a_w \in He_J$, where $u_w \in K$. Let $w_i \in J$. Then $aa_i = 0$, and so $\sum_{w \in W} u_w a_w a_i = 0$. Now

$$\sum_{w \in W} u_w a_w a_i = \sum_{w \in W, w(r_i) \in \Phi^+} u_w a_{ww_i} - \sum_{w \in W, w(r_i) \in \Phi^-} u_w a_w = 0.$$

That is,

$$\sum_{w \in W, w(r_i) \in \Phi^-} u_{ww_i} a_w - \sum_{w \in W, w(r_i) \in \Phi^-} u_w a_w = 0.$$

Since $\{a_w : w \in W\}$ form a basis of H, we have $u_{ww_i} = u_w$ for all $w \in W$ with $w(r_i) \in \Phi^-$. Hence $u_w = u_{ww_i}$ for all $w \in W$, with $w(r_i) \in \Phi^+$. Now if $w \in W$, w can be expressed uniquely in the form $w = yw_J$, where $y \in X_J$, $w_J \in W_J$ and $l(w) = l(y) + l(w_J)$. Write $w_J = w_{i_1} \dots w_{i_t}$, $w_{i_j} \in J$, $l(w_J) = t$. By the above, we have

$$u_y = u_{yw_{i_1}} = \dots = u_{yw_J} = u_w.$$

Hence $a = \sum_{y \in X_J} u_y a_y e_J$. Conversely, for each $y \in X_J$, $a_y e_J \in He_J$, and as $\{a_y e_J : y \in X_J\}$ is linearly independent and $\dim He_J = |X_J|$, $\{a_y e_J : y \in X_J\}$ is a basis of He_J.

Finally, $Ho_{\hat{J}} e_J \leqslant \{a \in H : aa_i = 0$ for all $w_i \in J$, $ae_L = 0$ for all $L \supset J\}$. Let $a = \sum_L \sum_{y \in Y_L} u_y a_y o_{\hat{L}} e_L$, $u_y \in K$, satisfy $aa_i = 0$ for all $w_i \in J$ and $ae_L = 0$ for all $L \supset J$. Since $a \in He_J$, $u_y = 0$ for all $y \in Y_L$ if $J \nsubseteq L$. So $a = \sum_{L \supseteq J} \sum_{y \in Y_L} u_y a_y o_{\hat{L}} e_L$. Set $S_J = \{w \in W : u_w \neq 0,\ w \in Y_L,\ L \supset J\}$. Suppose $S_J \neq \varnothing$. Then there exists an element y_0 of minimal length in S_J; suppose $y_0 \in Y_M$, $M \supset J$. Then $ae_M = 0$. Also $o_{\hat{J}} e_J e_M = 0$ as $M \supset J$. For other $L \supset J$, if $y \in Y_L$,

$$a_y o_{\hat{L}} e_L e_M = a_y e_L e_M = a_y + (\text{a combination of certain } a_w,$$

$$w \in W, \text{ with } l(w) > l(y)).$$

Then $ae_M = 0$ gives $\sum_{L\supset J}\sum_{y\in Y_L} u_y a_y o_{\hat{L}} e_L e_M = 0$. As y_0 is of minimal length in S_J, the coefficient of a_{y_0} in the left-hand side of the last equation is u_{y_0}. By the linear independence of $\{a_w : w\in W\}$, we have $u_{y_0} = 0$, which is a contradiction. Hence $S_J = \emptyset$ and $a = \sum_{y\in Y_J} u_y a_y o_{\hat{J}} e_J \in Ho_{\hat{J}} e_J$. Thus

$$Ho_{\hat{J}} e_J = \{a\in He_J : ae_L = 0 \text{ for all } L\supset J\}.$$

4.16 THEOREM. *For any* $J\subseteq R$,

$$Ho_J = \{a\in H : a(1+a_i) = 0 \textit{ for all } w_i\in J\}.$$

Ho_J *has basis* $\{a_w : w\in Y_{\hat{L}}, \hat{L}\subseteq \hat{J}\}$, *dimension* $|X_J|$ *and* $Ho_J = \sum^{\oplus}_{L\supseteq J} He_{\hat{L}} o_L$. *Finally,* $He_{\hat{J}} o_J = \{a\in Ho_J : ao_L = 0 \textit{ for all } L\supset J\}$.

PROOF. Similar to the proof of 4.15.

4.17 LEMMA. *Let* ψ_J *be the character of the representation of* H *on* $Ho_{\hat{J}} e_J$. *Then* ψ_J *takes values as follows: for each* $w\in W$, *let* $w = w_{i_1}\dots w_{i_t}$ *be a reduced expression for* w, *and set* $J(w) = \{w_{i_j} : 1\leqslant j\leqslant t\}$. *Then* $\psi_J(a_w) = (-1)^{l(w)} N_J(w)$, *where* $N_J(w)$ = *the number of elements* $y\in Y_J$ *such that* $y^{-1}(\Pi_{J(w)})\subseteq \Phi^-$.

PROOF. Use 4.10.

4.18 LEMMA. *Let* ϕ_J *be the character of the representation of* H *on* He_J. *Then* ϕ_J *takes values as follows: for* $w\in W$ *let* $w = w_{i_1}\dots w_{i_t}$ *be a reduced expression for* w. *Set* $J(w) = \{w_{i_j} : 1\leqslant j\leqslant t\}$. *Then* $\phi_J(a_w) = (-1)^{l(w)} M_J(w)$, *where* $M_J(w)$ = *the number of elements* $x\in X_J$ *such that* $x^{-1}(\Pi_{J(w)})\subseteq\Phi^-$. *Also,* $M_J(w) = \sum_{L\supseteq J} N_L(w)$.

PROOF. He_J has basis $\{a_w e_J : w\in X_J\}$. For any $w_i\in R$,

$$a_i a_w e_J = \begin{cases} -a_w e_J & \text{if } w^{-1}(r_i) < 0,\\ a_{w_i w} e_J, & \text{where } w_i w\in X_J \text{ if } w^{-1}(r_i) > 0, \text{ and}\\ & \quad w^{-1}(r_i)\neq r_j \text{ for any } r_j\in\Pi,\\ 0 & \text{if } w^{-1}(r_i) = r_j \text{ for some } r_j\in\Pi_J, \text{ for then}\\ & \quad a_i a_w = a_w a_j \text{ and } a_j e_J = 0. \end{cases}$$

The result now follows.

4.19 LEMMA. *Let* μ_J *be the character of the representation of* H *on* Ho_J. *Then* μ_J *takes values as follows: for each* $w\in W$, *let* $w = w_{i_1}\dots w_{i_t}$ *be a reduced expression for* w, *and set* $J(w) = \{w_{i_j} : 1\leqslant j\leqslant t\}$. *Then* $\mu_J(a_w) = (-1)^{l(w)} L_J(w)$, *where* $L_J(w)$ = *the number of elements* $z\in Z_J$ *such that* $z^{-1}(\Pi_{J(w)})\subseteq\Phi^-$, *and* $Z_J = \{w\in W : w(\Pi_J)\subseteq\Phi^-\}$. *Note that* $Z_J = \sum_{L\subseteq\hat{J}} Y_L$.

PROOF. Ho_J has basis $\{a_w\colon w\in Z_J\}$. For all $w_i\in R$,

$$a_i a_w = \begin{cases} -a_w & \text{if } w^{-1}(r_i)<0,\\ a_{w_i w} & \text{if } w^{-1}(r_i)>0.\end{cases}$$

If $w\in Z_J$, $w_i\in R$ and $w^{-1}(r_i)>0$, then $w_i w\in Z_J$, for if $r_j\in\Pi_J$, $w(r_j)=-s$ for some $s\in\Phi^+$, and $w_i(s)<0$ if and only if $s=r_i$. But if $s=r_i$, $w^{-1}(r_i)=-r_j$—impossible. The result now follows.

4.20 COROLLARY. (1) $\phi_J=\Sigma_{J\supseteq L}\psi_L$ *for all* $J\subseteq R$.
(2) $\mu_J=\Sigma_{J\supseteq L}\psi_{\hat{L}}$ *for all* $J\subseteq R$.

A direct sum decomposition of H into indecomposable left ideals is equivalent to expressing the identity of H as a sum of mutually orthogonal primitive idempotents. Let $1=\Sigma_{J\subseteq R}q_J$ and $1=\Sigma_{J\subseteq R}p_J$ be the decompositions of 1 corresponding to the decompositions $H=\Sigma^{\oplus}_{J\subseteq R}Ho_{\hat{J}}e_J$ and $H=\Sigma^{\oplus}_{J\subseteq R}He_J o_{\hat{J}}$ respectively, where $Hq_J=Ho_{\hat{J}}e_J$ and $Hp_J=He_J o_{\hat{J}}$. (There does not appear to be a specific expression for the q_J or the p_J in terms of $\{a_y o_{\hat{J}} e_J\colon y\in Y_J\}$ or $\{a_y e_J o_{\hat{J}}\colon y\in Y_J\}$ respectively).

4.21 THEOREM. *Let* $\{q_J\colon J\subseteq R\}$ *be a set of mutually orthogonal primitive idempotents with* $q_J\in Ho_{\hat{J}}e_J$ *for all* $J\subseteq R$ *such that* $1=\Sigma_{J\subseteq R}q_J$. *Then* $Ho_{\hat{J}}e_J=Hq_J$, *and if* N *is the nilpotent radical of* H, $No_{\hat{J}}e_J=Nq_J$ *is the unique maximal left ideal of* Hq_J, *and* $Hq_J/Nq_J\cong K$. Hq_J/Nq_J *affords the representation* λ_J *of* H *defined in* 3.1. *Finally,*

$$H/N\cong \sum_{J\subseteq R}^{\oplus} Hq_J/Nq_J\cong K\oplus K\oplus\dots\oplus K,\quad 2^n \text{ summands, where } n=|R|.$$

PROOF. By the theory of Artinian rings, Nq_J is the unique maximal left ideal of Hq_J, and $H/N\cong\Sigma^{\oplus}_{J\subseteq R}Hq_J/Nq_J$. Since $q_J\in Ho_{\hat{J}}e_J$, $Hq_J\leqslant Ho_{\hat{J}}e_J$. As

$$H=\sum_{J\subseteq R}^{\oplus}Hq_J=\sum_{J\subseteq R}^{\oplus}Ho_{\hat{J}}e_J,$$

we must have $Hq_J=Ho_{\hat{J}}e_J$ for all $J\subseteq R$. Then $Nq_J=NHq_J=NHo_{\hat{J}}e_J=No_{\hat{J}}e_J$ is the unique maximal left ideal of Hq_J. But

$$\{\sum_{y\in Y_J,\,y\neq w_{0\hat{J}}} u_y a_y o_{\hat{J}} e_J\colon u_y\in K\}$$

is a maximal left ideal of $Ho_{\hat{J}}e_J$ (see 4.10), and so

$$Nq_J=\{\sum_{y\in Y_J,\,y\neq w_{0\hat{J}}} u_y a_y o_{\hat{J}} e_J\colon u_y\in K\}.$$

Then Hq_J/Nq_J is a one-dimensional H-module generated by $a_{w_{0\hat{J}}} o_{\hat{J}} e_J + Nq_J$ which affords the representation λ_J of H, and since every element of Hq_J/Nq_J is of the form $ka_{w_{0\hat{J}}} o_{\hat{J}} e_J + Nq_J$ for some $k \in K$, $Hq_J/Nq_J \cong K$ for all $J \subseteq R$. Hence the result.

4.22 THEOREM. *Let $\{p_J : J \subseteq R\}$ be a set of mutually orthogonal primitive idempotents with $p_J \in He_J o_{\hat{J}}$ for all $J \subseteq R$ such that $1 = \sum_{J \subseteq R} p_J$. Then $He_J o_{\hat{J}} = Hp_J$, and if N is the nilpotent radical of H, $Ne_J o_{\hat{J}} = Np_J$ is the unique maximal left ideal of Hp_J, and $Hp_J/Np_J \cong K$. Hp_J/Np_J affords the representation λ_J of H defined in* 3.1. *Finally, $H/N \cong \sum_{J \subseteq R}^{\oplus} Hp_J/Np_J \cong K \oplus K \oplus \ldots \oplus K$, 2^n summands, where $n = |R|$.*

4.23 LEMMA. *$\{ka_{w_0 w_{0J}} o_{\hat{J}} e_J : k \in K\}$ and $\{ka_{w_0 w_{0J}} e_J o_{\hat{J}} : k \in K\}$ are minimal submodules of $Ho_{\hat{J}} e_J$ and $He_J o_{\hat{J}}$ respectively, where $w_0 w_{0J}$ is the unique element of maximal length in Y_J. These minimal left ideals both afford the representation $\lambda_{\bar{J}}$ of H, where $\bar{J} = \{w_i \in R : \text{there exists } w_j \in J \text{ with } w_0 w_j = w_i w_0\}$, or, alternatively, $\Pi_{\bar{J}}$ is defined by $w_0(\Pi_J) = -\Pi_{\bar{J}}$.*

4.24 NOTE. By the same methods, $H = \sum_{J \subseteq R}^{\oplus} e_J o_{\hat{J}} H$ and $H = \sum_{J \subseteq R}^{\oplus} o_{\hat{J}} e_J H$, both being direct sum decompositions of H into 2^n right ideals, where $n = |R|$. Further, $e_J o_{\hat{J}} H$ has K-basis $\{e_J o_{\hat{J}} a_y : y^{-1} \in Y_J\}$, and $o_{\hat{J}} e_J H$ has K-basis $\{o_{\hat{J}} e_J a_y : y^{-1} \in Y_J\}$. All the results for the left ideals He_J, Ho_J, $He_J o_{\hat{J}}$ and $Ho_{\hat{J}} e_J$ have analogues for the right ideals $e_J H$, $o_J H$, $o_{\hat{J}} e_J H$ and $e_J o_{\hat{J}} H$ respectively.

Let G be a finite group with a split (B, N) pair of rank n and characteristic p with Weyl group W, and let K be a field of characteristic p. Then the above decomposition of $H = H_K(G, B)$ gives a decomposition of 1_B^G, where 1_B is the principal character of the subgroup B of G, which will be discussed in a later paper.

5. The Cartan matrix of H

We have that $H = \sum_{J \subseteq R}^{\oplus} U_J$, where $U_J = Ho_{\hat{J}} e_J$ is an indecomposable left H-module. Thus $\{U_J : J \subseteq R\}$ are the principal indecomposable H-modules. $\{U_J/\operatorname{rad} U_J : J \subseteq R\}$, where $\operatorname{rad} U_J$ is the unique maximal submodule of U_J, are irreducible H-modules, such that $M_J = U_J/\operatorname{rad} U_J$ affords the representation λ_J of H.

DEFINITION. The *Cartan matrix* C of H, where H is of type (W, R), with $|R| = n$, is a $2^n \times 2^n$ matrix with rows and columns indexed by the subsets of R, and if we write $C = (c_{JL})$, then

$$c_{JL} = \text{the number of times } M_L \text{ is a composition factor of } U_J.$$

5.1 THEOREM. *For all $J, L \subseteq R$,*

$$c_{JL} = |Y_J \cap (Y_L)^{-1}| = |Y_L \cap (Y_J)^{-1}| = c_{LJ}.$$

Hence C is a symmetric matrix.

PROOF. U_J has K-basis $\{a_y o_{\hat{J}} e_J = a_y e_J : y \in Y_J\}$. Let $y_1, \ldots, y_s$ be all the elements of Y_J written in order of increasing length; if $i > j$ then $l(y_i) \geqslant l(y_j)$. Then set $U_J(i) = \{\sum_{j \geqslant i} k_{y_j} a_{y_j} e_J : k_{y_j} \in K\}$. $U_J(i)$ is a left ideal of H for all i, and $U_J(i) > U_J(i+1)$ for all i, $1 \leqslant i \leqslant s-1$. Then $U_J = U_J(1) > U_J(2) > \ldots > U_J(s) > 0$ is a composition series of U_J, with $U_J(i)/U_J(i+1)$ being an irreducible H-module with basis $a_{y_i} e_J + U_J(i+1)$ and affording the irreducible representation λ_L, defined in 3.1, where L is determined as follows: recall 4.10; let $w_j \in R$ and $y_i \in Y_J$. Then

$$a_j a_{y_i} e_J = \begin{cases} -a_{y_i} e_J & \text{if } y_i^{-1}(r_j) < 0, \\ 0 & \text{if } y_i^{-1}(r_j) = r_k \text{ for some } r_k \in \Pi, \\ a_{w_j y_i} e_J & \text{where } w_j y_i = y_l \text{ for some } y_l \in Y_J \text{ with } i < l, \text{ if} \\ & \quad y_i^{-1}(r_j) > 0 \text{ but } y_i^{-1}(r_j) \neq r_k \text{ for any } r_k \in \Pi. \end{cases}$$

Hence

$$\lambda_L : a_j \to \begin{cases} -1 & \text{if } y_i^{-1}(r_j) < 0, \\ 0 & \text{if } y_i^{-1}(r_j) > 0. \end{cases}$$

That is, $y_i^{-1} \in Y_L$.

Hence c_{JL} = the number of elements $y \in Y_J$ such that $y^{-1} \in Y_L$

$$= |Y_J \cap (Y_L)^{-1}| = |Y_L \cap (Y_J)^{-1}|$$

since if $y \in Y_J \cap (Y_L)^{-1}$, then $y^{-1} \in Y_L \cap (Y_J)^{-1}$.

5.2 THEOREM. *Let H be the 0-Hecke algebra over the field K of type (W, R), where W is indecomposable. Then if $|R| > 1$, H has three blocks. If $|R| = 1$, then H has two blocks.*

PROOF. If $|R| = 1$, then $W = W(A_1)$ and $H = H(1 + a_1) \oplus H(-a_1)$, where $R = \{w_1\}$. Both $(1 + a_1)$ and $(-a_1)$ are primitive idempotents as well as being central. Hence H has only two blocks.

Now suppose that $|R| > 1$. $e_R = [1 + a_{w_0}]$ and $(-1)^{l(w_0)} a_{w_0}$ are primitive and centrally primitive idempotents in H and so correspond to two distinct blocks.

The other primitive idempotents in H, that is, $\{q_J: J \neq \emptyset, R\}$ as in 4.21, determine at least one other block. We will show that provided W is indecomposable the Cartan matrix C' corresponding to the indecomposables U_J for $J \neq \emptyset$, R and the irreducibles M_L for $L \neq \emptyset$, R cannot be expressed in the form $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ (see Dornhoff (1972), Theorem 46.3).

Suppose that C' can be put in the form above. Let

$$S_1 = \{J \subset R: U_J \text{ and } M_J \text{ index the rows and columns of } C_1\},$$

$$S_2 = \{J \subset R: U_J \text{ and } M_J \text{ index the rows and columns of } C_2\}.$$

Suppose for some $J \subset R$, $|J| = n-1$ (where $n = |R|$), that $J \in S_1$. Then we show

(1) for all $L \subset R$ with $|L| = n-1$, $L \in S_1$,

(2) by decreasing induction on $|J|$ for all $J \neq \emptyset$, R that $J \in S_1$.

(a) Suppose $J = \{w_1, \ldots, \hat{w}_j, \ldots, w_n\}$ and $L = \{w_1, \ldots, \hat{w}_{j+1}, \ldots, w_n\}$, where the nodes corresponding to w_j and w_{j+1} in the graph of W are joined. Then the order of $w_j w_{j+1}$ is greater than 2. Now $w_{0\hat{J}} = w_j \in Y_J$ and $w_{0\hat{L}} = w_{j+1} \in Y_L$. Since the order of $w_j w_{j+1}$ is greater than 2, $w_{j+1} w_j \in Y_J$ and $w_j w_{j+1} \in Y_L$; that is, $w_{j+1} w_j \in Y_J \cap (Y_L)^{-1}$. Hence $J \in S_1$ if and only if $L \in S_1$.

Hence if there is some $J \in S_1$, with $|J| = n-1$, then all $L \subset R$ with $|L| = n-1$ are in S_1 by the above.

(b) Suppose that for all $J \subset R$ with $|J| > m$ that $J \in S_1$. Choose $L \subset R$ with $|L| = m$. We show $L \in S_1$. Suppose $L = \{w_{i_1}, \ldots, w_{i_m}\}$ with $1 \leqslant i_1 < \ldots < i_m \leqslant n$. Since W is indecomposable and $L \neq \emptyset$, R, then $|Y_L| > 1$. Choose some $w_{i_j} \in L$ and $w_k \in \hat{L}$ such that $w_{i_j} w_k$ has order r, where $r \geqslant 3$. Then $w_{i_j} w_{0\hat{L}} \in Y_L$ (as $w_{0\hat{L}}(r_{i_j}) \neq r_i$ for any $r_i \in \Pi_L$, for $w_{0\hat{L}}(r_{i_j}) = r_i$ for some $r_i \in \Pi_L$ implies that $r_{i_j} = r_i$ and $w_{0\hat{L}}$ is a product of reflections corresponding to roots orthogonal to r_{i_j}, and so for all $w_k \in \hat{L}$, $w_{i_j} w_k = w_k w_{i_j}$, which is a contradiction). Now consider $(w_{i_j} w_{0\hat{L}})^{-1} = w_{0\hat{L}} w_{i_j}$. Then suppose $w_{i_l} \in L$, $w_{i_l} \neq w_{i_j}$. Then $w_{0\hat{L}} w_{i_j}(r_{i_l}) \in \Phi^+$. Also $w_{0\hat{L}} w_{i_j}(r_{i_j}) \in \Phi^-$. Suppose $w_k \in \hat{L}$. Then

$$\begin{aligned} w_{0\hat{L}} w_{i_j}(r_k) &= w_{0\hat{L}}(r_k + u r_{i_j}) \quad \text{with } u \geqslant 0 \\ &= w_{0\hat{L}}(r_k) + u w_{0\hat{L}}(r_{i_j}). \end{aligned}$$

If $u = 0$, that is, if $w_{i_j} w_k = w_k w_{i_j}$, then $w_{0\hat{L}} w_{i_j}(r_k) \in \Phi^-$. If $u > 0$, as $w_{0\hat{L}}(r_k) = -r_i$ for some $r_i \in \Pi_{\hat{L}}$, and $w_{0\hat{L}}(r_{i_j}) \in \Phi^+$, $w_{0\hat{L}}(r_{i_j}) \neq r_{i_s}$ for any $r_{i_s} \in \Pi_L$, we have $w_{0\hat{L}} w_{i_j}(r_k) \in \Phi^+$. Hence $w_{0\hat{L}} w_{i_j} \in Y_M$, where

$$\begin{aligned} M &= \{L - \{w_{i_j}\}\} \cup \{w_k \in \hat{L}: w_{i_j} w_k \text{ has order } > 2\} \\ &= \{L - \{\{w_{i_j}\}\} \cup \{w_k \in \hat{L}: \text{the node corresponding to } w_k \text{ in the graph of} \\ &\qquad W \text{ is joined to that corresponding to } w_{i_j}\}. \end{aligned}$$

Now $|M|>|L|$ if the node corresponding to w_{i_j} is joined to at least two nodes corresponding to elements of $\hat{L}$, and then $L\in S_1$ by induction.

Let P_i be the node of the graph of W which corresponds to $w_i\in R$, $1\leqslant i\leqslant n$. Then suppose P_{i_j} is joined to only one P_k for all $w_k\in\hat{L}$. Then the above argument shows that $L=\{w_{i_1},\ldots,w_{i_m}\}$ and $M=\{w_{i_1},\ldots,\hat{w}_{i_j},\ldots,w_{i_m},w_k\}$ belong to the same S_i, where $i=1$ or $i=2$. Since $|L|\leqslant n-2$, $|\hat{L}|\geqslant 2$. Let w_{k_1} and w_{k_2} be any two elements of $\hat{L}$, such that there exists a sequence $P_{k_1}=P_{j_0},P_{j_1},\ldots,P_{j_r}=P_{k_2}$ of nodes such that P_{j_i} and $P_{j_{i+1}}$ are joined for all i, $0\leqslant i\leqslant r-1$, and P_{j_i} corresponds to an element of L for all i, $1\leqslant i\leqslant r-1$. If $r=1$, then P_{k_1} and P_{k_2} are joined. Without loss of generality, we may suppose there exists $w_{i_s}\in L$ such that P_{i_s} is joined to P_{k_1}. Then let $M=\{L-\{w_{i_s}\}\}\cup\{w_{k_1}\}$. M and L belong to the same S_i, and by the above, as M has an element w_{k_1} such that $w_{k_1}w_{i_s}$ and $w_{k_1}w_{k_2}$ both have order >2, where $w_{i_s},w_{k_2}\in\hat{M}$, $w_{i_s}\neq w_{k_2}$, then $M\in S_1$. If $r=2$, then L and M are in the same S_i, where $M=\{L-\{\{w_{j_1}\}\}\cup\{w_{k_1},w_{k_2}\}$, and by induction $M\in S_1$. If $r>2$, define

$$L_0=L,$$

$$L_1=\{L-\{w_{j_1}\}\}\cup\{w_{j_0}\},$$

$$\ldots$$

$$L_{r-2}=\{L_{r-3}-\{w_{j_{r-2}}\}\}\cup\{w_{j_{r-3}}\}.$$

Then $L_0,L_1,\ldots,L_{r-2}$ are all in the same S_i, and by the above, $L_{r-2}\in S_1$.

Hence $L\in S_1$. Then $S_2=\varnothing$, and so H has precisely three blocks.

5.3 Theorem. *Let H be a 0-Hecke algebra of type (W,R). Suppose W is decomposable, and let $W=W_1\times W_2\times\ldots\times W_r$, where each W_i is an indecomposable Coxeter group, and the corresponding Coxeter system is (W_i,R_i). Let H_i be the 0-Hecke algebra of type (W_i,R_i), and let m_i be the number of blocks of H_i. Then H has $m_1m_2\ldots m_r$ blocks.*

Proof. Suppose that $1=\sum_{i=1}^{t}e_i$ where the e_i are mutually orthogonal centrally primitive idempotents in H. Then the number of blocks of H is equal to t.

Now for all $w\in W_i$, $w'\in W_j$, where $1\leqslant i,j\leqslant r$ and $i\neq j$, we have that

$$a_w a_{w'}=a_{ww'}=a_{w'w}=a_{w'}a_w,$$

and so it follows that if f_i is a centrally primitive idempotent of H_i, then $f_1\ldots f_r$ is a centrally primitive idempotent of H. Suppose $1_{H_i}=\sum_{j=1}^{t(i)}f_{ij}$ where for a fixed i, $\{f_{ij}\colon 1\leqslant j\leqslant t(i)\}$ is a set of mutually orthogonal central primitive idempotents in H_i. Then $1_H=\sum_{j_1=1}^{t(1)}\cdots\sum_{j_r=1}^{t(r)}f_{1j_1}\cdots f_{rj_r}$, a sum of mutually orthogonal central primitive idempotents in H, and so H has $t(1)t(2)\ldots t(r)$ blocks, where $t(i)=m_i$.

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