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The Quiver of an Algebra Associated to the Mantaci-Reutenauer Descent Algebra and the Homology of Regular Semigroups

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Abstract We develop the homology theory of the algebra of a regular semigroup, which is a particularly nice case of a quasi-hereditary algebra in good characteristic. Directedness is characterized for these algebras, generalizing the case of semisimple algebras studied by Munn and Ponizovksy. We then apply homological methods to compute (modulo group theory) the quiver of a right regular band of groups, generalizing Saliola's results for a right regular band. Right regular bands of groups come up in the representation theory of wreath products with symmetric groups in much the same way that right regular bands appear in the representation theory of finite Coxeter groups via the Solomon-Tits algebra of its Coxeter complex. In particular, we compute the quiver of Hsiao's algebra, which is related to the Mantaci-Reutenauer descent algebra.

Keywords Quivers · Descent algebras · Regular semigroups · Representation theory

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1 Introduction

The algebras of (von Neumann) regular semigroups in good characteristic form a wide and natural class of quasi-hereditary algebras. Although this class of semigroup algebras may be unfamiliar to many representation theorists, they have surfaced in a number of papers over the past ten years [1, 8–10, 25, 28, 34–37, 40]. Although the fact that these are quasi-hereditary algebras was first pointed out quite late in the game by Putcha [35], in essence many of the properties of quasi-hereditary algebras were discovered quite early on in semigroup theory in this setting. For instance, the Munn-Ponizovsky description of the simple modules [12, 39] are exactly via construction of the standard modules and taking the maximal irreducible constituent in the partial order; the co-standard modules appear in the work of Rhodes and Zalcstein [39] (which was written in the 1960s); Nico [30, 31] computed early on the bound on the global dimension that one would get from the theory of quasi-hereditary algebras. The near matrix algebras of Du and Lin [18] are essentially the same thing as Munn algebras, introduced by W. D. Munn to study semigroup algebras [12, 39]; in the terminology of [18] semigroups algebras of regular semigroups have a bi-free standard system. In fact, regular semigroup algebras have a canonical quasi-hereditary structure coming from their semigroup structure via a principal series. Moreover, the associated semisimple algebras in the quasi-hereditary structure are group algebras over maximal subgroups. So the whole quasi-hereditary structure is already there at the semigroup level.

Just as ordinary group representation theory does not end immediately after observing that group algebras are semisimple, one should not close the door on the representation theory of regular semigroups after observing their algebras are quasi-hereditary. By passing to the algebra, without remembering the distinguished basis coming from the semigroup, one loses the information that is of interest to a semigroup theorist. With this philosophy in mind, we do not even give the formal definition of a quasi-hereditary algebra so that workers in semigroups and combinatorics who need to deal with semigroup algebras do not have to assimilate a number of technical definitions from the theory of finite dimensional algebras, which in the context of regular semigroups are quite clear. In this paper, we take very much the traditional viewpoint in semigroup theory that we want to answer questions modulo group theory. In particular, we consider, say, the quiver of a semigroup algebra to be computed if we can determine the vertices and arrows modulo being able to compute any representation theoretic fact we need concerning finite groups. Over algebraically closed fields of characteristic 0, this assumption is not unreasonable.

Throughout, we make extensive usage of homological methods rather than ring theoretic methods. This is because regular semigroup algebras behave extremely well homologically, especially with respect to taking quotients coming from semigroup ideals, whereas in general it is almost impossible to write down explicitly primitive idempotents for these algebras.

As an application of our techniques, we compute the quiver of a right regular band of groups. A right regular band is a semigroup satisfying the identities $x^2 = x$ and $xyx = yx$. The faces of a central hyperplane arrangement have the structure of a right regular band, something that was taken advantage of by Bidigare et al. [8] and

Brown [9, 10] to compute spectra of random walks on hyperplane arrangements. See [9, 10] for further examples of applications of right regular band algebras to probability. Bidigare also discovered that if one takes the reflection arrangement associated to a finite reflection group W, then the W-invariants of the algebra of the associated right regular band is precisely Solomon's descent algebra; see [10] for details. This led Aguiar et al. [1] to develop an approach to the representation theory of finite Coxeter groups via right regular bands. Saliola computed the quiver of a right regular band algebra and the projective indecomposables [40]; for the former he used homological methods, whereas in the latter case he computed primitive idempotents. Actually, all these papers consider the dual notion of left regular bands because they work with left modules.

A right regular band of groups is a regular semigroup in which each right ideal is two-sided. Intuitively, these are semigroups with a grading by a right regular band so that each homogeneous component is a group. Hsiao [24] associates a right regular band of groups $\mathcal{F}_n^G$ to each finite group G. The symmetric group S_n acts by automorphisms on the algebra of $\mathcal{F}_n^G$ and in the case G is abelian, Hsiao identifies the invariant algebra with the Mantaci-Reutenauer descent algebra [29] for the wreath product $G \wr S_n$; in the case G is non-abelian, he identifies the invariant algebra with an algebra recently introduced independently by Baumann and Hohlweg [6] and by Novelli and Thibon [32]. In this paper, we use homological methods to compute the quiver and the projective indecomposables for the algebra of a right regular band of groups in good characteristic. As an example, we compute the quiver for the algebra of Hsiao's semigroup $\mathcal{F}_n^G$ [24].

2 Preliminaries

The reader is referred [38, Appendix A] or [12, 27] for background on finite semigroups. Let S be a finite monoid (in this paper, all monoids and groups are assumed to be finite). Then S is called *regular* if, for all $s \in S$, there exists $t \in S$ with $sts = s$. Notice that st, ts are idempotents so in particular every principal left, right and two-sided ideal of a regular semigroup is generated by an idempotent. Green's preorders are defined by

- $s \leq_{\mathscr{J}} t$ if $SsS \subseteq StS$;
- $s \leq_{\mathscr{R}} t$ if $sS \subseteq tS$;
- $s \leq_{\mathscr{L}} t$ if $Ss \subseteq St$.

We write $s \mathscr{J} t$ if $SsS = StS$. Similar notation is used for $\mathscr{R}$ and $\mathscr{L}$. One writes $s \mathscr{H} t$ if $s \mathscr{L} t$ and $s \mathscr{R} t$. Of course $\leq_{\mathscr{J}}$ descends to a partial order on $S/\mathscr{J}$.

The set of idempotents of S is denoted $E(S)$. If e is an idempotent, we write G_e for the group of units of the monoid eSe; equivalently G_e is the $\mathscr{H}$-class of e. It is called the *maximal subgroup* of S at e. The following fact about finite semigroups is crucial [38, Appendix A].

Proposition 2.1 *Let S be a finite monoid and $e, f \in E(S)$ be $\mathscr{J}$-equivalent idempotents, i.e., $SeS = SfS$. Then $eSe \cong fSf$ and hence $G_e \cong G_f$. Moreover, eS, fS (Se, Sf) are isomorphic right (resp. left) S-sets.*

Another important property of finite semigroups is stability [38], which states that strictly comparable principal right (left) ideals cannot generate the same two-sided ideal.

Proposition 2.2 *Let s, x belong to a finite monoid S. Then*

$$sx \mathrel{\mathscr{J}} s \iff sx \mathrel{\mathscr{R}} s \text{ and } xs \mathrel{\mathscr{J}} s \iff xs \mathrel{\mathscr{L}} s.$$

It follows easily from this that if $e \in E(S)$ and J is the $\mathscr{J}$-class of e, then $J \cap eSe = G_e$, $eS \cap J$ is the $\mathscr{R}$-class of e and $Se \cap J$ is the $\mathscr{L}$-class of e. We use these and other consequences of stability throughout the paper without comment.

3 The Homological Theory of Regular Monoids

In this section we begin by studying homological aspects of the algebra of a regular monoid. Fix a field k and a regular monoid S. In characteristic zero, many of the results we present here were deduced by Putcha [35] as a consequence of the algebra being quasi-hereditary [13, 17]; see also [25, 28, 30, 31] for results on global dimension. In any characteristic, it is easy to see that kS is stratified in the sense of [14] via a principal series. However, things are better behaved in general for regular semigroups and there is no real need to work with principal series. Moreover, our basic philosophy is to reduce things to computations with groups. For these reasons, we provide complete proofs of results that can be deduced via other methods.

Let $J_1, \ldots, J_n$ be the collection of $\mathscr{J}$-classes of S. Assume that we have ordered them so that $J_i \leq_{\mathscr{J}} J_\ell$ implies $i \leq \ell$. Choose idempotents $e_1, \ldots, e_n$ with $e_i \in J_i$ and let G_i be the maximal subgroup at e_i. Define

$$J_i{}^{\downarrow} = \{s \in S \mid s <_{\mathscr{J}} e_i\}$$
$$J_i{}^{\Uparrow} = \{s \in S \mid s \ngeq_{\mathscr{J}} e_i\}.$$

Both $J_i{}^{\downarrow}$ and $J_i{}^{\Uparrow}$ are ideals of S. Notice that $J_i{}^{\downarrow} \subseteq J_i{}^{\Uparrow}$ and $e_iJ_i{}^{\downarrow} = e_iJ_i{}^{\Uparrow}$ (and dually).

A key property of regular semigroups is that if I is an ideal, then $I^2 = I$ since if $a \in I$ and $aba = a$, then $ba \in I$ and so $a = aba \in I^2$.

If A is an algebra, mod-A will denote the category of finitely generated right A-modules. The description of the simple modules for a finite semigroup are well known, see for instance [12, 20, 34, 35, 39]. We follow here the presentation and ideas of [20], which is the shortest and easiest accounting. First note that by stability, $e_i(kS/kJ_i{}^{\downarrow})e_i \cong kG_i$. For each $i = 1, \ldots, n$, define functors

$$\mathrm{Ind}_i, \mathrm{Coind}_i \colon \text{mod-}kG_i \to \text{mod-}kS/kJ_i{}^{\Uparrow} \subseteq \text{mod-}kS/kJ_i{}^{\downarrow}$$

by

$$\mathrm{Ind}_i(V) = V \otimes_{kG_i} e_i kS/kJ_i{}^{\Uparrow} = V \otimes_{kG_i} e_i kS/kJ_i{}^{\downarrow}$$

$$\mathrm{Coind}_i(V) = \mathrm{Hom}_{kG_i}\left(\left(kS/kJ_i{}^{\Uparrow}\right) e_i, V\right) = \mathrm{Hom}_{kG_i}\left(\left(kS/kJ_i{}^{\downarrow}\right) e_i, V\right).$$

These functors are exact and are the respective left and right adjoints of the restriction functor $M \mapsto Me_i$ from mod-$kS/kJ_i{}^{\downarrow} \to$ mod-kG_i (in fact $e_i kS/kJ_i{}^{\downarrow}$ and $(kS/kJ_i{}^{\downarrow})e_i$ are free kG_i-modules since G_i acts freely on $e_i S \cap J$ and dually). Also $\mathrm{Ind}_i(V)e_i \cong V \cong \mathrm{Coind}_i(V)e_i$. The functor Ind_i preserves projectivity and the functor Coind_i preserves injectivity as functors to mod-$kS/kJ_i{}^{\downarrow}$ (but not in general to mod-kS). Both functors preserve indecomposability.

If V is a simple kG_i-module, then it is known that $\mathrm{Ind}_i(V)$ has a unique maximal submodule $\mathrm{rad}(\mathrm{Ind}_i(V))$, which is in fact the largest submodule annihilated by e_i (or equivalently is the submodule of elements annihilated by J_i). The quotient $\widetilde{V} = \mathrm{Ind}_i(V)/\mathrm{rad}(\mathrm{Ind}_i(V))$ is then a simple kS-module and can be characterized as the unique simple kS-module M such that:

(1) e_i is $\leq_{\mathscr{J}}$-minimal with $Me_i \neq 0$;
(2) $Me_i \cong V$ as kG_i-modules.

Also one can show that $\widetilde{V}$ is the socle of $\mathrm{Coind}_i(V)$ and can be described as $\mathrm{Coind}_i(V)e_i kS$. One calls J_i the *apex* of the simple kS-module $\widetilde{V}$. It is known that every simple module for kS has an apex, i.e., is of the form $\widetilde{V}$ for a unique i and a unique simple kG_i-module V. See [20]. It is convenient to put a partial order on the simple kS-modules by setting $V \leq U$ if $V = U$ or the apex of U is strictly $\mathscr{J}$-below the apex of V. Assume that the characteristic of k does not divide the order of any maximal subgroup of S. Then we call this the *canonical quasi-hereditary structure* on kS. One can show that kS is indeed quasi-hereditary [13, 17, 35] with respect to this partial ordering and that the modules of the form $\mathrm{Ind}_i(V)$ are the standard modules, whereas the modules $\mathrm{Coind}_i(V)$ are the co-standard modules [35].

If A is an algebra and I an ideal, then mod-A/I is a full subcategory of mod-A. The inclusion has left and right adjoints given by $M \mapsto M \otimes_A A/I$ and $M \mapsto \mathrm{Hom}_A(A/I, M)$ respectively. If MI denotes the submodule of M generated by all elements ma with $m \in M$ and $a \in I$, then $M \otimes_A A/I \cong M/MI$. On the other hand $\mathrm{Hom}_A(A/I, M)$ consists of those elements of M annihilated by I. It is easily verified that the left adjoint preserves projectivity, the right adjoint preserves injectivity and both are the identity on mod-A/I [5, 7]. If I is an ideal of a semigroup S and M is a kS-module, we write MI instead of MkI to ease notation.

To compute the quiver of kS, we would like to work with the induced and coinduced modules rather than with projective covers, which we do not know how to compute in general. We can do this thanks to the following well-known lemma concerning idempotent ideals [5], which we prove for completeness.

Lemma 3.1 *Let I be an idempotent ideal of an algebra A and suppose that $M, N \in$ mod-A/I. Then*

$$\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_{A/I}(M, N).$$

Proof Let P be a projective A-module. Then the exact sequence of A-modules

$$0 \longrightarrow PI \longrightarrow P \longrightarrow P/PI \longrightarrow 0$$

gives rise to the exact sequence

$$\mathrm{Hom}_A(PI, N) \longrightarrow \mathrm{Ext}^1_A(P/PI, N) \longrightarrow 0.$$

But $\mathrm{Hom}_A(PI, N) \cong \mathrm{Hom}_{A/I}(PI/PI^2, N) = 0$ as $I^2 = I$. It follows that $\mathrm{Ext}^1_A(P/PI, N) = 0$.

Now we can find a short exact sequence of A/I-modules

$$0 \longrightarrow K \longrightarrow (A/I)^m \longrightarrow M \longrightarrow 0.$$

By the above, $\mathrm{Ext}^1_A((A/I)^m, N) = 0 = \mathrm{Ext}^1_{A/I}((A/I)^m, N)$. Using long exact Ext-sequences, both $\mathrm{Ext}^1_A(M, N)$ and $\mathrm{Ext}^1_{A/I}(M, N)$ can be identified as the cokernel of the map $\mathrm{Hom}_A((A/I)^m, N) \to \mathrm{Hom}_A(K, N)$. This proves the lemma. □

The following lemma is from [5].

Lemma 3.2 *Let A be an algebra and I an idempotent ideal that is projective as a right A-module. Then, for any A/I-modules M, N, there is an isomorphism* $\mathrm{Ext}^n_A(M, N) \cong \mathrm{Ext}^n_{A/I}(M, N)$ *all $n \geq 0$.*

Proof First using the previous lemma, the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

and the projectivity of I and A, we obtain $\mathrm{Ext}^n_A(A/I, N) = 0$ for $n \geq 1$ from the long exact Ext-sequence. Indeed, the case $n = 1$ follows from Lemma 3.1. In general, we have an exact sequence

$$0 = \mathrm{Ext}^n_A(I, N) \longrightarrow \mathrm{Ext}^{n+1}_A(A/I, N) \longrightarrow \mathrm{Ext}^{n+1}_A(A, N) = 0.$$

The lemma is now proved by induction on n, the case $n = 0$ being trivial and the case $n = 1$ following from Lemma 3.1. Suppose the lemma holds for n. Again choose a short exact sequence of A/I-modules

$$0 \longrightarrow K \longrightarrow (A/I)^m \longrightarrow M \longrightarrow 0.$$

The long exact Ext-sequence and what we just proved then yield dimension shifts $\mathrm{Ext}^n_A(K, N) \cong \mathrm{Ext}^{n+1}_A(M, N)$ and $\mathrm{Ext}^n_{A/I}(K, N) \cong \mathrm{Ext}^{n+1}_{A/I}(M, N)$. Application of the inductive hypothesis to K completes the proof. □

Our next lemma is a variant on a result of [5] where filtrations are considered. If J is a $\mathscr{J}$-class and R is an $\mathscr{R}$-class of S contained in J, we can make kJ and kR into kS-modules by identifying them with the isomorphic vector spaces $(kJ + kJ^{\uparrow})/kJ^{\uparrow}$ and $(kR + kJ^{\uparrow})/kJ^{\uparrow}$, respectively.

Lemma 3.3 *Let I be an ideal of the regular monoid S. Then we have* $\mathrm{Ext}^n_{kS}(M, N) \cong \mathrm{Ext}^n_{kS/kI}(M, N)$ *for any kS/kI-modules M, N and $n \geq 0$.*

Proof We proceed by induction on the number of $\mathscr{J}$-classes of I. For convenience, we allow in this proof $I = \emptyset$, in which case the conclusion is vacuous. Suppose it is true when the ideal has m $\mathscr{J}$-classes. Let I be an ideal with $m + 1$ $\mathscr{J}$-classes and let J be a maximal $\mathscr{J}$-class of I. Then $I' = I \setminus J$ is an ideal of S with m $\mathscr{J}$-classes. Let M, N be kS/kI-modules. Then they are also kS/kI'-modules and so by induction $\mathrm{Ext}^n_{kS}(M, N) \cong \mathrm{Ext}^n_{kS/kI'}(M, N)$ for all n. Set $A = kS/kI'$ and $C = kI/kI'$. Then $A/C \cong kS/kI$ and C is an idempotent ideal of A. Moreover, $C \cong kJ$ and if e is an idempotent of J, then $eA = ekJ$ is projective. Observe that $ekJ = kR$, where R is the $\mathscr{R}$-class of e. Now Green's Lemma [12, 21, 27, 38] implies kJ is isomorphic to a direct sum of copies of kR as an A-module, one for each $\mathscr{L}$-class of J, and so $C \cong kJ$ is projective. Lemma 3.2 then yields $\mathrm{Ext}^n_{kS/kI'}(M, N) \cong \mathrm{Ext}^n_{kS/kI}(M, N)$, for all n, completing the proof. □

In the terminology of [5], this result says that kI is a strong idempotent ideal of kS. Let us state a variant of the well-known Eckmann-Shapiro lemma from homological algebra [7, 23]. We sketch the idea of the proof.

Lemma 3.4 (Eckmann-Shapiro) *Let A be an algebra, $e \in A$ an idempotent and B a subalgebra of eAe. Let M be a B-module and N an A-module. If eA is a flat left B-module, then* $\mathrm{Ext}^n_A(M \otimes_B eA, N) \cong \mathrm{Ext}^n_B(M, Ne)$. *If Ae is a projective right B-module,* $\mathrm{Ext}^n_A(N, \mathrm{Hom}_B(Ae, M)) \cong \mathrm{Ext}^n_B(Ne, M)$. *Moreover, these isomorphisms are natural.*

Proof Since the functor $(-) \otimes_B eA$ is exact (by flatness) and preserves projectives (being left adjoint of the exact functor $V \mapsto Ve$), it takes a projective resolution of a B-module M to a projective resolution of the A-module $M \otimes_B eA$. Applying the functor $\mathrm{Hom}_A(-, N)$ to this projective resolution of $M \otimes_B eA$ and using the adjunction gives an isomorphism of the chain complexes computing the Ext-vector spaces $\mathrm{Ext}^n_A(M \otimes_B eA, N)$ and $\mathrm{Ext}^n_B(M, Ne)$. The second isomorphism is proved similarly. □

As a consequence we obtain the following natural isomorphisms stemming from induction and coinduction.

Lemma 3.5 *Let M be a $kS/kJ_i{}^{\downarrow}$-module and V a kG_i-module, then*

$$\mathrm{Ext}^n_{kS}(\mathrm{Ind}_i(V), M) \cong \mathrm{Ext}^n_{kG_i}(V, Me_i)$$

$$\mathrm{Ext}^n_{kS}(M, \mathrm{Coind}_i(V)) \cong \mathrm{Ext}^n_{kG_i}(Me_i, V)$$

for all $n \geq 0$. Moreover, the isomorphisms are natural. Consequently, the global dimension of kG_i is bounded by the global dimension of kS.

Proof We handle just the first isomorphism as the second is dual. By Lemma 3.3, $\mathrm{Ext}^n_{kS}(\mathrm{Ind}_i(V), M) \cong \mathrm{Ext}^n_{kS/kJ_i{}^{\downarrow}}(\mathrm{Ind}_i(V), M)$. Since $e_i kS/kJ_i{}^{\downarrow}$ is a free left kG_i-module, the Eckmann-Shapiro Lemma implies

$$\mathrm{Ext}^n_{kS/kJ_i{}^{\downarrow}}(\mathrm{Ind}_i(V), M) \cong \mathrm{Ext}^n_{kG_i}(V, Me_i).$$

The final statement is clear since, for any kG_i-module W,

$$\mathrm{Ext}^n_{kG_i}(V, W) = \mathrm{Ext}^n_{kG_i}(V, \mathrm{Ind}_i(W)e_i) \cong \mathrm{Ext}^n_{kS}(\mathrm{Ind}_i(V), \mathrm{Ind}_i(W)).$$

This completes the proof. □

Since a group algebra is well known to have finite global dimension if and only if the characteristic of the field does not divide the order of the group, it follows that if kS has finite global dimension then the characteristic of the field does not divide the order of any maximal subgroup of S. In fact, Nico [30, 31] showed that kS has finite global dimension if and only if the characteristic of k does not divide the order of any maximal subgroup. More precisely he proved the following theorem.

Theorem 3.6 (Nico) *Let S be a regular monoid and suppose that the characteristic of k does not divide the order of any maximal subgroup of S. If J is a $\mathscr{J}$-class, define*

$$\sigma(J) = \begin{cases} 0 & kJ^0 \text{ has an identity} \\ 1 & kJ^0 \text{ has a one-sided identity only} \\ 2 & \text{else.} \end{cases}$$

If $\mathscr{C}$ is a chain of $\mathscr{J}$-classes, define $\tau(\mathscr{C}) = \sum_{J\in\mathscr{C}} \sigma(J)$. Then the global dimension is bounded by the maximum of $\tau(\mathscr{C})$ over all chains of $\mathscr{J}$-classes of S. In particular, it is bounded by $2(m-1)$ where m is the length of the longest chain of non-zero $\mathscr{J}$-classes of S.

Here J^0 is the semigroup with underlying set $J \cup \{0\}$ with multiplication given, for $x, y \in J$, by

$$x \cdot y = \begin{cases} xy & xy \in J \\ 0 & \text{else} \end{cases}$$

and where 0 is a multiplicative zero. The final statement of the theorem can also be obtained from the general theory of quasi-hereditary algebras [13, 17, 35].

The following theorem will allow us to describe to some extent the quiver of a regular monoid. One could derive at least a part of it from the theory of stratified algebras [14]. For the characteristic zero case, Putcha [35] deduced some of these results from the theory of quasi-hereditary algebras.

Theorem 3.7 *Let S be a regular monoid and k a field. Suppose $\widetilde{U}$ and $\widetilde{V}$ are simple kS-modules with apexes J_i, J_ℓ, respectively. Let $N = \mathrm{rad}(\mathrm{Ind}_i(U))$ and $N' = \mathrm{Coind}_i(U)/\widetilde{U}$. Then:*

(1) *If J_i, J_ℓ are $\leq_{\mathscr{J}}$-incomparable, then $\mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V}) = 0$;*
(2) *If $J_i <_{\mathscr{J}} J_\ell$, then*

$$\begin{aligned}\mathrm{Ext}^1_{kS}\left(\widetilde{U}, \widetilde{V}\right) &\cong \mathrm{Hom}_{kS/kJ_\ell{}^{\downarrow}}\left(N/NJ_\ell{}^{\downarrow}, \widetilde{V}\right) \\ &\cong \mathrm{Hom}_{kG_\ell}\left(\left[\left(N/NJ_\ell{}^{\downarrow}\right)/\mathrm{rad}\left(N/NJ_\ell{}^{\downarrow}\right)\right]e_\ell, V\right);\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ext}^1_{kS}\left(\widetilde{V}, \widetilde{U}\right) &\cong \mathrm{Hom}_{kS/kJ_\ell{}^{\downarrow}}\left(\widetilde{V}, \mathrm{Hom}_{kS}\left(kS/kJ_\ell{}^{\downarrow}, N'\right)\right) \\ &\cong \mathrm{Hom}_{kG_\ell}\left(V, \mathrm{Soc}\left(\mathrm{Hom}_{kS}\left(kS/kJ_\ell{}^{\downarrow}, N'\right)\right)e_\ell\right);\end{aligned}$$

(3) *If $J_i = J_\ell$, then $\mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V})$ embeds in $\mathrm{Ext}^1_{kG_i}(U, V)$, and in particular is 0 if the characteristic of k does not divide $|G_i|$.*

Proof Suppose that $J_i \not\geq_{\mathscr{J}} J_\ell$. Then the long exact Ext-sequence derived from the short exact sequence $0 \to N \to \mathrm{Ind}_i(U) \to \widetilde{U} \to 0$ yields that

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{kS}\left(\widetilde{U}, \widetilde{V}\right) \longrightarrow \mathrm{Hom}_{kS}\left(\mathrm{Ind}_i(U), \widetilde{V}\right) \longrightarrow \mathrm{Hom}_{kS}\left(N, \widetilde{V}\right) \\ &\longrightarrow \mathrm{Ext}^1_{kS}\left(\widetilde{U}, \widetilde{V}\right) \longrightarrow \mathrm{Ext}^1_{kS}\left(\mathrm{Ind}_i(U), \widetilde{V}\right) \end{aligned}$$

is exact. The first non-zero map is an isomorphism since $\widetilde{V}$ is simple and $\widetilde{U}$ is the top of $\mathrm{Ind}_i(U)$. Now $\mathrm{Hom}_{kS}(N, \widetilde{V}) = \mathrm{Hom}_{kS}(N/NJ_\ell{}^{\downarrow}, \widetilde{V})$. But then

$$\begin{aligned} \mathrm{Hom}_{kS}\left(N/NJ_\ell{}^{\downarrow}, \widetilde{V}\right) &= \mathrm{Hom}_{kS}\left(\left(N/NJ_\ell{}^{\downarrow}\right)/\mathrm{rad}\left(N/NJ_\ell{}^{\downarrow}\right), \widetilde{V}\right) \\ &= \mathrm{Hom}_{kS}\left(\left(N/NJ_\ell{}^{\downarrow}\right)/\mathrm{rad}\left(N/NJ_\ell{}^{\downarrow}\right), \mathrm{Coind}_\ell(V)\right) \\ &= \mathrm{Hom}_{kG_\ell}\left(\left[\left(N/NJ_\ell{}^{\downarrow}\right)/\mathrm{rad}\left(N/NJ_\ell{}^{\downarrow}\right)\right] e_\ell, V\right). \end{aligned}$$

Since $J_i \not\geq_{\mathscr{J}} J_\ell$, $\widetilde{V}$ is a $kS/kJ_i{}^{\downarrow}$-module. Lemma 3.5 then provides the isomorphism $\mathrm{Ext}^1_{kS}(\mathrm{Ind}_i(U), \widetilde{V}) \cong \mathrm{Ext}^1_{kG_i}(U, \widetilde{V}e_i)$.

Suppose first that $J_i \neq J_\ell$. Then $\widetilde{V}e_i = 0$ (as $J_i \not\geq_{\mathscr{J}} J_\ell$) and so

$$0 \longrightarrow \mathrm{Hom}_{kS}\left(N/NJ_\ell{}^{\downarrow}, \widetilde{V}\right) \longrightarrow \mathrm{Ext}^1_{kS}\left(\widetilde{U}, \widetilde{V}\right) \longrightarrow 0$$

is exact. Now if J_i and J_ℓ are incomparable, then $Ne_\ell = 0$ since N is a $kS/kJ_i{}^{\Uparrow}$-module. This proves (1) and the first statement of (2). The second statement of (2) is dual.

Assume now $J_i = J_\ell$. Then recalling that N is annihilated by e_i, it follows that we have an exact sequence

$$0 \to \mathrm{Ext}^1_{kS}\left(\widetilde{U}, \widetilde{V}\right) \longrightarrow \mathrm{Ext}^1_{kS}\left(\mathrm{Ind}_i(U), \widetilde{V}\right) \cong \mathrm{Ext}^1_{kG_i}\left(U, \widetilde{V}e_i\right) = \mathrm{Ext}^1_{kG_i}(U, V)$$

This completes the proof. □

In particular, if the characteristic of k does not divide the order of any maximal subgroup of S, then we have the following result, originally derived by Putcha when the characteristic of k is zero from the theory of quasi-hereditary algebras [35].

Corollary 3.8 *Let S be a regular monoid and k a field whose characteristic does not divide the order of any maximal subgroup of S. Suppose $\widetilde{U}$ and $\widetilde{V}$ are simple kS-modules with apexes J_i and J_ℓ, respectively. Then* $\mathrm{Ext}_{kS}(\widetilde{U}, \widetilde{V}) \neq 0$ *implies that either $J_i <_{\mathscr{J}} J_\ell$ or $J_\ell <_{\mathscr{J}} J_i$.*

In the case of characteristic p, if $\widetilde{U}, \widetilde{V}$ are simple modules with the same apex J_i, it is not necessarily true that $\dim \mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V}) = \dim \mathrm{Ext}^1_{kG_i}(U, V)$. For instance, if $C = \{e, g\}$ is a cyclic group of order 2 and k is an algebraically closed field of characteristic 2, then it is easy to verify that the trivial module k is the unique simple kC-module and $\dim \mathrm{Ext}^1_{kC}(k, k) = 1$. On the other hand, let S be the Rees matrix semigroup

$$\mathscr{M}\left(C, 2, 2, \begin{pmatrix} e & e \\ e & g \end{pmatrix}\right)$$

with an adjoined identity. Direct computation shows $\dim \mathrm{Ext}^1_{kS}(\widetilde{k}, \widetilde{k}) = 0$.

4 Projective Indecomposables and Directedness

A quasi-hereditary algebra is said to be *directed* if all its standard modules are projective. This depends on the ordering of the simple modules in general. With this as motivation, we shall define a regular semigroup S to be *directed* with respect to a field k (or say that kS is *directed*), if the characteristic of k does not divide the order of any maximal subgroup of S and each induced module $\mathrm{Ind}_i(V)$, with V a simple kG_i-module, is projective (i.e., kS is directed with respect to its canonical quasi-hereditary structure). In this case, it follows that the $\mathrm{Ind}_i(V)$ are the projective indecomposables of kS since Ind_i preserves indecomposability and $\mathrm{Ind}_i(V)$ has simple top $\widetilde{V}$. Our aim is to show that kS is directed if and only if the sandwich matrix [12, 27, 38] of each $\mathscr{J}$-class J_i is left invertible over kG_i. To do this, we prove that S is directed with respect to k if and only if $\widetilde{V} = \mathrm{Coind}_i(V)$ for all i and all V a simple kG_i-module. This is a standard fact in the theory of quasi-hereditary algebras, but we give a proof for completeness.[1]

Proposition 4.1 *Suppose that S is a regular monoid and the characteristic of k does not divide the order of any maximal subgroup of S. Then S is directed with respect to k if and only if all the coinduced modules* $\mathrm{Coind}_i(V)$ *with V a simple kG_i-module are simple kS-modules (for all i).*

Proof Suppose first that $\widetilde{V} = \mathrm{Coind}_i(V)$ for all simple kG_i-modules V and all i. A standard homological argument shows that a module M over a finite dimensional algebra A is projective if and only if $\mathrm{Ext}^1_A(M, W) = 0$ for all simple A-modules W.

So suppose that W is a simple kS-module and V is a simple kG_i-module. First assume that the apex J_ℓ of W is not strictly $\mathscr{J}$-below J_i. Then W is a $kS/kJ_i^{\downarrow}$-module and so Lemma 3.5 yields

$$\mathrm{Ext}^1_{kS}(\mathrm{Ind}_i(V), W) = \mathrm{Ext}^1_{kG_i}(V, We_i) = 0,$$

where the latter equality follows since kG_i is semisimple and hence V is projective.

Thus we may assume that $J_\ell <_{\mathscr{J}} J_i$. By hypothesis, $W = \mathrm{Coind}_\ell(We_\ell)$. Since $\mathrm{Ind}_i(V)$ is a $kS/kJ_\ell^{\downarrow}$-module an application of Lemma 3.5 implies

$$\begin{aligned}\mathrm{Ext}^1_{kS}(\mathrm{Ind}_i(V), W) &= \mathrm{Ext}^1_{kS}(\mathrm{Ind}_i(V), \mathrm{Coind}_\ell(We_\ell))\\ &\cong \mathrm{Ext}^1_{kG_\ell}(\mathrm{Ind}_i(V)e_\ell, We_\ell) = 0\end{aligned}$$

where the last equality follows since $\mathrm{Ind}_i(V)e_\ell = 0$ as $J_\ell <_{\mathscr{J}} J_i$ (or by semisimplicity of kG_ℓ). This proves that $\mathrm{Ind}_i(V)$ is projective.

Suppose conversely that for all i and all simple kG_i-modules V, one has that $\mathrm{Ind}_i(V)$ is projective. Then these are precisely the projective indecomposables of kS, as discussed above. Let U be a simple kG_ℓ-module and set M equal to the cokernel of the inclusion $\widetilde{U} \to \mathrm{Coind}_\ell(U)$. If $M = 0$, then we are done. So assume $M \neq 0$. Since $\mathrm{Coind}_\ell(U)$ is a $kS/kJ_\ell{}^{\Uparrow}$-module and $\mathrm{Coind}_\ell(U)e_\ell kS = \widetilde{U}$, it follows that the apex

[1] We are grateful to Vlastimil Dlab for pointing out to us that this equivalence is known.

of any composition factor of M is strictly $\mathscr{J}$-above J_ℓ. Consequently, the projective cover P of M is a direct sum of modules of the form $\mathrm{Ind}_i(V)$ with $J_i >_{\mathscr{J}} J_\ell$. We shall obtain a contradiction by showing that $\mathrm{Hom}_{kS}(\mathrm{Ind}_i(V), M) = 0$ for $J_i >_{\mathscr{J}} J_\ell$. Indeed, by projectivity of $\mathrm{Ind}_i(V)$ we have an exact sequence

$$\mathrm{Hom}_{kS}(\mathrm{Ind}_i(V), \mathrm{Coind}_\ell(U)) \longrightarrow \mathrm{Hom}_{kS}(\mathrm{Ind}_i(V), M) \longrightarrow 0.$$

Because $\mathrm{Ind}_i(V)$ is a $kS/kJ_\ell{}^{\downarrow}$-module the leftmost term of the above sequence is isomorphic to $\mathrm{Hom}_{kG_\ell}(\mathrm{Ind}_i(V)e_\ell, U) = 0$ as $J_i >_{\mathscr{J}} J_\ell$ implies $\mathrm{Ind}_i(V)e_\ell = 0$. □

To check the criterion in the above proposition, we need to make explicit how the simple modules of kS sit in the coinduced modules. What we are about to do is essentially give a coordinate-free argument for the results of Rhodes and Zalcstein [39].

Proposition 4.2 *Let V be a simple kG_i-module. Then there is a natural isomorphism* $\mathrm{Hom}_{kS}(\mathrm{Ind}_i(V), \mathrm{Coind}_i(V)) \cong \mathrm{Hom}_{kG_i}(V, V) \neq 0$. *Moreover, if* $\varphi \in \mathrm{Hom}_{kS}(\mathrm{Ind}_i(V), \mathrm{Coind}_i(V))$ *is non-zero, then* $\ker\varphi = \mathrm{rad}(\mathrm{Ind}_i(V))$ *and* $\mathrm{Im}\,\varphi = \widetilde{V} = \mathrm{Soc}(\mathrm{Coind}_i(V))$.

Proof First note that since $\mathrm{Ind}_i(V)$, $\mathrm{Coind}_i(V)$ are $kS/kJ_i^{\downarrow}$-modules, the adjunction yields

$$\mathrm{Hom}_{kG_i}(V, V) = \mathrm{Hom}_{kG_i}(\mathrm{Ind}_i(V)e_i, V) \cong \mathrm{Hom}_{kS}(\mathrm{Ind}_i(V), \mathrm{Coind}_i(V)).$$

Suppose now that $\varphi\colon \mathrm{Ind}_i(V) \to \mathrm{Coind}_i(V)$ is a non-zero homomorphism. Because $\mathrm{Ind}_i(V)e_ikS = \mathrm{Ind}_i(V)$ by construction, it follows that

$$\varphi(\mathrm{Ind}_i(V)) = \varphi(\mathrm{Ind}_i(V))e_ikS \subseteq \mathrm{Coind}_i(V)e_ikS = \widetilde{V}.$$

Since $\varphi \neq 0$, it follows by simplicity of $\widetilde{V}$, that $\varphi(\mathrm{Ind}_i(V)) = \widetilde{V}$. As $\mathrm{Ind}_i(V)$ has a unique maximal submodule, we conclude $\ker\varphi = \mathrm{rad}(\mathrm{Ind}_i(V))$. □

For a kG_i-module V, we set $V^* = \mathrm{Hom}_{kG_i}(V, kG_i)$; it is a left kG_i-module. It is a standard fact that $\mathrm{Hom}_{kG_i}(V, W)$ is naturally isomorphic to $W \otimes_{kG_i} V^*$ for finitely generated kG_i-modules (when the characteristic of k does not divide $|G_i|$) [7]. The isomorphism sends $w \otimes \varphi$ to the map $v \mapsto w\varphi(v)$. To prove the isomorphism, one first observes that it is trivial for $V = kG_i$ since both modules are isomorphic to W. One then immediately obtains the isomorphism for all finitely generated projective modules and hence all finitely generated kG_i-modules since kG_i is semisimple.

Let L_i and R_i denote the $\mathscr{L}$-class and $\mathscr{R}$-class of G_i, respectively. Observe that as vector spaces, we have $kL_i = (kS/kJ_i{}^{\downarrow})e_i$ and $kR_i = e_ikS/kJ_i{}^{\downarrow}$ by stability. Moreover, the corresponding kG_i-kS-bimodule structure on kR_i is induced by left multiplication by elements of G_i and by the right Schützenberger representation of S on R_i [12, 27, 38] (i.e., the action of S on R_i by partial functions obtained via restriction of the regular action). To simplify notation, we will use kR_i and kL_i for the rest of this section. Then

$$\mathrm{Ind}_i(V) = V \otimes_{kG_i} kR_i$$

$$\mathrm{Coind}_i(V) = \mathrm{Hom}_{kG_i}(kL_i, V) = V \otimes_{kG_i} kL_i^*.$$

Multiplication in the semigroup induces a non-zero homomorphism

$$C_i\colon kR_i \otimes_{kS} kL_i \cong e_i kS/kJ_i{}^{\downarrow} \otimes_{kS} \left(kS/kJ_i{}^{\downarrow}\right) e_i \to e_i \left(kS/kJ_i{}^{\downarrow}\right) e_i \cong kG_i \tag{4.1}$$

which moreover is a map of kG_i-bimodules. From the isomorphism

$$\mathrm{Hom}_{kG_i}(kR_i \otimes_{kS} kL_i, kG_i) \cong \mathrm{Hom}_{kS}(kR_i, \mathrm{Hom}_{kG_i}(kL_i, kG_i))$$

we obtain a corresponding non-zero kS-linear map $C_i\colon kR_i \to kL_i^*$ (abusing notation). In fact, C_i is a morphism of kG_i-kS-bimodules since Eq. 4.1 respects both the left and right kG_i-module structures.

Let $T \subseteq R_i$ be a complete set of representatives of the $\mathscr{L}$-classes of J_i and $T' \subseteq L_i$ be a complete set of representatives of the $\mathscr{R}$-classes of J_i. Then G_i acts freely on the left of R_i and T is a transversal for the orbits and dually T' is a transversal for the orbits of the free action of G_i on the right of L_i; see [38, Appendix A]. Thus kR_i is a free left kG_i-module with basis T and kL_i is a free right kG_i-module with basis T'. The dual basis to T' is then a basis for the free left kG_i-module kL_i^*. It is instructive to verify that the associated matrix representation of S on kR_i is the classical right Schützenberger representation by row monomial matrices and the representation of S on kL_i is the left Schützenberger representation by column monomial matrices [12, 27, 38, 39]. Hence if $\ell_i = |T|$ and $r_i = |T'|$, then as kG_i-modules we have $kR_i \cong kG_i^{\ell_i}$ and $kL_i \cong kG_i^{r_i}$. Thus C_i is the bilinear form given by the $\ell_i \times r_i$-matrix (also denoted C_i) with

$$(C_i)_{ba} = \begin{cases} \lambda_b \rho_a & \lambda_b \rho_a \in J \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

where $\lambda_b \in T$ represents the $\mathscr{L}$-class b and $\rho_a \in T'$ represents the $\mathscr{R}$-class a. Note that $(C_i)_{ba} \in G_i \cup \{0\}$ by stability and C_i is just the usual sandwich (or structure) matrix of the $\mathscr{J}$-class J_i coming from the Green-Rees structure theory [12, 27, 38]. The reader may take Eq. 4.2 as the definition of the sandwich matrix if he/she so desires. In particular, using T as a basis for kR_i and the dual basis to T' as a basis for kL_i^*, we can view the sandwich matrix of J_i as the matrix of the map $C_i\colon kR_i \to kL_i^*$. The fact that the sandwich matrix gives a morphism of kG_i-kS-bimodules translates exactly into the so-called linked equations of [27, 38].

Putting together the above discussion, we obtain the following module-theoretic version of a result of Rhodes and Zalcstein [39].

Theorem 4.3 *Let V be a simple kG_i-module and suppose that V is flat. Then the simple kS-module $\widetilde{V}$ is the image of the morphism*

$$V \otimes C_i\colon \mathrm{Ind}_i(V) = V \otimes_{kG_i} kR_i \to V \otimes_{kG_i} kL_i^* = \mathrm{Coind}_i(V)$$

where C_i is the sandwich matrix for J_i. This holds in particular if the characteristic of k does not divide $|G_i|$.

Proof Since C_i is not the zero matrix and V is flat, $V \otimes C_i$ is a non-zero homomorphism. Proposition 4.2 then implies the desired conclusion. □

We now characterize when kS is directed. This generalizes the result of Munn and Ponizovsky characterizing semisimplicity of kS in terms of invertibility of the structure matrices since a quasi-hereditary algebra is semisimple if and only if both it and its opposite algebra are directed with respect to a fixed quasi-hereditary structure.

Theorem 4.4 *Let S be a regular monoid and k a field such that the characteristic of k does not divide the order of any maximal subgroup of S. Then kS is directed if and only if the sandwich matrix of each $\mathscr{J}$-class of S is left invertible. In this case, the global dimension of kS is bounded by $m-1$ where m is the length of the longest chain of non-zero $\mathscr{J}$-classes of S.*

Proof The second statement is immediate from Nico's Theorem. For the first, we use Proposition 4.1. Suppose first that all the sandwich matrices are left invertible. Let C_i be the sandwich matrix for J_i. By assumption

$$kR_i \xrightarrow{C_i} kL_i^* \longrightarrow 0 \tag{4.3}$$

is exact. Let V be a simple kG_i-module. Since the tensor product is right exact, tensoring V with Eq. 4.3 yields the exact sequence

$$\mathrm{Ind}_i(V) = V \otimes_{kG_i} kR_i \longrightarrow V \otimes_{kG_i} kL_i^* = \mathrm{Coind}_i(V) \longrightarrow 0.$$

Since any non-zero homomorphism $\mathrm{Ind}_i(V) \to \mathrm{Coind}_i(V)$ has image $\widetilde{V}$ by Proposition 4.2, it follows that $\widetilde{V} = \mathrm{Coind}_i(V)$. This shows that kS is directed.

Suppose conversely that kS is directed. Let C_i be the structure matrix of J_i. Since we are dealing with finite dimensional algebras, to show that C_i is left invertible, it suffices to show that $C_i \colon kR_i \to kL_i^*$ is onto. Let V be a simple kG_i-module. Since V is projective and hence flat, Theorem 4.3 implies that the image of $V \otimes C_i$ is $\widetilde{V} = \mathrm{Coind}_i(V)$, that is $V \otimes C_i$ is onto. Because kG_i is semisimple, it follows that if $V_1, \ldots, V_s$ are the simple kG_i-modules and $m_1, \ldots, m_s$ are their corresponding multiplicities in kG_i, then $C_i = kG_i \otimes C_i = (m_1 V_1 \otimes C_i) \oplus \cdots \oplus (m_s V_s \otimes C_i)$ and hence is onto. This completes the proof. □

The following corollary will be useful for computing quivers of directed semigroup algebras.

Corollary 4.5 *Suppose that kS is directed. Let $\widetilde{U}, \widetilde{V}$ be simple modules with respective apexes $J_i \geq_{\mathscr{J}} J_\ell$. Then $\mathrm{Ext}^n_{kS}(\widetilde{U}, \widetilde{V}) = 0$ all $n \geq 1$.*

Proof Since $\widetilde{U}$ is a $kS/kJ_\ell{}^{\downarrow}$-module and $\widetilde{V} = \mathrm{Coind}_\ell(V)$, Lemma 3.5 yields

$$\mathrm{Ext}^n_{kS}\left(\widetilde{U}, \widetilde{V}\right) = \mathrm{Ext}^n_{kS}\left(\widetilde{U}, \mathrm{Coind}_\ell(V)\right) \cong \mathrm{Ext}^n_{kG_\ell}\left(\widetilde{U}e_\ell, V\right) = 0,$$

for all $n \geq 1$, where the last equality follows from the semisimplicity of kG_ℓ. □

Putting together Theorem 3.7 and Corollary 4.5, we see that if kS is directed, then $\mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V}) \neq 0$ implies that the apex of $\widetilde{V}$ must be strictly $\mathscr{J}$-above the apex of $\widetilde{U}$.

We can also compute the Cartan invariants in the case kS is directed, modulo group theory. Let k be an algebraically closed field. The *Cartan matrix* C of a finite dimensional k-algebra A is the matrix with entries indexed by the projective indecomposables and with $C_{PQ} = \dim \mathrm{Hom}_A(P, Q)$, or equivalently the multiplicity of $P/\mathrm{rad}(P)$ as a composition factor of Q [7].

Theorem 4.6 *Let S be a regular monoid and k an algebraically closed field. Assume that kS is directed. Let V be a simple kG_i-module and W a simple kG_ℓ-module. Set $P = \mathrm{Ind}_i(V)$, $Q = \mathrm{Ind}_\ell(W)$. Then*

$$C_{PQ} = \begin{cases} \dim \mathrm{Hom}_{kG_i}(V, \mathrm{Hom}_{kS}(kS/kJ_i{}^{\downarrow}, Q)e_i) & J_i >_{\mathscr{J}} J_\ell \\ 1 & V = W \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, the Cartan matrix C of kS is a unipotent matrix.

Proof First note that since Q is a $kS/kJ_\ell{}^{\Uparrow}$-module, all of its composition factors have apex $\geq_{\mathscr{J}} J_\ell$. Suppose that $J_i = J_\ell$. Then $\mathrm{Hom}_{kS}(P, Q) = \mathrm{Hom}_{kG_i}(V, Qe_i) = \mathrm{Hom}_{kG_i}(V, W)$ and so in this case $C_{P,Q} = 1$ if $V = W$ and 0 otherwise. Finally, suppose that $J_i >_{\mathscr{J}} J_\ell$. Then since P is a $kS/kJ_i{}^{\downarrow}$-module

$$\begin{aligned} \mathrm{Hom}_{kS}(P, Q) &\cong \mathrm{Hom}_{kS/kJ_i{}^{\downarrow}}(\mathrm{Ind}_i(V), \mathrm{Hom}_{kS}(kS/kJ_i{}^{\downarrow}, Q)) \\ &\cong \mathrm{Hom}_{kG_i}(V, \mathrm{Hom}_{kS}(kS/kJ_i{}^{\downarrow}, Q)e_i) \end{aligned}$$

as required. The final statement follows since we can define a partial order on the projective indecomposables by $\mathrm{Ind}_i(V) \leq \mathrm{Ind}_\ell(W)$ if and only if $J_i = J_\ell$ and $V = W$ or $J_i >_{\mathscr{J}} J_\ell$. Hence the Cartan matrix is unitriangular with respect to an appropriate ordering of the projective indecomposables. □

This theorem will be the starting point for a more detailed computation of the Cartan invariants for the algebra of a right regular band of groups (Theorem 5.6).

5 Right Regular Bands of Groups

In this section we specialize our results to right regular bands of groups. A *right regular band of groups* (which we shall call an RRBG) is a regular semigroup for which each right ideal is a two-sided ideal, or equivalently Green's relations $\mathscr{J}$ and $\mathscr{R}$ coincide. In particular, RRBGs include groups, right regular bands, commutative regular semigroups and right simple semigroups.

5.1 The Structure of Right Regular Bands of Groups

Here we record the basic structural properties of right regular bands of groups. For basic facts about finite semigroups, the reader is referred to [38, Appendix A] or to [2, 12, 27]. If s belongs to a finite semigroup S, then s^ω denotes the unique idempotent positive power of s.

Proposition 5.1 *A semigroup S is a right regular band of groups if and only if it satisfies*

(1) $s^{\omega}s = s$
(2) $s^{\omega}ts^{\omega} = ts^{\omega}$

all $s, t \in S$.

Proof Suppose first that S satisfies the above two properties. By (1), each element of S generates a cyclic group and so S is clearly regular. Suppose R is a right ideal of S and let $r \in R$ and $s \in S$. Let $a \in S$ be such that $rar = r$. Then ra is an idempotent, and so by (2) $sr = srar = rasrar \in R$. Thus R is a two-sided ideal.

Conversely, suppose S is an RRBG. Since S is regular, $s = sts$ for some $t \in S$. The element $e = ts$ is an idempotent $\mathscr{L}$-equivalent to s. On the other hand, $eS = SeS = SsS = sS$ and so $e \mathscr{R} s$. Thus the $\mathscr{H}$-class of s is a group [27, 38] and so $s^{\omega}s = s$.

For (2), choose $n > 1$ so that $x^n = x^{\omega}$ all $x \in S$. Then we have

$$ts^{\omega} = (ts^{\omega})^{\omega}ts^{\omega} = (ts^{\omega})^{n-1}t(s^{\omega}ts^{\omega})$$

and so $s^{\omega}ts^{\omega} \mathscr{L} ts^{\omega}$. Since $\mathscr{J} = \mathscr{R}$, it follows $s^{\omega}ts^{\omega} \mathscr{R} ts^{\omega}$. But then $ts^{\omega} \in s^{\omega}S$ and so $s^{\omega}ts^{\omega} = ts^{\omega}$, as required. □

In particular, it follows that the class of RRBGs is closed under product, subsemigroups and quotients. Also a band is an RRBG if and only if it is a right regular band in the sense that is satisfies the identities $x^2 = x$ and $xyx = yx$. Let us remark that one can replace (2) by $s^{\omega}ts = ts$. Recall that $E(S)$ denotes the idempotent set of S.

Corollary 5.2 *Let S be an RRBG. Then $E(S)$ is a right regular band.*

Proof By Proposition 5.1, if e, f are idempotents, then $efef = eef = ef$. Thus $E(S)$ is a subsemigroup and hence a right regular band. □

Corollary 5.3 *Let S be an RRBG and $e \in E(S)$, then the map $s \mapsto se$ gives a retraction from S to $Se = eSe$.*

Proof Indeed, from $ete = te$ we obtain $ste = sete$. □

The next lemma shows that the set of principal right ideals of an RRBG S is a meet semilattice grading S in a natural way. For the case of a right regular band (monoid), Brown and Saliola call the dual of this lattice the support lattice [9, 10, 40]. The lemma is a special case of a result of Clifford [12, Chapter 4].

Lemma 5.4 *Let S be a right regular band of groups. Then $aS \cap bS = abS$. Hence the principal right ideals form a meet semilattice Λ and the map $a \mapsto aS$ is a surjective homomorphism $S \to \Lambda$ whose fibers are the $\mathscr{R}$-classes.*

Proof Let $e = a^{\omega}$ and $f = b^{\omega}$. Then $eS = aS$, $fS = bS$ and $abS = SabS = SefS = efS$. Now by Proposition 5.1, $efS = fefS$ and so $efS \subseteq eS \cap fS$. On the other hand, if $x \in eS \cap fS$, then $ex = x$, $fx = x$ and so $efx = x$ and hence $x \in efS$. Thus $abS = efS = eS \cap fS = aS \cap bS$. □

Consequently, we have the following well-known result.

Corollary 5.5 *Let S be a RRBG and let J be $\mathscr{J}$-class. Then $S \setminus J^{\uparrow}$ is a subsemigroup of S.*

5.2 The Algebra of a Right Regular Band of Groups

Fix a monoid S which is an RRBG and let k be an algebraically closed field such that the characteristic of k does not divide the order of any maximal subgroup of S. We retain the notation from the previous sections. Since each $\mathscr{J}$-class J_i of S consists of a single $\mathscr{R}$-class, the sandwich matrix of each $\mathscr{J}$-class J_i consists of a single column with non-zero entries from G_i and hence is automatically left invertible over kG_i. Therefore, kS is directed by Theorem 4.4. Hence the global dimension of kS is bounded by the longest chain of non-zero $\mathscr{J}$-classes minus 1 and the projective indecomposables are precisely the induced modules $\mathrm{Ind}_i(V)$ with V a simple kG_i-module. In this case, they have a particularly simple form. If J is a $\mathscr{J}$-class of S with maximal subgroup G and V is a simple kG-module, then the associated projective indecomposable is $V \otimes_{kG} kJ$ where $s \in S$ acts on kJ as multiplication by s if $J \subseteq SsS$ and as the zero map if $J \nsubseteq SsS$. The simple modules are the coinduced modules.

5.2.1 The Semisimple Quotient

For a right regular band, the semisimple quotient of its algebra is the algebra of a semilattice cf. [9, 10]. In the case of a right regular band of groups, one can replace the semilattice with a semilattice of groups, as was observed in [3, 43, 44]. The notion of a semilattice of groups is extremely important in semigroup representation theory. For instance, it is shown in [3] that a semigroup S has a basic algebra over an algebraically closed field k if and only if it admits a homomorphism to a semilattice of abelian groups inducing the semisimple quotient on the level of their semigroup algebras. All this will be made explicit here for the case of RRBGs as we will need the details in order to compute the Cartan invariants explicitly. We begin by recalling the notion of a semilattice of groups, a construction going back to Clifford [12].

Let Λ be a meet semilattice and $G\colon \Lambda^{op} \to \mathbf{Grp}$ be a presheaf of groups on Λ, that is, a contravariant functor from Λ (viewed as a poset) to the category of groups. If $e \leq f$, and $g \in G(f)$, then we write $g|_e$ for the image of g under restriction map $G(f) \to G(e)$. Then one can place a semigroup structure on $T = \coprod_{e\in\Lambda} G(e)$ by defining the product of $g \in G(e)$ and $h \in G(f)$ by $gh = g|_{ef}h|_{ef}$ [12]. Such a semigroup is called a *semilattice of groups*. Clifford showed that semilattices of groups are precisely the regular semigroups with central idempotents, or alternatively the inverse semigroups with central idempotents [12, Chapter 4]. Recall that a semigroup S is called *inverse* if, for all $s \in S$, there exists a unique $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$; equivalently S is inverse if it is regular and has commuting idempotents [12]. It is a well known result of Munn and Ponizovsky that if S is an inverse semigroup and k is a field such that the characteristic of k divides the order of no maximal subgroup of S, then kS is semisimple [12]. In the case that T is a semilattice of groups as above, it can be shown that $kT \cong \prod_{e\in\Lambda} kG(e)$ using a Möbius inversion argument [43, 44] .

Now let S be again our fixed right regular band of groups and k our algebraically closed field of good characteristic. Let $\Lambda = S/\mathscr{J} = S/\mathscr{R}$ be the lattice of $\mathscr{J}$-classes, which is isomorphic to the lattice of principal right ideals cf. Lemma 5.4. We retain the notation of the previous sections: so $\Lambda = \{J_1, \ldots, J_n\}$ and we have fixed idempotents $e_1, \ldots, e_n$ representing the $\mathscr{J}$-classes with corresponding maximal subgroups $G_1, \ldots, G_n$. Define $i \wedge \ell$, for $i, \ell \in \{1, \ldots, n\}$, by the equation $J_i \wedge J_\ell = J_{i\wedge\ell}$. We form a semilattice of groups $F\colon \Lambda^{op} \to \mathbf{Grp}$ by setting $F(J_i) = G_i$ and defining the restriction $G_i \to G_\ell$, for $J_\ell \leq_{\mathscr{J}} J_i$ by $g|_{J_\ell} = ge_\ell$. It is immediate from stability and Corollary 5.3 that this restriction map is a homomorphism and that if $i = \ell$, then it is the identity map. If $J_i \geq_{\mathscr{J}} J_\ell \geq_{\mathscr{J}} J_m$, then $e_\ell \geq_{\mathscr{R}} e_m$ and so $e_\ell e_m = e_m$. Thus $(g|_{J_\ell})|_{J_m} = ge_\ell e_m = ge_m = g|_{J_m}$, establishing functoriality. Let $T = \coprod_{J_i \in \Lambda} G_i$ be the corresponding inverse monoid.

There is a surjective homomorphism $\varphi\colon S \to T$ given by $\varphi(s) = se_i$ if $s \in J_i$. Indeed, from Lemma 5.4, one has if $s \in J_i$ and $t \in J_\ell$, then $st \in J_{i\wedge\ell}$. Moreover, denoting by $\cdot$ the product in T and using $e_i, e_\ell \geq_{\mathscr{R}} e_{i\wedge\ell}$, we obtain

$$\varphi(s) \cdot \varphi(t) = se_i \cdot te_j = se_i e_{i\wedge\ell} te_\ell e_{i\wedge\ell} = se_{i\wedge\ell} te_{i\wedge\ell} = ste_{i\wedge\ell} = \varphi(st)$$

establishing that φ is a homomorphism. Suppose that $\varphi(s) = e_i$ for $s \in S$. Then $s \in J_i$ and $se_i = e_i$. Since $e_i \mathrel{\mathscr{R}} s^\omega$, we have $s = ss^\omega = se_i s^\omega = e_i s^\omega = s^\omega$. Thus $\varphi^{-1}(e_i)$ is the right zero semigroup $E(J_i)$. It follows from [3, Theorem 3.5] that the induced surjective map $\Phi\colon kS \to kT$ has nilpotent kernel. Since $kT \cong kG_1 \times \cdots \times kG_n$ is semisimple, we conclude that Φ is the semisimple quotient. One can describe the semisimple quotient directly by defining $\psi\colon kS \to kG_1 \times \cdots \times kG_n$ by $\psi(s) = (g_1, \ldots, g_n)$ where

$$g_i = \begin{cases} se_i & s \geq_{\mathscr{J}} J_i \\ 0 & \text{otherwise.} \end{cases}$$

See [43, 44] for details. Notice that kS is basic if and only if each of its maximal subgroups is abelian. In [3], the semigroups with basic algebras were determined for any field.

5.2.2 The Cartan Invariants

Assume now that k is an algebraically closed field of characteristic 0. Using the character formulas for multiplicities from [44], we compute the Cartan invariants of kS. We retain the above notation. Let μ be the Möbius function [42] for the lattice Λ of $\mathscr{J}$-classes. Let V be a simple kG_i-module and let M be any kS-module. Let χ_V be the character of V and θ the character of M. Then since kT is the semisimple quotient of kS, it follows that θ factors through φ as $\chi\varphi$ with χ the character of $M/\mathrm{rad}(M)$ as a representation of T. Observing that the semilattice of idempotents $E(T)$ of T is isomorphic to Λ, it follows from the formula in [44] for multiplicities of irreducible constituents in representations of inverse semigroups that the multiplicity of $\widetilde{V}$ as a composition factor of M is given by the formula

$$\frac{1}{|G_i|} \sum_{g \in G_i} \chi_V(g^{-1}) \sum_{J_m \leq_{\mathscr{J}} J_i} \theta(ge_m)\mu(J_m, J_i). \tag{5.1}$$

We apply Eq. 5.1 to compute the Cartan invariants for kS. Let P and Q be projective indecomposables for kS. We already know from Theorem 4.6 that $P = \mathrm{Ind}_i(V)$ and $Q = \mathrm{Ind}_\ell(W)$ for appropriate simple kG_i and kG_ℓ-modules V and W since kS is directed. Moreover, the entry C_{PQ} of the Cartan matrix is 0 unless

$J_i \geq_{\mathscr{J}} J_\ell$ and that if $J_i = J_\ell$, then $C_{PQ} = 1$ if $V = W$ and 0 if $V \neq W$. All that remains then is to compute C_{PQ} in the case $J_i >_{\mathscr{J}} J_\ell$.

Theorem 5.6 *Let S be a right regular band of groups and k an algebraically closed field of characteristic 0. Let V be a simple kG_i-module and W a simple kG_ℓ-module with respective characters χ_V, χ_W. Set $P = \mathrm{Ind}_i(V)$ and $Q = \mathrm{Ind}_\ell(W)$. Denote by μ the Möbius function for the lattice of principal right ideals of S. Let C be the Cartan matrix of kS. Then*

$$C_{PQ} = \begin{cases} * & J_i >_{\mathscr{J}} J_\ell \\ 1 & V = W \\ 0 & \textit{otherwise.} \end{cases}$$

where

$$* = \frac{1}{|G_i|} \sum_{g \in G_i} \chi_V(g^{-1}) \sum_{J_\ell \leq_{\mathscr{J}} J_m \leq_J J_i} \mu(J_m, J_i) \sum_{e \in E(J_\ell), (ege_m)^\omega = e} \chi_W(ege_\ell).$$

Proof It only remains to consider the case that $J_i >_{\mathscr{J}} J_\ell$ by the remarks before the theorem. The Cartan invariant C_{PQ} is exactly the multiplicity of $\widetilde{V}$ as a composition factor of the module Q. Let θ be the character of Q. Since $Q = W \otimes_{kG} kJ_\ell$, it follows that if $s \not\geq_{\mathscr{J}} J_\ell$, then $\theta(s) = 0$. Hence in our setting, the second sum in Eq. 5.1 can be taken over those J_m with $J_\ell \leq_{\mathscr{J}} J_m \leq_{\mathscr{J}} J_i$. To compute θ, we observe that we can take $E(J_\ell)$ as a set of representatives of the $\mathscr{L}$-classes of J_ℓ. Then kJ_ℓ is a free left kG_ℓ-module with basis $E(J_\ell)$. Let $b = |E(J_\ell)|$. The isomorphism of kJ_ℓ with kG_ℓ^b sends $x \in J$ to the row vector with $xe_\ell \in G_\ell$ in the coordinate indexed by x^ω and 0 in all other coordinates. The associated matrix representation of S over kG_ℓ takes $s \geq_J J_\ell$ to the row monomial matrix $RM(s)$ which has its unique non-zero entry in the row corresponding to $e \in E(G_\ell)$ in the column corresponding to $(es)^\omega$ and this entry is $ese_\ell \in G_\ell$. This is just the Schützenberger representation by row monomial matrices over G_ℓ [12, 27, 38, 39]. Now if $\rho\colon G_\ell \to GL(k)$ is the irreducible representation afforded by W, then the matrix $\rho \otimes RM(s)$ for s acting on $W \otimes_{kG_\ell} kJ_\ell$ is obtained by applying ρ-entrywise to $RM(s)$ [39]. The block row e has a diagonal entry if and only if $(es)^\omega = e$ and this entry is $\rho(ese_\ell)$. Hence the trace of $\rho \otimes RM(s)$, that is $\theta(s)$, is given by

$$\sum_{e \in E(J_\ell), (es)^\omega = e} \chi_W(ese_\ell).$$

The formula for C_{PQ} now follows from Eq. 5.1 and the observation $J_m \geq_{\mathscr{J}} J_\ell$ implies $e_m \geq_{\mathscr{R}} e_\ell$ and so $e_m e_\ell = e_\ell$. □

In the case that S is a right regular band, the modules V and W are trivial and the above formula reduces to Saliola's result [40].

5.3 The Quiver of a Right Regular Band of Groups

Set $A = kS$ and let $e \in E(S)$. From Proposition 5.1, we easily deduce the following lemma.

Lemma 5.7 *Let $e \in E(S)$ and $a \in A$. Then $eae = ae$. In particular, $eAe = Ae$ and the map $a \mapsto ae$ is a retraction from A onto $eAe = Ae$.*

Let us recall that the restriction functor $M \mapsto Me$ from mod-A to mod-eAe admits a right adjoint, the functor $V \mapsto \mathrm{Hom}_{eAe}(Ae, V) = \widehat{V}$. Since $eAe = Ae$ by the above lemma, as a vector space $\mathrm{Hom}_{eAe}(Ae, V)$ is just V (identify f with $f(e)$). The computation

$$(fa)(e) = f(ae) = f(eae) = f(e)eae$$

shows that the module action of A on V is given by $va = veae$, i.e., one makes V an A-module via the retraction in Lemma 5.7. In particular, this action extends the action of eAe and hence V is simple if and only if $\widehat{V}$ is simple. Evidently the functor $V \mapsto \widehat{V}$ is exact and $\widehat{V}e = V$. The following is a special case of Green's theory [22, Chapter 6].

Lemma 5.8 *Let $A = kS$ with S an RRBG and let $e \in E(S)$. Up to isomorphism, the simple A-modules M with $Me \neq 0$ are the modules of the form $\widehat{V}$ with V a simple eAe-module. More precisely, if M is a simple A-module with $Me \neq 0$, then Me is a simple eAe-module and $M = \widehat{Me}$.*

Proof Assume M is a simple A-module with $Me \neq 0$. Let $m \in Me$ be non-zero. Then $meA = mA = M$, so $meAe = Me$. Thus Me is simple. Now $\mathrm{Hom}_A(M, \widehat{Me}) = \mathrm{Hom}_{eAe}(Me, Me) \neq 0$. Thus $M \cong \widehat{Me}$ by simplicity. □

Since $Ae = eAe$ is a free eAe-module, the Eckmann-Shapiro Lemma (our Lemma 3.4) immediately yields.

Lemma 5.9 *If M is an A-module and N is an eAe-module, then there is an isomorphism $\mathrm{Ext}^n_A(M, \widehat{N}) \cong \mathrm{Ext}^n_{eAe}(Me, N)$ for all $n \geq 0$.*

In particular, we obtain the following corollary.

Corollary 5.10 *Let $A = kS$ with S an RRBG and let $e \in E(S)$. Suppose M, N are A-modules with N simple and $Ne \neq 0$. Then $\mathrm{Ext}^n_A(M, N) \cong \mathrm{Ext}^n_{eAe}(Me, Ne)$.*

Proof By Lemma 5.8, $N \cong \widehat{Ne}$ and so $\mathrm{Ext}^n_A(M, N) = \mathrm{Ext}^n_A(M, \widehat{Ne}) \cong \mathrm{Ext}^n_{eAe}(Me, Ne)$, the last isomorphism coming from Lemma 5.9. □

Assume now that k is an algebraically closed field. The (Gabriel) quiver of kS is the directed graph with vertex set the simple kS-modules and with $\dim \mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V})$ arrows from $\widetilde{U}$ to $\widetilde{V}$. Suppose $\widetilde{U}$ and $\widetilde{V}$ have apexes J_i, J_ℓ, respectively. By Corollaries 3.7 and 4.5, there are no arrows $\widetilde{U} \to \widetilde{V}$ unless $J_i <_{\mathscr{J}} J_\ell$. Since $e_i <_{\mathscr{J}} e_\ell$ and $\widetilde{U}e_i \neq 0$, it follows that $\widetilde{U}e_\ell \neq 0$. Also $\widetilde{V}e_\ell \neq 0$. Hence Corollary 5.10 implies that to compute the number of arrows between $\widetilde{U}$ and $\widetilde{V}$ (in both directions) we may assume that S is a monoid and $\widetilde{V}$ has apex the unit group. By Corollary 5.5, $T = S \setminus J_i{}^{\Uparrow}$ is a submonoid of S (necessarily a RRBG) and clearly $kT \cong kS/kJ_i{}^{\Uparrow}$. As a consequence of Lemma 3.3, it follows that to compute $\mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V})$, we may assume that J_i is the minimal ideal of S (note that we may have to replace e_i with the idempotent e_ie_ℓ in our computations when cutting down to Se_ℓ).

Summarizing, we have reduced the computation of the quiver of a right regular band of groups S (in good characteristic) to the following situation: we have a simple module $\widetilde{U}$ with apex the minimal ideal J of S and a simple module $\widetilde{V}$ with apex the unit group G of S where, moreover, $G \neq J$. Our goal is to compute $\mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V})$.

Let e be an idempotent in J and set $H = eJe$; it is the maximal subgroup at e. Then $eS = J$ and $Se = H$, that is, J is the $\mathscr{R}$-class of H and H is its own $\mathscr{L}$-class. We write Ind and Coind for the induction and coinduction functors adjoint to the restriction mod-$kS \to$ mod-kH given by $M \mapsto Me$. As $kH = kSe$, we have $kH^* = \mathrm{Hom}_{kH}(kSe, kH) = \mathrm{Hom}_{eAe}(Ae, kH) = \widehat{kH}$. Hence we can identify kH with kH^* as a vector space. The kH-kS-bimodule structure is given by the regular left action of kH and by the right action $xa = xeae$ for $x \in kH$ and $a \in kS$.

Proposition 5.11 *The map $f\colon kJ \to kH$ given by $f(x) = xe$ is a surjective homomorphism of kH-kS-bimodules. The kernel N has basis consisting of all differences of the form $x - xe$ with $x \in J \setminus H$.*

Proof It follows immediately from Lemma 5.7 that f is a morphism of bimodules. Clearly it is onto since it restricts to the identity on kH. Moreover, $kJ \cong N \oplus kH$ and the splitting $kJ \to N$ sends x to $x - xe$. In particular, the elements of the form $x - xe$ with $x \in J \setminus H$ span N. Since $\dim N = |J| - |H|$, the above set is indeed a basis. □

Remark 5.12 The map f is easily verified to be the map given by the sandwich matrix of J if we choose to represent each $\mathscr{L}$-class of J by its idempotent.

Let us set $I = S \setminus G$. Then I is an ideal of S. Our next goal is to identify N/NI as a kH-kG bimodule. The approach is inspired by Saliola [40]. Define a relation $\smile$ on J by $x \smile x'$ if:

(1) $xe = x'e$; and
(2) there exists $w \in I \setminus J$ with $xw = x$ and $x'w = x'$.

Let $\approx$ be the equivalence relation generated by $\smile$. It follows easily that if $x \approx x'$, then $xe = x'e$. In particular, distinct elements of H are never equivalent. Notice that if $S = J \cup G$, then $\smile$ relates no elements. We remark that in (2), one may always assume w is idempotent by replacing it with w^ω.

Proposition 5.13 *If $g \in G$ and $x \smile x'$, then $xg \smile x'g$. If $h \in H$, then $hx \smile hx'$*

Proof Suppose $x \smile x'$. By definition $xe = x'e$ and $xw = x$, $x'w = x'$ some $w \in I \setminus J$. Then $xge = xege = x'ege = x'ge$. Also, since $g^{-1}wg \in I \setminus J$, we have $xg(g^{-1}wg) = xwg = xg$ and $x'g(g^{-1}wg) = x'g$. For the second statement, evidently $hxe = hx'e$ and $hxw = hx$, $hx'w = hx'$. This establishes the proposition. □

Let us set $X = J/{\approx}$. It follows from Proposition 5.13 that X admits a left action of H and a right action of G that commute and hence kX is a kH-kG bimodule. There is an epimorphism $\epsilon\colon kX \to kH$ of kH-kG bimodules given by $[x] \mapsto xe$ where $[x]$ denotes the $\approx$-class of x. Surjectivity comes about as $\epsilon([h]) = h$ for $h \in H$; it is a morphism of right kG-modules since $xge = xege$. Let $M = \ker \epsilon$. Then M is also a kH-kG bimodule. By Corollary 5.3, there is a homomorphism $\psi\colon G \to H$ given by $\psi(g) = ge$. The character of M as a kG-module is the permutation character associated to the action of G on X minus the character of kH viewed as a right kG-module via ψ. It is easy to see that M has a basis consisting of the elements of the form $[x] - [xe]$ where $[x]$ runs over all equivalence classes not containing an element of H. In particular, $\dim M = |X| - |H|$. Let N be as in Proposition 5.11.

Lemma 5.14 *As kH-kG bimodules, $N/NI \cong M$.*

Proof Let Ω be a transversal to $\approx$ containing H and let $\overline{x}$ be the element of Ω equivalent to x for $x \in J$. Define a linear map $T\colon N \to M$ on the basis by $x - xe \mapsto [x] - [xe]$. This is clearly a surjective map since the elements of the form $[x] - [xe]$ span M. Notice that if $x, x' \in J$ with $xe = x'e$, then $T(x - x') = T(x - xe) + T(x'e - x') = [x] - [xe] + [x'e] - [x'] = [x] - [x']$. Consequently, if $h \in H$ and $g \in G$, then as $ge = ege$

$$T(h(x - xe)g) = T(hxg - hxeg) = [hxg] - [hxeg] = h[x]g - h[xe]g$$

so T is a bimodule homomorphism.

Next, we establish $NI \subseteq \ker T$. Indeed, if $w \in I$, then we claim that, for any $x \in J$, either $xw = xew$ or $xw \smile xew$. Clearly $xwe = xewe$. Suppose that $w \in J$. Then $xw \mathscr{H} xew$ by stability and so $(xw)^{\omega} = (xew)^{\omega}$, whence

$$xw = xw(xw)^{\omega} = xw(xew)^{\omega} = xew(xew)^{\omega} = xew$$

where the penultimate equality uses $ze = eze$ all $z \in S$. If $w \notin J$, then $xw = xww^{\omega}$, $xew = xeww^{\omega}$ and $w^{\omega} \in I \setminus J$, establishing $xw \smile xew$. Thus $T((x - xe)w) = [xw] - [xew] = 0$. Elements of the form $(x - xe)w$ with $x \in J, w \in I$ span NI, yielding the inclusion $NI \subseteq \ker T$. To show that the induced map $T\colon N/NI \to M$ is an isomorphism, we show that $\dim N/NI \leq \dim M$. To this effect, we show that N/NI is spanned by elements of the form $x - xe + NI$ with $x \in \Omega \setminus H$. The number of such elements is $\dim M$.

We begin by showing that if $x \approx x'$, then $x - x' \in NI$. Suppose first $x \smile x'$. Then since $xe = x'e$, we have $x - x' \in N$. Also there exists $w \in I \setminus J$ so that $xw = x, x'w = x'$. Then $x - x' = (x - x')w \in NI$. In general, there exist $x = x_0 \smile x_1 \smile x_2 \smile \cdots \smile x_n = x'$. Then

$$x - x' = (x_0 - x_1) + (x_1 - x_2) + \cdots + (x_{n-1} - x_n) \in NI,$$

as required. Next suppose $x \approx x'$ and $u \approx u'$. Then we claim $x - u + NI = x' - u' + NI$. Indeed, $x - u = x - x' + x' - u' + u' - u$ and $x - x', u' - u \in NI$, as was already observed. In particular, for $x \in J$ we have $x - xe + NI = \overline{x} - \overline{x}e + NI$ (since $\overline{x}e = xe$). Because elements of the form $x - xe + NI$ span N/NI, we are done. □

It follows that $U \otimes_{kH} N/NI \cong U \otimes_{kH} M$ as kG-modules. We are now ready to finish up our computation of the quiver of an RRBG.

Theorem 5.15 *Retaining the previous notation,* $\dim \mathrm{Ext}^1_{kS}(\widetilde{U}, \widetilde{V})$ *is the multiplicity of V as a composition factor in the kG-module $U \otimes_{kH} M$.*

Proof We have an exact sequence of kH-kS bimodules

$$0 \longrightarrow N \longrightarrow kJ \longrightarrow kH \longrightarrow 0.$$

Since kH is semisimple, all kH-modules are projective and hence flat and so there results an exact sequence of kS-modules

$$0 \longrightarrow U \otimes_{kH} N \longrightarrow U \otimes_{kH} kJ \longrightarrow U \otimes_{kH} kH \longrightarrow 0.$$

Recall that H is its own $\mathscr{L}$-class, J is the $\mathscr{R}$-class of H and we are identifying kH^* with kH. Thus the middle term is $\mathrm{Ind}(U)$, whereas the rightmost term is $\mathrm{Coind}(U) = \widetilde{U}$ (since kS is directed). Because $\mathrm{Ind}(U)$ has simple top, it follows that $\mathrm{rad}(\mathrm{Ind}(U)) = U \otimes_{kH} N$. Theorem 3.7 implies in our setting

$$\begin{aligned}
\mathrm{Ext}^1_{kS}\left(\widetilde{U}, \widetilde{V}\right) &= \mathrm{Hom}_{kG}(\mathrm{rad}(\mathrm{Ind}(U))/\mathrm{rad}(\mathrm{Ind}(U))I, V) \\
&= \mathrm{Hom}_{kG}(\mathrm{rad}(\mathrm{Ind}(U)) \otimes_{kS} kS/kI, V) \\
&= \mathrm{Hom}_{kG}(U \otimes_{kH} N \otimes_{kS} kS/kI, V) \\
&= \mathrm{Hom}_{kG}(U \otimes_{kH} N/NI, V) \\
&\cong \mathrm{Hom}_{kG}(U \otimes_{kH} M, V)
\end{aligned}$$

which is the desired multiplicity as kG is semisimple and k is algebraically closed. □

6 Examples

From the results of the previous section it follows that one can in principle compute the quiver over the complex numbers of a right regular band of groups provided one has the character tables of its maximal subgroups. The algorithm reduces to computing the number of arrows from a simple module with apex the minimal ideal to a simple module with apex the group of units. This section provides a number of examples. We retain throughout the notation of the previous section.

6.1 Right Regular Bands

Our first example is the case of a right regular band S. In this case, if U is a simple module with apex the minimal ideal and V is a simple module with apex the singleton $\mathscr{J}$-class of the identity, then all the groups involved are trivial and the representations are trivial. Thus the number of arrows from U to V is just $\dim M = |X| - 1$. This is exactly Saliola's result [40].

6.2 Permutation Groups with Adjoined Constant Maps

Next suppose that $G \leq S_n$ is a permutation group of degree n. Let $\overline{G}$ consist of G along with the constant maps on $\{1, \ldots, n\}$. Then $\overline{G}$ is an RRBG with group of units G and minimal ideal J consisting of the constant maps. The Krohn-Rhodes Theorem [19, 26, 27, 38] implies that every finite semigroup is a quotient of a subsemigroup of an iterated wreath product of semigroups of the form $\overline{G}$ with G a finite simple group. Putcha computed the quiver of any regular monoid with exactly 2 $\mathscr{J}$-classes in terms of decomposing group representations via his method of monoid quivers [35] and so the examples in this subsection could also be handled via his methods.

Let k be an algebraically closed field such that the characteristic of k does not divide the order of G. The simple modules for $\overline{G}$ are the trivial module k and the simple kG-modules $V_1, \ldots, V_s$, which become $k\overline{G}$-modules by having the constants act as the zero map. Assume that V_1 is the trivial kG-module. Since $\overline{G} = G \cup J$, the relation $\smile$ is empty and hence $X = J$. So kX can be identified with the permutation module of G coming from the embedding $G \leq S_n$. Then as a kG-module, M is just the result of removing one copy of V_1 from the permutation module. So if the permutation module kX decomposes as $kX = \bigoplus_{i=1}^{s} m_i V_i$, then there are $m_1 - 1$ arrows from k to V_1 and m_i arrows from k to V_i, for $i > 1$. Since there are no directed paths of length 2 or more in the quiver, it follows that $k\overline{G}$ is a hereditary algebra.

Assume now that k has characteristic zero. We provide a complete characterization of the representation type of $k\overline{G}$ in the case that G acts transitively. Recall that the *rank* of G, denoted $\operatorname{rk} G$, is the number of orbitals of G, or equivalently the number of orbits of a point stabilizer [11, 16]. It is well known that the rank of G is the sum of the squares of the multiplicities of the irreducible constituents of the associated permutation module [11]. In particular, if the rank is at most 4, the permutation module must be multiplicity-free. In fact, if the rank of G is 5, then also the permutation module must be multiplicity-free. Otherwise, one has $\theta = 1 + 2\chi$ where θ is the permutation character, 1 is the trivial character and χ is some irreducible character of G. Let $g \in G$ be a fixed-point-free element [11, 16]. Then $0 = 1 + 2\chi(g)$, which implies that $-1/2$ is an algebraic integer, a contradiction.[2] By Gabriel's Theorem [4, 7], a hereditary algebra has finite representation type if and only if each connected component of the underlying graph of its quiver is Dynkin of type A, D or E; the algebra has tame representation type if and only if each component is a Euclidean diagram of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$.

Let's consider the structure of our quiver. By transitivity of G, the trivial representation of G appears exactly once in the permutation module and so there are no arrows from k to V_1. In general, the quiver consists of isolated points and the connected component C of the trivial $k\overline{G}$-module k, which is a star (possibly with multiple edges if $\operatorname{rk} G \geq 6$). From our description of the quiver, it follows that if G has rank 1 (and so $n = 1$), then C is A_1; if G has rank 2, then C is A_2; if G has rank 3, then C is A_3; if G has rank 4, then C is D_4; if the rank of G is 5, then C is $\widetilde{D}_4$; and if the rank of G is at least 6, then C is neither Dynkin nor Euclidean of the above types. We have thus proved:

Theorem 6.1 *Let G be a transitive permutation group and k an algebraically closed field of characteristic zero. Then $k\overline{G}$ is hereditary and*

(1) *$k\overline{G}$ is of finite representation type if and only if* $\operatorname{rk} G \leq 4$*;*
(2) *$k\overline{G}$ is of tame representation type if and only if* $\operatorname{rk} G = 5$*;*
(3) *$k\overline{G}$ is of wild representation type if and only if* $\operatorname{rk} G \geq 6$.

Since G is 2-transitive if and only if it has rank 2, we deduce the following corollary, first proved by Ponizovsky [33].

[2]We thank John Dixon for this supplying this argument.

Corollary 6.2 *Let G be a 2-transitive permutation group. Then $k\overline{G}$ is of finite representation type.*

Notice that computing the quiver of $k\overline{G}$ is equivalent to decomposing the permutation module kX into irreducibles and so one cannot hope to do better than compute the quiver of a right regular band of groups modulo decomposing group representations.

6.3 Hsiao's Algebra

Next we want to compute the quiver of Hsiao's algebra [24]. To each finite group G, Hsiao associates a left regular band of groups $\mathcal{F}_n^G$ (i.e., a regular semigroup in which each left ideal is two-sided). The group S_n acts by automorphisms on $\mathcal{F}_n^G$ and Hsiao showed that if G is abelian, then the invariant algebra $\mathbb{C}\mathcal{F}_n^{S_n}$ is anti-isomorphic to the descent algebra for $G \wr S_n$ of Mantaci and Reutenauer [29], a generalization of Bidigare's result for S_n [8, 10]; see [6, 24, 32] for the non-abelian case. Our goal is to compute the quiver for the opposite algebra to $k\mathcal{F}_n^G$ when k has good characteristic. To be consistent with [8–10, 24, 40] we shall work with $k\mathcal{F}_n^G$ and left modules. So the vertices of our quiver will be the simple left $k\mathcal{F}_n^G$-modules and the number of arrows from U to V wil l be $\dim \mathrm{Ext}^1_{k\mathcal{F}_n^G}(U, V)$, where now we work in the category of finitely generated left $k\mathcal{F}_n^G$-modules. Of course, this quiver is the usual quiver for algebra of the opposite semigroup of $\mathcal{F}_n^G$, which is a right regular band of groups, hence the results of the previous section apply. We remind the reader of the construction of Hsiao's left regular band of groups $\mathcal{F}_n^G$.

Fix a finite group G with identity 1_G and an integer $n \geq 1$. Set $[n] = \{1, \ldots, n\}$. An *ordered G-partition* of n consists of a sequence

$$\tau = ((P_1, g_1), \ldots, (P_r, g_r)) \tag{6.1}$$

where $\mathscr{P} = \{P_1, \ldots, P_r\}$ is a set partition of $\{1, \ldots, n\}$ and $g_1, \ldots, g_r \in G$. The monoid $\mathcal{F}_n^G$ consists of all ordered G-partitions of n with multiplication given by

$$((P_1, g_1), \ldots, (P_r, g_r))((Q_1, h_1), \ldots, (Q_s, h_s)) =$$
$$((P_1 \cap Q_1, g_1h_1), (P_1 \cap Q_2, g_1h_2), \ldots, (P_r \cap Q_1, g_rh_1), \ldots, (P_r \cap Q_s, g_rh_s))$$

where empty intersections are omitted. In fact, in [24] Hsiao writes h_ig_j instead of g_jh_i, but it is easy to see that our semigroup is isomorphic to his using the inversion in the group G. The identity element of $\mathcal{F}_n^G$ is $([n], 1_G)$. One can compute directly that

$$((P_1, g_1), \ldots, (P_r, g_r)) \mathscr{J} ((Q_1, h_1), \ldots, (Q_s, h_s))$$

if and only if $r = s$ and the set partitions $\{P_1, \ldots, P_r\}$ and $\{Q_1, \ldots, Q_r\}$ are equal; so $\mathscr{J}$-classes are in bijection with set partitions of n. We write $J_{\mathscr{P}}$ for the $\mathscr{J}$-class corresponding to a set partition $\mathscr{P}$. In fact, the $\mathscr{J}$-order is precisely the usual refinement order on set partitions: $\mathscr{P} \leq \mathscr{Q}$ if and only if each block of $\mathscr{P}$ is contained in a block of $\mathscr{Q}$. Let us write $\mathscr{P} \prec \mathscr{Q}$ if $\mathscr{P}$ is covered by $\mathscr{Q}$ in this ordering, that is, one can obtain $\mathscr{Q}$ from $\mathscr{P}$ by joining together two blocks.

An element τ as per Eq. 6.1 is idempotent if and only if $g_1 = \cdots = g_r = 1_G$. The maximal subgroup at τ in this case is isomorphic to G^r. The semigroup $\mathcal{F}_n^G$ is a left regular band of groups satisfying the identities $x^{|G|+1} = x$ and $xyx^{|G|} = xy$ [24]. In particular, if k is an algebraically closed field whose characteristic does not divide $|G|$, then we can apply our techniques to $k\mathcal{F}_n^G$. So from now on we assume that k is such a field.

Let X be a set. Define an *X-labelled set partition* of n to be a subset $\{(P_1, x_1), \ldots, (P_r, x_r)\}$ of $2^{[n]} \times X$ such that $\{P_1, \ldots, P_r\}$ is a set partition of n. We are now ready to describe the quiver of $k\mathcal{F}_n^G$.

Theorem 6.3 *Let G be a finite group and k an algebraically closed field such that the characteristic of k does not divide $|G|$. Denote by $\mathrm{Irr}(G)$ the set of simple left kG-modules. Then, for $n \geq 1$, the quiver of $k\mathcal{F}_n^G$ has vertex set the $\mathrm{Irr}(G)$-labelled set partitions of n. Let $\{(P_1, V_1), \ldots, (P_r, V_r)\}$ be an $\mathrm{Irr}(G)$-labelled set partition. We specify the outgoing arrows from this vertex as follows. Let $U \in \mathrm{Irr}(G)$ and $1 \leq i \neq j \leq r$. Then there are $\dim \mathrm{Hom}_{kG}(U, V_i \otimes_k V_j)$ (i.e., the multiplicity of U as a composition factor in $V_i \otimes_k V_j$) arrows from $\{(P_1, V_1), \ldots, (P_r, V_r)\}$ to*

$$\{(P_i \cup P_j, U)\} \cup \{(P_1, V_1), \ldots, (P_r, V_r)\} \setminus \{(P_i, V_i), (P_j, V_j)\}.$$

Proof In what follows we do not distinguish between a set partition of n and the corresponding equivalence relation on $[n]$. The $\mathscr{J}$-classes of $\mathcal{F}_n^G$ are in bijection with set partitions. If $\mathscr{P} = \{P_1, \ldots, P_r\}$ is a set partition, the maximal subgroup of the corresponding $\mathscr{J}$-class $J_{\mathscr{P}}$ is isomorphic to the group $G_{\mathscr{P}} = G^{[n]/\mathscr{P}}$. If $e = ((P_{i_1}, 1_G), \ldots, (P_{i_r}, 1_G))$ is an idempotent of $J_{\mathscr{P}}$, then the isomorphism $G_e \to G_{\mathscr{P}}$ takes $((P_{i_1}, g_{i_1}), \ldots, (P_{i_r}, g_{i_r}))$ to $f\colon [n]/P \to G$ given by $f(P_i) = g_i$.

Now the simple $kG_{\mathscr{P}}$-modules are in bijection with $\mathrm{Irr}(G)$-labelled set partitions via the correspondence

$$\{(P_1, V_1), \ldots, (P_r, V_r)\}\} \longmapsto V_1 \otimes_k \cdots \otimes_k V_r$$

with the tensor product action: $f \cdot v_1 \otimes \cdots \otimes v_r = f(P_1)v_1 \otimes \cdots \otimes f(P_r)v_r$; see [15]. It follows that the vertices of the quiver of $k\mathcal{F}_n^G$ can be identified with $\mathrm{Irr}(G)$-labelled partitions. In this proof, it will be convenient to work with this "coordinate-free" model of the maximal subgroups of $\mathcal{F}_n^G$, rather than fixing an idempotent from each $\mathscr{J}$-class. We will then change idempotent representatives of the $\mathscr{J}$-classes as is convenient and always use our fixed isomorphisms to the $G_{\mathscr{P}}$ to translate between simple modules from a given maximal subgroup and $\mathrm{Irr}(G)$-labelled set partitions.

Let us write π for the projection $2^{[n]} \times \mathrm{Irr}(G) \to 2^{[n]}$. So π takes an $\mathrm{Irr}(G)$-labelled set partition to its "underlying" partition. Fix $\mathrm{Irr}(G)$-labelled set partitions P and Q and let $\mathscr{P} = \pi(P)$, $\mathscr{Q} = \pi(Q)$. Since the $\mathscr{J}$-order is given by $J_{\mathscr{P}} \leq_{\mathscr{J}} J_{\mathscr{Q}}$ if and only if $\mathscr{P} \leq \mathscr{Q}$ in the refinement order, the results of the previous section show there are no arrows $P \to Q$ unless $\mathscr{P} < \mathscr{Q}$. Fix idempotents $e_{\mathscr{P}} \in J_{\mathscr{P}}$ and $e_{\mathscr{Q}} \in J_{\mathscr{Q}}$. Replacing $e_{\mathscr{P}}$ by $e_{\mathscr{Q}}e_{\mathscr{P}}$ if necessary, we may assume that $e_{\mathscr{P}} \leq_{\mathscr{R}} e_{\mathscr{Q}}$ and hence $e_{\mathscr{P}} \in e_{\mathscr{Q}}J_{\mathscr{P}}$. Because $e_{\mathscr{P}} <_{\mathscr{L}} e_{\mathscr{Q}}$, as the $\mathscr{J}$ and $\mathscr{L}$-orders coincide in a left regular band of groups, we have in fact $e_{\mathscr{P}} < e_{\mathscr{Q}}$ (recall that the idempotents of any semigroup are partially ordered by $e \leq f$ if and only if $ef = fe = e$ [12, 38]).

Suppose first that $\mathscr{P} < \mathscr{Q}$ but $\mathscr{P} \not\prec \mathscr{Q}$. Let us assume that $\mathscr{P} = \{P_1, \ldots, P_r\}$ and $\mathscr{Q} = \{Q_1, \ldots, Q_s\}$. Let $H = G_{e_{\mathscr{P}}}$ be the maximal subgroup at the idempotent $e_{\mathscr{P}}$. We show that the equivalence relation $\approx$ on $e_{\mathscr{Q}}J_{\mathscr{P}}$ has $|H|$ classes. It will then follow that the bimodule M constructed in the previous section is zero, as it has dimension $|X| - |H|$ where $X = e_{\mathscr{Q}}J/{\approx}$, and hence there are no arrows in the quiver from any vertex associated to $J_{\mathscr{P}}$ to any vertex associated to $J_{\mathscr{Q}}$. To do this, we show that for any $\gamma \in e_{\mathscr{Q}}J$, one has that $\gamma \approx e_{\mathscr{P}}\gamma$. This will show that each element of $e_{\mathscr{Q}}J$ is equivalent to an element of H. Since distinct elements of H are never identified under $\approx$, this will establish that $\approx$ has $|H|$ classes.

In the proof, we shall need the following notation. If $\sigma \in S_r$, then set

$$\sigma((B_1, g_1), \ldots, (B_r, g_r)) = ((B_{\sigma(1)}, g_{\sigma(1)}), \ldots, (B_{\sigma(r)}, g_{\sigma(r)})). \tag{6.2}$$

Without loss of generality we may assume $e_{\mathscr{P}} = ((P_1, 1_G), \ldots, (P_r, 1_G))$. The elements of H are then precisely the elements τ of the form Eq. 6.1. For $\gamma \in J_{\mathscr{P}}$, it is easy to see that $e_{\mathscr{P}}\gamma = \tau$, with τ as per Eq. 6.1, if and only if there is a permutation $\sigma \in S_r$ so that $\gamma = \sigma\tau$ using the notation of Eq. 6.2. The proof that $\approx$ has $|H|$ classes relies on two claims.

Claim 1 Suppose $\gamma_1, \gamma_2 \in J_{\mathscr{P}}$ are such that $e_{\mathscr{P}}\gamma_1 = e_{\mathscr{P}}\gamma_2$ and there exists $\rho \in \mathcal{F}_n^G$ with $\rho \not\geq_{\mathscr{J}} J_{\mathscr{Q}}$ and $\rho\gamma_i = \gamma_i$, $i = 1, 2$. Then $e_{\mathscr{Q}}\gamma_1 \approx e_{\mathscr{Q}}\gamma_2$.

Proof Because $e_{\mathscr{P}} < e_{\mathscr{Q}}$, if $e_{\mathscr{P}}\gamma_1 = e_{\mathscr{P}}\gamma_2$, then $e_{\mathscr{P}}e_{\mathscr{Q}}\gamma_1 = e_{\mathscr{P}}\gamma_1 = e_{\mathscr{P}}\gamma_2 = e_{\mathscr{P}}e_{\mathscr{Q}}\gamma_2$. Next observe that $e_{\mathscr{Q}}\rho <_{\mathscr{J}} J_{\mathscr{Q}}$. Because $e_{\mathscr{Q}}\gamma_i = e_{\mathscr{Q}}\rho\gamma_i = e_{\mathscr{Q}}\rho e_{\mathscr{Q}}\gamma_i$, for $i = 1, 2$, in the case $e_{\mathscr{Q}}\rho \notin J_{\mathscr{P}}$ it is immediate that $e_{\mathscr{Q}}\gamma_1 \smile e_{\mathscr{Q}}\gamma_2$. If $e_{\mathscr{Q}}\rho \in J_{\mathscr{P}}$, then $e_{\mathscr{Q}}\gamma_i = e_{\mathscr{Q}}\rho\gamma_i \mathrel{\mathscr{R}} e_{\mathscr{Q}}\rho$ by stability. Since $J_{\mathscr{P}}$ has a unique $\mathscr{L}$-class, it follows that $e_{\mathscr{Q}}\gamma_1 \mathrel{\mathscr{H}} e_{\mathscr{Q}}\gamma_2$. Let $f = (e_{\mathscr{Q}}\gamma_1)^{\omega} = (e_{\mathscr{Q}}\gamma_2)^{\omega}$. Then $f \mathrel{\mathscr{L}} e_{\mathscr{P}}$ and so $fe_{\mathscr{P}} = f$. Thus we have

$$e_{\mathscr{Q}}\gamma_1 = fe_{\mathscr{Q}}\gamma_1 = fe_{\mathscr{P}}e_{\mathscr{Q}}\gamma_1 = fe_{\mathscr{P}}e_{\mathscr{Q}}\gamma_2 = fe_{\mathscr{Q}}\gamma_2 = e_{\mathscr{Q}}\gamma_2.$$

This proves the claim. □

Claim 2 Given $(m\ m+1) \in S_r$ and $\alpha \in J_{\mathscr{P}}$, there exists $\rho \in \mathcal{F}_n^G$ with $\rho \not\geq_{\mathscr{J}} J_{\mathscr{Q}}$ and $\rho\alpha = \alpha$, $\rho(m\ m+1)\alpha = (m\ m+1)\alpha$ where we follow the notation of Eq. 6.2.

Proof Suppose that $\alpha = ((P_{j_1}, g_1), \ldots, (P_{j_r}, g_r))$. Let $\mathscr{P}'$ be the partition obtained from $\mathscr{P}$ by joining P_{j_m} and $P_{j_{m+1}}$. Since $\mathscr{P} \prec \mathscr{P}'$ and $\mathscr{P} \not\prec \mathscr{Q}$, it follows that $J_{\mathscr{P}'} \not\geq_{\mathscr{J}} J_{\mathscr{Q}}$. Routine computation shows that

$$\rho = ((P_{j_1}, 1_G), \ldots, (P_{j_m} \cup P_{j_{m+1}}, 1_G), \ldots, (P_{j_r}, 1_G))$$

does the job. □

To complete the proof in the case $\mathscr{P} \not\prec \mathscr{Q}$, we must show that if τ is as in Eq. 6.1 and $\gamma = \sigma\tau \in e_{\mathscr{Q}}J_{\mathscr{P}}$ with $\sigma \in S_r$, then $\gamma \approx \tau$. Since S_r is generated by the consecutive transpositions, we can connect τ to γ by a sequence $\tau = \alpha_0, \alpha_1, \ldots, \alpha_n = \gamma$ where $\alpha_{i+1} = (m_i\ m_i+1)\alpha_i$ for some m_i, all $i = 0, \ldots, n-1$. Then by Claims 1 and 2, we have $\tau = e_{\mathscr{Q}}\alpha_0 \approx e_{\mathscr{Q}}\alpha_1 \approx \cdots \approx e_{\mathscr{Q}}\alpha_n = \gamma$.

Next suppose that $\mathscr{P} \prec \mathscr{Q}$. Then we can order our $\mathrm{Irr}(G)$-labelled partitions P and Q so that $\mathscr{P} = \{P_1, \ldots, P_r\}$ and $\mathscr{Q} = \{P_1, \ldots, P_{r-2}, P_{r-1} \cup P_r\}$. To fix notation, we write

$$P = \{(P_1, V_1), \ldots, (P_r, V_r)\}$$
$$Q = \{(P_1, U_1), \ldots, (P_{r-2}, U_{r-2}), (P_{r-1} \cup P_r, U)\}.$$

We choose as representatives of $J_{\mathscr{P}}$ and $J_{\mathscr{Q}}$ the respective idempotents $e_{\mathscr{P}} = ((P_1, 1_G), \ldots, (P_r, 1_G))$ and $e_{\mathscr{Q}} = ((P_1, 1_G), \ldots, (P_{r-2}, 1_G), (P_{r-1} \cup P_r, 1_G))$. Notice that $e_{\mathscr{P}} < e_{\mathscr{Q}}$. Then under our isomorphisms $G_{\mathscr{P}} \cong G_{e_{\mathscr{P}}}$ and $G_{\mathscr{Q}} \cong G_{e_{\mathscr{Q}}}$ the simple $kG_{e_{\mathscr{P}}}$-module corresponding to P is $V_1 \otimes_k \cdots \otimes_k V_r$ and the simple $kG_{e_{\mathscr{Q}}}$-module corresponding to Q is $U_1 \otimes_k \cdots \otimes_k U_{r-2} \otimes_k U$.

Since $J_{\mathscr{Q}}$ covers $J_{\mathscr{P}}$ in the $\mathscr{J}$-order, the equivalence relation $\approx$ identifies no elements. Therefore, $X = ke_{\mathscr{Q}}J_{\mathscr{P}}$. Now it is not hard to compute that $e_{\mathscr{Q}}J_{\mathscr{P}}$ consists of all elements of the form τ and $(r-1\ r)\tau$ with τ as per Eq. 6.1. Moreover, $e_{\mathscr{P}}\tau = \tau = e_{\mathscr{P}}(r-1\ r)\tau$. Therefore, the bimodule M from the previous section has a basis consisting of the elements of the form $(r-1\ r)\tau - \tau$ with τ as above. It is immediate that as a right $kG_{e_{\mathscr{P}}}$-module, M is isomorphic to the regular module since if $\rho \in G_{e_{\mathscr{P}}}$ then $((r-1\ r)\tau)\rho = (r-1\ r)(\tau\rho)$. On the other hand if

$$\lambda = ((P_1, h_1), \ldots, (P_{r-2}, h_{r-2}), (P_{r-1} \cup P_r, h)) \in G_{e_{\mathscr{Q}}}, \tag{6.3}$$

then $\lambda\big((r-1\ r)\tau - \tau\big)$ is $(r-1\ r)\beta - \beta$ where

$$\beta = ((P_1, h_1g_1), \ldots, (P_{r-2}, h_{r-2}g_{r-2}), (P_{r-1}, hg_{r-1}), (P_r, hg_r)) \tag{6.4}$$

as can be verified by direct computation.

Since M is regular as a right $kG_{e_{\mathscr{P}}}$-module,

$$M \otimes_{kG_{e_{\mathscr{P}}}} (V_1 \otimes_k \cdots \otimes_k V_r) \cong V_1 \otimes_k \cdots \otimes_k V_r$$

as a vector space. By Eq. 6.4, the action of $\lambda \in G_{e_{\mathscr{Q}}}$ on $V_1 \otimes_k \cdots \otimes_k V_r$ is given on elementary tensors by

$$\lambda \cdot v_1 \otimes \cdots \otimes v_r = h_1v_1 \otimes \cdots \otimes h_{r-2}v_{r-2} \otimes hv_{r-1} \otimes hv_r$$

with λ as per Eq. 6.3.

Since the tensor product distributes over sums, it follows that the multiplicity of $U_1 \otimes_k \cdots \otimes_k U_{r-2} \otimes U$ as a composition factor of $V_1 \otimes_k \cdots \otimes_k V_r$ is 0 unless $U_i = V_i$ for $1 \leq i \leq r-2$, in which case it is the multiplicity of U as a constituent in $V_{r-1} \otimes_k V_r$. This completes the proof. □

For example, if G is trivial, then the quiver of $k\mathcal{F}_n^G$ is just the Hasse diagram of the lattice of set partitions of n, as was first proved by Saliola [40] and Schocker [41]. Suppose now that G is a finite abelian group and the characteristic of k does not divide $|G|$. Let $\widehat{G}$ be the dual group of G, that is, $\mathrm{Hom}_{\mathbb{Z}}(G, k^*)$. Of course, $\widehat{G} \cong G$ and $\mathrm{Irr}(G) \cong \widehat{G}$. The tensor product of representations corresponds to the product of the characters. Thus the quiver of $k\mathcal{F}_n^G$ has vertices all $\widehat{G}$-labelled partitions of n. A vertex $\{(P_1, \chi_1), \ldots, (P_r, \chi_r)\}$ has $\binom{r}{2}$ outgoing arrows: for $1 \leq i \neq j \leq r$, there is an arrow from $\{(P_1, \chi_1), \ldots, (P_r, \chi_r)\}$ to $\{(P_i \cup P_j, \chi_i \cdot \chi_j)\} \cup \{(P_1, \chi_1), \ldots, (P_r, \chi_r)\} \setminus \{(P_i, \chi_i), (P_j, \chi_j)\}$. In particular, the quiver is acyclic with no multiple arrows between vertice s.

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