HYPERPLANE ARRANGEMENTS AND DESCENT ALGEBRAS

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ABSTRACT. We will define an algebra on the faces of a hyperplane arrangement and explain how the descent algebra embeds in this algebra when the hyperplane arrangement is the reflection arrangement of a finite Coxeter group. We will use the structure of the former algebra to prove results about the latter algebra.

1. THE FACE SEMIGROUP ALGEBRA OF A HYPERPLANE ARRANGEMENT

1.1. **The Braid Arrangement.** The main reference for the theory of hyperplane arrangements is the book [Orlik and Terao, 1992]. An introductory treatment is the lecture notes by Richard Stanley posted on his website.

A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{R}^d$ is a finite set of hyperplanes in $\mathbb{R}^d$. We will restrict our attention to *central* hyperplane arrangements where all the hyperplanes contain the origin.

Through these notes we will be interested specifically in the *braid arrangement* $\mathcal{B}$. It consists of the hyperplanes $H_{ij} = \{v \in \mathbb{R}^d : v_i = v_j\}$ where $1 \leq i < j \leq d$. Since all the hyperplanes intersect in the one dimensional subspace $v_1 = v_2 = \cdots = v_n$, the braid arrangement gives a hyperplane arrangement in a $n-1$ dimensional vector space by intersecting the hyperplanes in the arrangement by the hyperplane $v_1+v_2+v_3 = 0$. The arrangement for $n = 3$ can be pictured as in Figure 1. Figure 2 shows the resulting arrangement for $n = 4$ intersected with one hemisphere of the unit sphere of $\mathbb{R}^3$. A few notes about the image: the image shows only one hemisphere of the sphere; the equator does not correspond to a hyperplane in the arrangement so it is denoted by

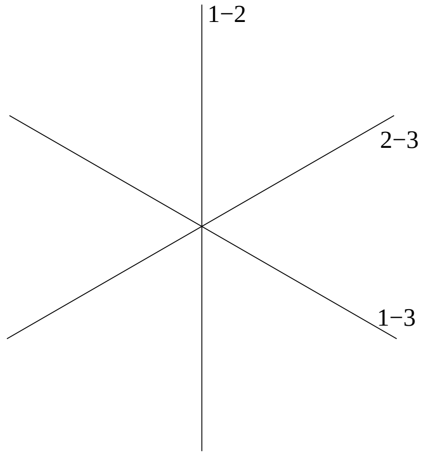

FIGURE 1. The Braid Arrangement for $n = 3$.

a dotted line; the great circle corresponding to the hyperplane H_{ij} is labelled i-j; I stole the image from Ken Brown's paper [Brown, 2000] and modified it.

The symmetric group S_n on n elements acts on the vector space $\mathbb{R}^n$ by permuting coordinates: for $\omega \in S_n$ and $v \in \mathbb{R}^n$, let

$$\omega(v) = \omega((v_1, \ldots, v_n)) = (v_{\omega^{-1}(1)}, \ldots, v_{\omega^{-1}(n)}).$$

Then the transpositions (i, j) where $1 \leq i < j \leq n$, which generate S_n, act on $\mathbb{R}^n$ by reflecting about the hyperplanes H_{ij}. Any finite group isomorphic to a finite group generated by a set of reflections of $\mathbb{R}^n$ is called a *finite reflection group* or a *finite Coxeter group*. The results presented here generalize to all finite Coxeter groups *mutatis mutandis*.

1.2. **The Faces of an Arrangement.** Let $\mathcal{A}$ denote a hyperplane arrangement. Each hyperplane $H \in \mathcal{A}$ determines two open half-spaces of $\mathbb{R}^n$ denoted H^+ and H^-. The choice of which half-space to label $+$ or $-$ is arbitrary, but fixed.

A *face* of $\mathcal{A}$ is a nonempty intersection of the form

$$x = \bigcap_{H \in \mathcal{A}} H^{\sigma_H(x)},$$

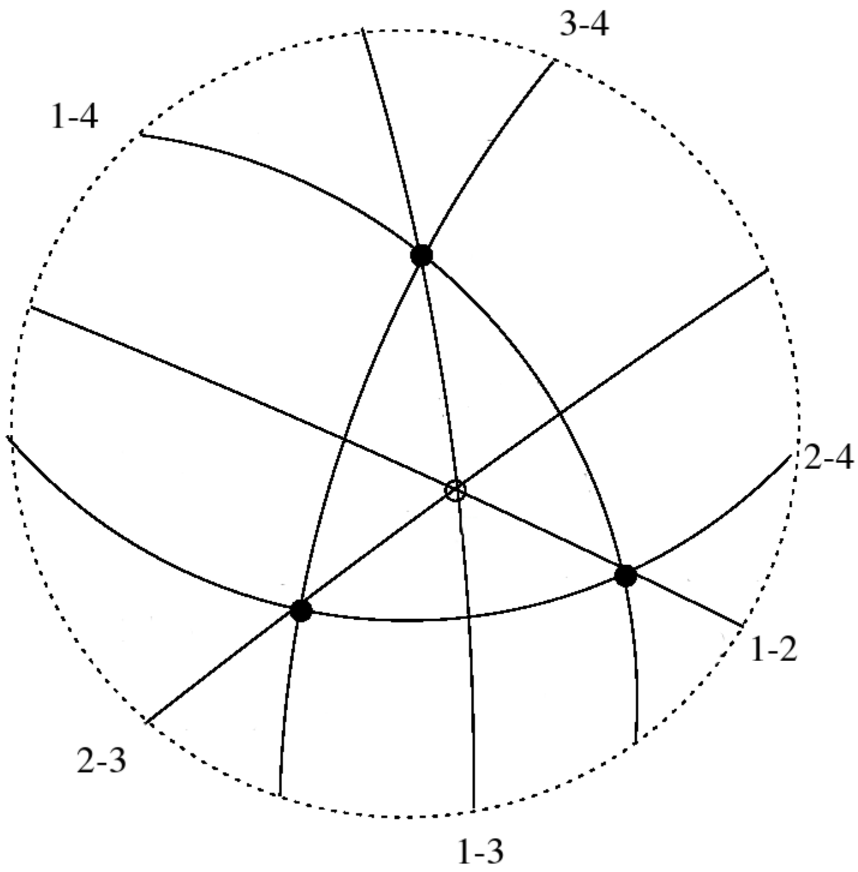

FIGURE 2. The Braid Arrangement for $n = 4$.

where $\sigma_H(x) \in \{+, -, 0\}$ and $H^0 = H$. Note that x is a relatively open subset of $\mathbb{R}^n$. If x is a face, then the vector $\sigma(x) = (\sigma_H(x))_{H\in\mathcal{A}}$ is the *sign vector* of x. In Figure 3 the faces of the braid arrangement for $n = 3$ are labelled by their sign vectors.

A *chamber* is a face that is the nonempty intersections of the open half spaces determined by the hyperplanes $H \in \mathcal{A}$. Equivalently, the chambers are the faces c such that $\sigma_H(c) \neq 0$ for all $H \in \mathcal{A}$. Note that the chambers are the connected components of the complement $\mathbb{R}^n - \cup_{H\in\mathcal{A}} H$.

Partially order the faces $\mathcal{F}$ by $x \leq y$ iff for each $H \in \mathcal{A}$ either $\sigma_H(x) = 0$ or $\sigma_H(x) = \sigma_H(y)$. Equivalently, $x \leq y$ iff $x \subset \bar{y}$. This

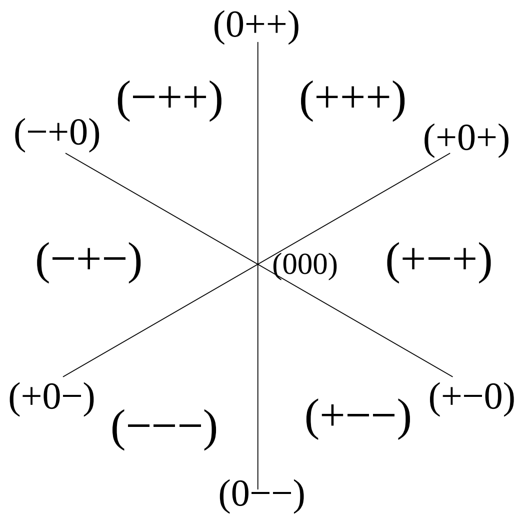

FIGURE 3. Sign vectors on the faces of the braid arrangement for $n = 3$.

partial order is called the *face relation*, and if $x \leq y$, then we say x *is a face of* y. This terminology comes from the fact that the closure of a chamber c is a polyhedron and that the closure of the faces (in the above sense) of c are the polyhedral faces of the polyhedron. Note that the chambers are the maximal elements in this partial order.

1.2.1. *The Faces of the Braid Arrangement.* Let $\mathcal{A}$ denote the braid arrangement. Let $v \in \mathbb{R}^n$ be a vector in a chamber of $\mathcal{A}$. Then v is not on any of the hyperplanes H_{ij}, so all the coordinates of v are distinct. Therefore, there exists $\omega \in S_n$ such that

$$v_{\omega(1)} < \cdots < v_{\omega(n)}.$$

All vectors in the chamber satisfy this identity, so the chamber can be identified with the permutation ω of the set $[n] = \{1, \ldots, n\}$. (This is true for any finite reflection group: the chambers of the reflection arrangement are in bijective correspondence with the elements of the group.) The faces of the chamber are obtained by changing some of the inequalities above to equalities. So the faces $\mathcal{F}$ of $\mathcal{A}$ can be identified with *set compositions* (ordered set partitions) of $[n]$. For example,

$$(23, 4, 1) \leftrightarrow \{v \in V : v_2 = v_3 < v_4 < v_1\}.$$

Here we have concatenated the elements of each block to simplify notation: $(23, 4, 1)$ denotes the set composition $(\{2, 3\}, \{4\}, \{1\})$. In Figure 4 the faces of the braid arrangement for $n = 4$ are labelled by the corresponding set compositions.

It is straightforward to verify that the partial order (the face relation) on set compositions is given by $(B_1, \ldots, B_m) \leq (C_1, \ldots, C_l)$ iff $(C_1, \ldots, C_l)$ consists of a set composition of B_1, followed by a set composition of B_2, and so forth. The action of S_n on $\mathcal{F}$ is given by $\omega((B_1, \ldots, B_r)) = (\omega(B_1), \ldots, \omega(B_r))$. The poset of faces for the braid arrangement for $n = 3$ is depicted in Figure 5.

1.3. **The Support Map and the Intersection Lattice.** The *support* $\mathrm{supp}(x)$ of a face $x \in \mathcal{F}$ is the the intersection of the hyperplanes in $\mathcal{A}$ containing x.

$$\mathrm{supp}(x) = \bigcap_{\substack{H \in \mathcal{A} \\ \sigma_H(x)=0}} H.$$

The set $\mathcal{L} = \mathrm{supp}(\mathcal{F})$ of supports of faces of $\mathcal{A}$ is a graded lattice ordered by inclusion, called the *intersection lattice* of $\mathcal{A}$. (Some authors order the intersection lattice by *reverse* inclusion, so some care is needed while reading the literature.) For $X, Y \in \mathcal{L}$ the *meet* $X \wedge Y$ of X and Y is the intersection $X \cap Y$ and the *join* $X \vee Y$ of X and Y is $X + Y$, the smallest subspace of $\mathbb{R}^d$ containing both X and Y. The top element $\hat{1}$ of $\mathcal{L}$ is the ambient vector space $\mathbb{R}^d$ and the bottom element $\hat{0}$ is the intersection of all hyperplanes in the arrangement $\bigcap_{H\in\mathcal{A}} H$. The rank of $X \in \mathcal{L}$ is the dimension of the subspace $X \subset \mathbb{R}^d$.

The chambers are the faces of support $\hat{1}$. Since $\mathrm{supp}(x) \leq \mathrm{supp}(y)$ if $x \leq y$, the support map $\mathrm{supp} : \mathcal{F} \to \mathcal{L}$ is an order-preserving poset surjection.

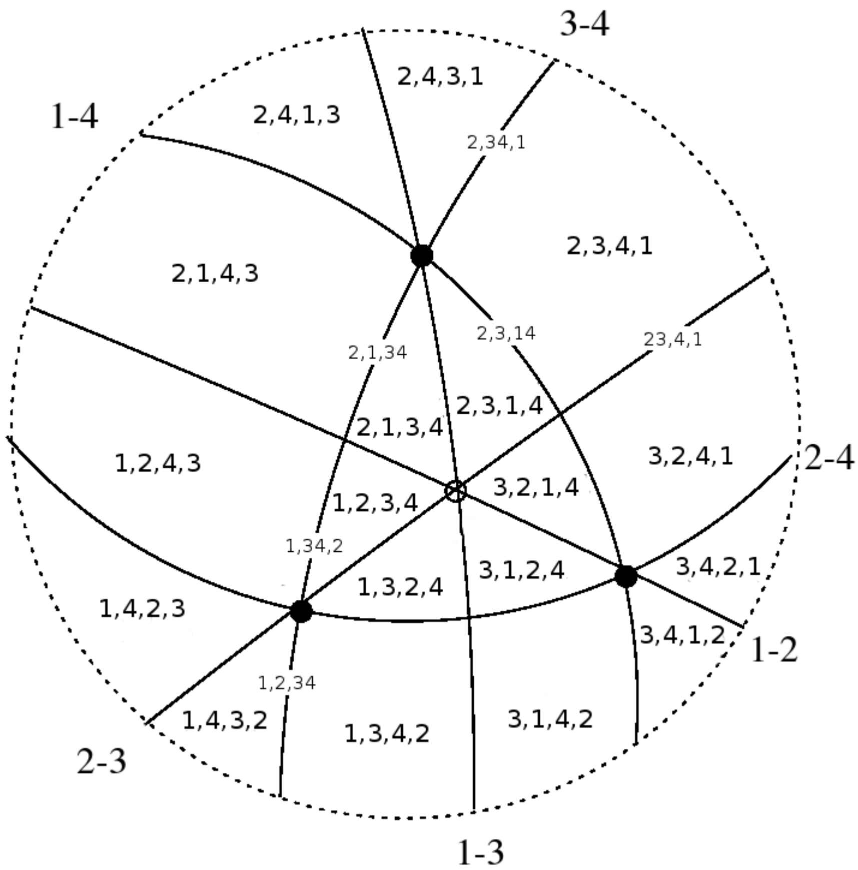

Figure 4. Set compositions on the braid arrangement when $n = 4$.

1.3.1. *The Support Map and the Intersection Lattice of the Braid Arrangement.* As we saw above, the faces of the braid arrangement correspond to set compositions of $[n]$. Under this identification, the support map just forgets the order of the set composition, giving a set partition of $[n]$.

$$\operatorname{supp}\Big((B_1, \ldots, B_r)\Big) = \{B_1, \ldots, B_r\},$$

where $\{B_1, \ldots, B_r\}$ is a set partition of $[n]$. Explicitly, this identification between set partitions of $[n]$ and the intersection lattice of the braid

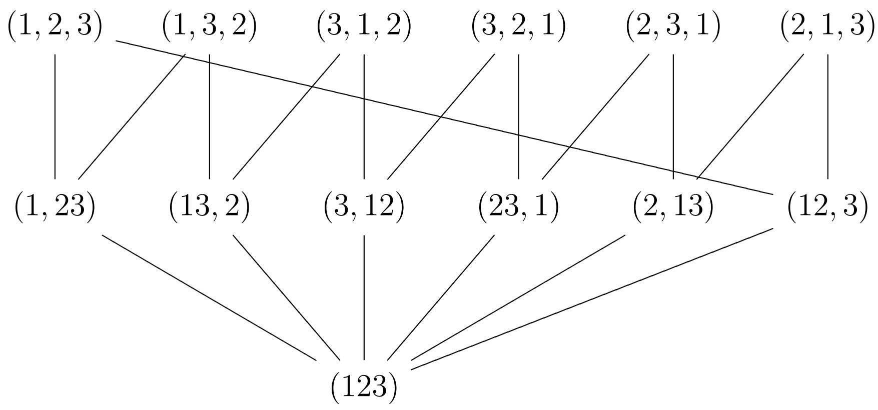

FIGURE 5. The poset of faces (set compositions) of the braid arrangement for $n = 3$.

arrangement is given by the following.

$$\{B_1, \dots, B_r\} \leftrightarrow$$
$$\Big\{v \in V : v_i = v_j \text{ if } \exists h \text{ such that } i, j \in B_h\Big\} = \bigcap_{h=1}^{r} \left(\bigcap_{i,j \in B_h} H_{ij} \right),$$

where $\{B_1, \dots, B_r\}$ is a set partition of $[n]$. If B and C are set partitions of $[n]$, then $B \lessdot C$ iff B is obtained from C by merging two blocks of C. The action of S_n on $\mathcal{L}$ is given by $\omega(\{B_1, \dots, B_r\}) = \{\omega(B_1), \dots, \omega(B_r)\}$.

1.4. **The Face Semigroup.** For $x, y \in \mathcal{F}$ the product xy is the face of $\mathcal{A}$ with sign vector

$$\sigma_H(xy) = \begin{cases} \sigma_H(x), & \text{if } \sigma_H(x) \neq 0, \\ \sigma_H(y), & \text{if } \sigma_H(x) = 0. \end{cases}$$

Proposition 1.1. *The product xy of two faces x and y is a face of $\mathcal{A}$.*

Proof. We need to show that the intersection determined by the sign vector $\sigma(xy)$ is nonempty. Let

$$A = \bigcap_{\sigma_H(x) \neq 0} H^{\sigma_H(x)}, \qquad B = \bigcap_{\sigma_H(x) = 0} H^{\sigma_H(y)}.$$

Then $xy = A \cap B$. If $v \in x$, then v is contained in A and in the closure of B. Since A is open, it follows that $A \cap B$ is nonempty because any open set containing v intersects B. $\square$

It is straightforward to check that this product is associative, noncommutative and that the identity element is the intersection of all the hyperplanes in the arrangement $1 = \bigcap_{H \in \mathcal{A}} H$. Note that the support of the identity element 1 is $\hat{0}$ (and not $\hat{1}$).

The support $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ satisfies $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$ for all $x, y \in \mathcal{F}$. Therefore, supp is a semigroup surjection, where $\mathcal{L}$ is considered a semigroup with product given by the join $\vee$, as well as an ordering-preserving poset surjection.

Exercise 1. Let $x, y \in \mathcal{F}$. Prove that if v_x and v_y are two points in x and y, respectively, then xy is the face that contains $v_x + (v_y - v_x)\epsilon$ for all sufficiently small $\epsilon > 0$. (*Hint*: If $\sigma_H(x) \neq 0$, then $H^{\sigma_H(x)}$ is an open set containing v_x. Thus there exists an $\epsilon_H > 0$ such that $v_x + (v_y - v_x)\epsilon \in H^{\sigma_H(x)}$ for all $0 \leq \epsilon < \epsilon_H$. Let $\epsilon' = \min_{\sigma_H(x) \neq 0}(\epsilon_H)$. Show that for $v_x + (v_y - v_x)\epsilon \in H^{\sigma_H(xy)}$ for all $H \in \mathcal{A}$ and all $0 < \epsilon < \epsilon'$.)

Exercise 2. For all $x, y \in \mathcal{F}$,

(1) $x^2 = x$,
(2) $xyx = xy$,
(3) $xy = y$ iff $x \leq y$,
(4) If $x \leq y$, then $\operatorname{supp}(x) \leq \operatorname{supp}(y)$.
(5) $xy = x$ iff $\operatorname{supp}(y) \leq \operatorname{supp}(x)$,
(6) $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$.

Remark 1.2. Conditions (1) and (2) of the proposition say that $\mathcal{F}$ belongs to a class of semigroups known as *left regular bands*.

Exercise 3. Let $\mathcal{C}$ denote the set of chambers of a hyperplane arrangement $\mathcal{A}$. Define $d : \mathcal{C} \times \mathcal{C} \to \mathbb{N}$ for $c, c' \in \mathcal{C}$ by setting $d(c, c')$ equal to the number of hyperplanes that separate c from c'.

(1) Prove that d is a metric.

(2) If x is a face of the arrangement $\mathcal{A}$ and $c \in \mathcal{C}$ is a chamber, then show that there is a unique face c' with $x \leq c'$ minimizing $d(c, c')$. Prove that $c' = xc$.

1.4.1. *The Product of Faces in the Braid Arrangement.* Recall that the faces of the braid arrangement correspond to set compositions of $[n]$. Let $B = (B_1, \ldots, B_l)$ and $C = (C_1, \ldots, C_m)$ denote two set compositions of $[n]$. Let $B(i)$ denote the block of B that contains i, and let $B(i) < B(j)$ denote that the block $B(i)$ appears before the block $B(j)$ in B. For each $1 \leq i < j \leq n$ define $H_{ij}^+ = \{v \in \mathbb{R}^n : v_i < v_j\}$ and $H_{ij}^- = \{v \in \mathbb{R}^n : v_i > v_j\}$. Then the sign vector of B is given by

$$\sigma_{H_{ij}}(B) = \begin{cases} 0, & B(i) = B(j), \\ +, & B(i) < B(j), \\ -, & B(i) > B(j). \end{cases} \qquad \sigma_{H_{ij}}(C) = \begin{cases} 0, & C(i) = C(j), \\ +, & C(i) < C(j), \\ -, & C(i) > C(j). \end{cases}$$

Therefore, the sign vector of the product BC is given by

$$\sigma_{H_{ij}}(BC) = \begin{cases} 0, & B(i) = B(j) \text{ and } C(i) = C(j), \\ +, & B(i) < B(j), \text{ or } B(i) = B(j) \text{ and } C(i) < C(j), \\ -, & B(i) > B(j), \text{ or } B(i) = B(j) \text{ and } C(j) < C(i). \end{cases}$$

From this it follows that the product of set compositions B and C is

$$\begin{aligned} &(B_1, \ldots, B_l)\ (C_1, \ldots, C_m) \\ &\qquad = (B_1 \cap C_1, \ldots, B_1 \cap C_m,\ \cdots,\ B_l \cap C_1, \ldots, B_l \cap C_m)^{✄}, \end{aligned}$$

where ✄ means "delete empty intersections".

Here are some examples.

Example 1. $(2467, 931, 58)(34, 1256, 789) = (4, 26, 7, 3, 1, 9, 5, 8)$. Try it at home; it's fun.

Example 2. $(5, 1234678)(2, 4, 6, 7, 3, 1, 5, 8) = (5, 2, 4, 6, 7, 3, 1, 8)$. In this example, a chamber c is multiplied on the left by a set composition of the form $(i, [n] - i)$. This has the effect of moving i to the beginning (the top) of the set composition c. This process is known as the *random-to-top* shuffle.

Example 3. $(137, 24568)(2, 4, 6, 7, 3, 1, 5, 8) = (7, 3, 1, 2, 4, 6, 5, 8)$. In this example a chamber c is multiplied on the left by a set composition of the form $(S, [n] - S)$ where $S \subset [n]$. This has the effect of moving the elements of S, in the order they appeared in the composition c, to the beginning of the composition. This is precisely the inverse of riffle shuffling a deck of cards. When you riffle shuffle a deck of cards, you divide the set in half and shuffle the cards together. Here we pull out a subset of the cards and place the cards on top.

Example 4. $(137, 245, 8, 6)(1, 2, 3, 4, 5, 6, 7, 8) = (1, 3, 7, 2, 4, 5, 8, 6)$. Here a face set composition $B = (B_1, \ldots, B_m)$ of $[n]$ is multiplied on the right by the chamber $(1, 2, \ldots, n)$. The resulting set composition has singleton blocks, and is obtained by listing the elements of B_1 in numerical order, followed by the elements of B_2 in numerical order, and so forth. Let $\omega \in S_n$ be the permutation corresponding to this composition. In this example $\omega = (1, 3, 7, 2, 4, 5, 8, 6)$. Then the set of indices i for which $\omega(i) > \omega(i + 1)$ is a subset of the $\{|B_1|, |B_1| + |B_2|, \ldots, |B_1| + \cdots + |B_{m-1}|\}$ since $\omega(i) < \omega(i + 1)$ if the i-th element and the $i+1$-th element appear in the same block B_j.

Proposition 1.3. *Let $(B_1, \ldots, B_m)$ be a set composition of $[n]$. Then the product $(B_1, \ldots, B_m)(1, 2, \ldots, n)$ is the set composition formed by listing the elements of B_1 in (numerical) order, then listing the elements of B_2 in order, and so forth. Explicitly, $(B_1, \ldots, B_m)(1, 2, \ldots, n)$ is*

$$\Big(b_1^{(1)}, b_2^{(1)}, \ldots, b_{|B_1|}^{(1)}, \ \cdots, \ b_1^{(m)}, b_2^{(m)}, \ldots, b_{|B_m|}^{(m)}\Big).$$

where $b_1^{(i)} < b_2^{(i)} < \ldots < b_{|B_i|}^{(i)}$ are the elements of the block B_i. Moreover,

$$\{i : \omega(i) > \omega(i+1)\} \subseteq \{|B_1|, |B_1| + |B_2|, \ldots, |B_1| + \cdots + |B_{m-1}|\}.$$

1.5. **The Face Semigroup Algebra.** Let $\mathcal{A}$ denote a hyperplane arrangement in $\mathbb{R}^n$ and let k denote an arbitrary field. The *face semigroup* *algebra* of $\mathcal{A}$ with coefficients in the field k is the semigroup algebra $k\mathcal{F}$ of the face semigroup $\mathcal{F}$ of $\mathcal{A}$. It consists of linear combinations of elements of $\mathcal{F}$ with multiplication induced by the product of $\mathcal{F}$. The face semigroup algebra $k\mathcal{F}$ is a finite dimensional associative algebra with identity $1 = \bigcap_{H\in\mathcal{A}} H$. Unless explicitly stated otherwise, no assumptions will be made on the characteristic of the field k.

1.6. **Random Walks on the Chambers of a Hyperplane Arrangement.** Let $\mathcal{A}$ denote a hyperplane arrangement. It follows from the definition that the product of a face with a chamber, in either order, is a chamber. Therefore, the set $\mathcal{C}$ of chambers is an ideal of the semigroup $\mathcal{F}$. We can therefore run a random walk on this ideal using the product of $\mathcal{F}$.

Let $p = \{p_x\}_{x\in\mathcal{F}}$ denote a probability distribution on the faces $\mathcal{F}$. Therefore, the p_x are nonnegative real numbers satisfying $\sum_x p_x = 1$. If at the i-th stage of the random walk we are at a chamber c, then pick a face x with probability p_x and move to the chamber xc. To be explicit, xc is the product of the faces x and c.

As noted in the previous section, if $\mathcal{A}$ is the braid arrangement and if p_x is nonzero only on the faces of the form $(i, [n]-i)$, where $1 \leq i \leq n$, and zero otherwise, then the resulting random walk is the random-to-top card shuffling process. Similarly, one obtains the inverse riffle shuffle by assigning weights $1/2^n$ to the faces $(S, [n]-S)$ with $\emptyset \subsetneq S \subsetneq [n]$ and $2/2^n$ to the one block partition $([n])$.

The main tool for studying a random walk is the *transition matrix* T of the random walk. It is the matrix indexed by the "states" of the random walk and with (s,t)-entry the probability of moving to the state s from the state t. For the random walks on the chambers of a hyperplane arrangement, the states are the chambers and the probability of moving from chamber c to chamber d is the sum of the probabilities p_x on the faces x satisfying $d = xc$. Therefore, the (c,d)-entry of T is $T(c,d) = \sum_{xc=d} p_x$. We can now state a remarkable theorem that describes the eigenvalues of the transition matrix.

Theorem 1.4 ([Brown and Diaconis, 1998]). *Let $\mathcal{A}$ be a hyperplane arrangement in $V = \mathbb{R}^n$, let $\mathcal{F}$ be the set of faces, left $\mathcal{L}$ be the intersection lattice and let p be a probability measure on $\mathcal{F}$. Then the transition matrix T of the random walk defined above is diagonalizable. For each $X \in \mathcal{L}$ there is one eigenvalue*

$$\lambda_X = \sum_{\substack{x \in \mathcal{F} \\ \operatorname{supp}(x) \leq X}} p_x,$$

with multiplicity

$$m_X = |\mu(X, V)| = (-1)^{\operatorname{codim}(X)} \mu(X, V),$$

where μ is the Möbius function of $\mathcal{L}$.

This result was extended in [Brown, 2000] to the entire class of semigroups known as left regular bands, and later partly generalized to semigroups known as bands [Brown, 2004]. These papers are based on the following observation. Let $\{p_x\}_{x \in \mathcal{F}}$ denote a probability measure on the faces $\mathcal{F}$, and let $p = \sum_{x \in \mathcal{F}} p_x x$ denote the element of the semigroup algebra $\mathbb{R}\mathcal{F}$ of $\mathcal{F}$ with the coefficient of x given by p_x. (The semigroup algebra is defined in the next section.) Then for any element $a = \sum_{c \in \mathcal{F}} a_c c$ of $\mathbb{R}\mathcal{C}$,

$$pa = \sum_x p_x x \sum_c a_c c = \sum_d \left(\sum_{\substack{x,c \\ xc = d}} a_c p_x \right) d = \sum_d \left(\sum_c a_c T(c, d) \right) d,$$

where T is the transition matrix of the random walk. Therefore, left multiplication by p corresponds to right multiplication by T on row vectors $(a_c)_{c \in \mathcal{C}}$. This allows one to study the random walk by using algebraic techniques. For example, Brown shows that the subalgebra $\mathbb{R}[p]$ of $\mathbb{R}\mathcal{F}$ generated by $p = \sum_x p_x x$, where $p_x \geq 0$, is *split semisimple*, which implies that the action of p on any $\mathbb{R}[p]$-module is diagonalizable. Since $k\mathcal{C}$ is a $\mathbb{R}[p]$-module, the diagonalizability result follows. The eigenvalues with their multiplicities are obtained using the irreducible representations of the semigroup algebra $\mathbb{R}\mathcal{F}$.

2. The Descent Algebra in the Face Semigroup Algebra of a Reflection Arrangement

In this section we will prove that the descent algebra of S_n is anti-isomorphic to a subalgebra of the face semigroup algebra $k\mathcal{F}$ of the braid arrangement. We begin by defining the descent algebra of S_n.

2.1. **The Descent Algebra of S_n.** Let $\omega \in S_n$. The *descent set* $\operatorname{des}(\omega)$ of ω is the set of indices i for which $\omega(i) > \omega(i+1)$.

$$\operatorname{des}(\omega) = \{i \in [n-1] : \omega(i) > \omega(i+1)\}.$$

For $J \subset [n-1]$ let x_J denote the element of the group algebra kS_n of S_n that is the sum of all elements of S_n with descent set contained in J.

$$x_J = \sum_{\operatorname{des}(\omega) \subseteq J} \omega.$$

Solomon proved that the elements x_J for $J \subset [n-1]$ form a basis of a subalgebra $\mathcal{D}(S_n)$ of kS_n, called the *descent algebra* [Solomon, 1976]. He showed that such an algebra can be constructed for each finite Coxeter group.

2.2. **An Action of S_n on Set Compositions.** Suppose A is an algebra. Let G be a group. The group G is said to *act* on A if there is a homomorphism of G into the group of endomorphisms of A (recall that an endomorphism of A is an algebra isomorphism $A \xrightarrow{\cong} A$). We denote the action of $g \in G$ on $a \in A$ by writing $g(a)$. Let A^G denote the set of elements $a \in A$ such that $g(a) = a$ for all $g \in G$. Then A^G is a subalgebra of A: if $a, a' \in A$, then $g(a) = a$ and $g(a') = a'$ for all $g \in G$; hence $g(aa') = g(a)g(a') = aa'$ for all $g \in G$. The algebra A^G is called the *G-invariant subalgebra* of A.

We now show that there is an action of S_n on the algebra $k\mathcal{F}$. Recall that a face of an arrangement is a nonempty intersection of the form $\bigcap_H H^{\sigma_H}$, where σ_H is either 0, $+$ or $-$. Since the action of S_n on $\mathbb{R}^n$ permutes the set of hyperplanes in the braid arrangement, it follows that this action of S_n on $\mathbb{R}^n$ permutes the set of faces $\mathcal{F}$ of the braid arrangement. By identifying the faces with set compositions this action of S_n is given by: for $\omega \in S_n$ and a set composition $(B_1, \ldots, B_m)$,

$$\omega\Big((B_1, \ldots, B_m)\Big) = \big(\omega(B_1), \ldots, \omega(B_m)\big).$$

Exercise 4. Show that this action of S_n on set compositions of $[n]$ agrees with the action of S_n on $\mathbb{R}^n$ after identifying the faces of the braid arrangement with set compositions of $[n]$.

2.3. **The Descent Algebra as a Subalgebra of the Face Semigroup Algebra of the Braid Arrangement.**

Theorem 2.1 ([Bidigare, 1997])**.** *The descent algebra $\mathcal{D}(S_n)$ of S_n is anti-isomorphic to the subalgebra $(k\mathcal{F})^{S_n}$.*

Proof. Let $\mathcal{C}$ denote the set of chambers in the braid arrangement and let $k\mathcal{C}$ denote the k-vector space spanned by the chambers $\mathcal{C}$. Since the product of a face with a chamber is always a chamber, it follows that $k\mathcal{C}$ is a two-sided ideal of the face semigroup algebra $k\mathcal{F}$. This allows us to view $k\mathcal{C}$ as a (left) $k\mathcal{F}$-module. The action of $a \in k\mathcal{F}$ on any element $b \in k\mathcal{C}$ is given by multiplying b by a on the left: $a \cdot b = ab$.

Since $k\mathcal{C}$ is a $k\mathcal{F}$-module and $(k\mathcal{F})^{S_n}$ is a subalgebra of $k\mathcal{F}$, it follows that $k\mathcal{C}$ is also a $(k\mathcal{F})^{S_n}$-module. Therefore, each element $a \in (k\mathcal{F})^{S_n}$ gives an endomorphism f_a of $k\mathcal{C}$ via the multiplication in $k\mathcal{F}$:

$$f_a : k\mathcal{C} \to k\mathcal{C}$$
$$f_a(b) = ab.$$

Note that f_a commutes with the action of S_n: if $\omega \in S_n$, then

$$\omega(f_a(b)) = \omega(ab) = \omega(a)\omega(b) = a\omega(b) = f_a(\omega(b)),$$

by using the fact that $w(a) = a$ for all $\omega \in S_n$. This gives an algebra homomorphism $(k\mathcal{F})^{S_n} \to \mathrm{End}_{kS_n}(k\mathcal{C})$, where $\mathrm{End}_{kS_n}(k\mathcal{C})$ denotes the k-algebra of kS_n-endomorphisms of $k\mathcal{C}$.

Recall that $\mathcal{C}$ is the set of set compositions of $[n]$ into singleton blocks. This gives an isomorphism $\psi : kS_n \to k\mathcal{C}$ of S_n-modules by mapping the permutation $\omega \in S_n$ to the chamber $(\omega(1), \dots, \omega(n))$. This in turn gives a k-algebra isomorphism $\mathrm{End}_{kS_n}(k\mathcal{C}) \cong \mathrm{End}_{kS_n}(kS_n)$:

$$\begin{aligned} \mathrm{End}_{S_n}(k\mathcal{C}) &\to \mathrm{End}_{S_n}(kS_n) \\ f &\mapsto \psi \circ f \circ \psi^{-1}. \end{aligned}$$

Recall that $\mathrm{End}_{kS_n}(kS_n)$ is the algebra of homomorphisms $g : kS_n \to kS_n$ that commute with the action of S_n. Therefore, for any $\omega \in S_n$, we have

$$g(\omega) = g(\omega 1) = \omega(g(1)).$$

That is, any endomorphism g of kS_n that commutes with the action of S_n is given by right multiplication by an element of kS_n (multiplication by $g(1)$). This gives an isomorphism

$$\begin{aligned} \mathrm{End}_{kS_n}(kS_n) &\to (kS_n)^{op} \\ g &\mapsto g(1), \end{aligned}$$

with inverse that sends an element $d \in kS_n$ to the endomorphism $g(a) = ad$. Here $(kS_n)^{op}$ denotes the algebra obtained from kS_n by reversing its multiplication since the above map is a product reversing isomorphism.

Combining all these maps gives the homomorphism

$$\begin{aligned} \xi : (k\mathcal{F})^{S_n} &\to (kS_n)^{op} \\ \xi(a) &= \big(\psi \circ f_a \circ \psi^{-1}\big)(1). \end{aligned}$$

Next we apply ξ to elements of $(k\mathcal{F})^{S_n}$. Let $B = (B_1, \dots, B_m)$ denote a set composition of $[n]$ and let a_B denote the sum of all compositions in the S_n-orbit of B. Then $a_B \in (k\mathcal{F})^{S_n}$. Moreover, any element of $(k\mathcal{F})^{S_n}$ is a linear combination of elements of this form. We have,

$$\begin{aligned} \xi(a_B) &= \big(\psi \circ f_{a_B} \circ \psi^{-1}\big)(1) \\ &= \big(\psi \circ f_{a_B}\big)((1, 2, \dots, n)) \\ &= \psi\big(a_B(1, 2, \dots, n)\big). \end{aligned}$$

But $a_B(1,2,\ldots,n)$ is the sum of all the set compositions into singleton blocks where the corresponding permutation has descent set contained in $J := \{|B_1|, |B_1|+|B_2|, \ldots, |B_1|+\cdots+|B_{m-1}|\}$ (this follows from Proposition 1.3). Therefore, $\xi(a_B)$ is the sum of all the permutations in S_n with descent set contained in J. That is, $\xi(a_B) = x_J$.

Since the elements a_B span $(k\mathcal{F})^{S_n}$, it follows that the image of ξ is the descent algebra. This proves that $\mathcal{D}(S_n)$ is indeed an algebra, being the homomorphic image of an algebra morphism. To show that we have an isomorphism it is enough to count dimensions. The descent algebra is of dimension 2^{n-1} since the elements x_J are linearly independent and there are 2^{n-1} of them (one for each $J \subset [n-1]$). On the other hand, there are 2^{n-1} faces of the chamber $(1,2,\ldots,n)$, and the sum of the elements in the orbits of each of these faces form a basis of $(k\mathcal{F})^{S_n}$. $\square$

2.4. **Generalization to all finite Coxeter Groups.** This section outlines how to adapt the above proof to any finite Coxeter group.

A *finite Coxeter group* W (or a *finite reflection group*) is a finite group generated by a set of reflections in a real vector space V. The *reflection arrangement* $\mathcal{A}(W)$ of W is the hyperplane arrangement consisting of the hyperplanes fixed by some reflection in W. The Coxeter group W permutes the hyperplanes in $\mathcal{A}(W)$, so W acts on the intersection lattice $\mathcal{L}(W)$ of $\mathcal{A}(W)$ and on the faces $\mathcal{F}(W)$ of $\mathcal{A}(W)$. The action of W on $\mathcal{F}(W)$ extends linearly to an action of W on the semigroup algebra $k\mathcal{F}(W)$. When the Coxeter group W is clear from the context we will write $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{A}$ for $\mathcal{F}(W)$, $\mathcal{L}(W)$ and $\mathcal{A}(W)$, respectively.

Let c denote a chamber in the reflection arrangement $\mathcal{A}$ of W. If $x \leq c$ is a codimension one face of c, then the hyperplane $\operatorname{supp}(x)$ is called a *wall* of c. Let $S \subset W$ denote the set of reflections in the walls of c. Then S is a generating set of W [Brown, 1989, §I.5A] and there is a well-defined notion of the *length* of an element in W when expressed as a word in the generators S.

For $J \subset S$ let $W_J = \langle J \rangle$ denote the subgroup of W generated by the elements in J. Each coset of W_J in W contains a unique element of minimal length [Humphreys, 1990, Proposition 1.10(c)]. Let W^J denote the set of these *minimal length coset representatives*. Let $w_J = \sum_{w \in W^J} w \in kW$ denote the sum of the minimal length coset representatives of W_J, where kW is the group algebra kW of W with coefficients in some field k. The k-vector space $\mathcal{D}(W)$ spanned by the elements w_J for $J \subset S$ is a subalgebra of the group algebra kW called the *descent algebra* of W. It was introduced by Solomon [Solomon, 1976]. The last statement of the following theorem was first noticed by [Bidigare, 1997].

Theorem 2.2 ([Brown, 2000]). *Let W be any finite Coxeter group. The following composition is injective with image the descent algebra $\mathcal{D}(W)$ of W.*

$$(k\mathcal{F}(W))^W \hookrightarrow \mathrm{End}_{kW}(k\mathcal{C}) \cong \mathrm{End}_{kW}(kW) \cong (kW)^{op}.$$

Therefore, $(k\mathcal{F}(W))^W$ is anti-isomorphic to $\mathcal{D}(W)$.

Proof. Let $\mathcal{C}$ denote the set of chambers in the reflection arrangement $\mathcal{A}$. The k-vector space $k\mathcal{C}$ spanned by the chambers $\mathcal{C}$ is a two-sided ideal of the face semigroup algebra $k\mathcal{F}$ of $\mathcal{A}$. Therefore, it is a $k\mathcal{F}$-module and hence a $(k\mathcal{F})^W$-module, where $(k\mathcal{F})^W$ denotes the subalgebra of $k\mathcal{F}$ consisting of elements invariant under the action of W. The action of W on $k\mathcal{C}$ commutes with the action of $(k\mathcal{F})^W$ on $k\mathcal{C}$, so there is an algebra morphism $(k\mathcal{F})^W \to \mathrm{End}_{kW}(k\mathcal{C})$, where $\mathrm{End}_{kW}(k\mathcal{C})$ denotes the k-algebra of kW-endomorphisms of $k\mathcal{C}$. There is an isomorphism $k\mathcal{C} \cong kW$ of kW-modules given by identifying $w(c)$ with w for all $w \in W$ (for some fixed chamber c). This gives a k-algebra isomorphism $\mathrm{End}_{kW}(k\mathcal{C}) \cong \mathrm{End}_{kW}(kW)$. Since any kW-endomorphism commuting with the action of W is given by right multiplication by an element of kW, there is an isomorphism $\mathrm{End}_{kW}(kW) \cong (kW)^{op}$, where $(kW)^{op}$ is the k-algebra obtained from kW be reversing the multiplication in kW.

Recall that any face $x \in \mathcal{F}$ of the reflection arrangement is in the W-orbit of a unique face y of c [Humphreys, 1990, Theorem 1.12]; and that the stabilizer of $y \leq c$ is W_J, where J is the set of reflections in the walls of c containing y [Humphreys, 1990, Theorem 1.15]. Therefore, the elements $\sum_{w \in W^J} w(y)$ for $y \leq c$ form a basis of $(k\mathcal{F})^W$. The above anti-isomorphism sends $\sum_{w \in W^J} w(y)$ to $\sum_{w \in W^J} w$. □

2.5. **Coefficients.** This section discusses how to expand a product of basis elements in the basis recovering a well-known formula of Garsia and Remmel [Garsia and Remmel, 1985].

Let $B = (B_1, \ldots, B_m)$ be a set composition of $[n]$. The *shape* $\lambda(B)$ of B is the integer composition of n given by the sizes of the blocks of B: $\lambda(B) = (|B_1|, \ldots, |B_m|)$. Each integer composition λ of n gives an element $x_\lambda = \sum_{\lambda(B)=\lambda} B$ in $(k\mathcal{F})^{S_n}$. Moreover, these elements are linearly independent, giving a basis of $(k\mathcal{F})^{S_n}$. This result describes how to multiply basis elements.

Proposition 2.3. *Let α, β and γ denote integer compositions of n. Define elements $c_{\alpha\beta\gamma}$ by*

$$x_\alpha x_\beta = \sum_\gamma c_{\alpha\beta\gamma}\ x_\gamma.$$

Then $c_{\alpha\beta\gamma}$ is the number of ways that some fixed set composition C of shape γ can be written as the product AB where A and B have shapes α and β, respectively.

Proof. Suppose C is a set composition of shape $\lambda(C) = \gamma$. Then the coefficient of C in the right hand side of the above equation is precisely $c_{\alpha\beta\gamma}$ since all set compositions of shape γ have the same coefficient. The coefficient of C is exactly the number of ways that it can be written as the product AB where A and B are set compositions of shapes α and β, respectively. □

As a corollary we obtain a method to determine the coefficients $c_{\alpha\beta\gamma}$ due to Garsia and Remmel.

Corollary 2.4 ([Garsia and Remmel, 1985]). *The coefficients $c_{\alpha\beta\gamma}$ in the previous proposition count the number of matrices with row sums equal α, column sums equal to β and whose nonzero entries, read from left to right, top to bottom, is the composition γ.*

Proof. Suppose $\gamma = (\gamma_1, \dots, \gamma_k)$ is an integer composition of n. Then the set composition $C = \Big(\{1, \dots, \gamma_1\}, \cdots, \{\sum_{i=1}^{k-1} \gamma_i, \dots, \sum_{i=1}^{k} \gamma_i\}\Big)$ has shape γ. From the previous result $c_{\alpha\beta\gamma}$ is the number of different ways of writing C as the product AB of set compositions A and B of shapes α and β, respectively.

The product of $A = (A_1, \dots, A_m)$ and $B = (B_1, \dots, B_l)$ is obtained by reading the entries of the following matrix from left to right, top to bottom, discarding any empty sets.

$$\begin{bmatrix} A_1 \cap B_1 & A_1 \cap B_2 & \cdots & A_1 \cap B_l \\ A_2 \cap B_1 & A_2 \cap B_2 & \cdots & A_2 \cap B_l \\ \vdots & \vdots & & \vdots \\ A_m \cap B_1 & A_m \cap B_2 & \cdots & A_m \cap B_l \end{bmatrix}.$$

Replacing each set in the matrix by its cardinality gives a matrix whose i-th row sums to α_i, whose j-th column sums to β_j and whose nonzero entries, read from left to right, top to bottom, give the integer composition γ.

Conversely, given such a matrix, replace the zero entries by an empty set, and replace the i-th nonzero entry, read from left to right, top to bottom, by $\{1 + \gamma_1 + \cdots + \gamma_{i-1}, \dots, \gamma_1 + \cdots + \gamma_i\}$. Let A denote the set composition with i-th block equal to the union of the sets in the i-th row of the matrix, and let B denote the set composition with j-th block equal to the union of the sets in the j-th column of the matrix. Note that size of the i-th block of A is α_i, so A has shape α. Similarly, B has shape β. And the product $AB = C$, where C is defined above. $\square$

Exercise 5. Recover Solomon's original formula for the coefficients. For $J \subset [n-1]$, let $\lambda(J) = (j_1, j_2 - j_1, j_3 - j_2, \dots, n - j_i)$, where $j_1 < \cdots < j_i$ are the elements of J. Then $\lambda(J)$ is an integer composition on n. Also, let X_J denote the set of elements $\omega \in S_n$ with $\operatorname{des}(\omega) \subset J$. Prove that $c_{\lambda(K),\lambda(J),\lambda(L)} = |\{\omega \in X_J^{-1} \cap X_K : L = K \cap \omega^{-1}(J)\}|$.

3. Idempotents

This section demonstrates how to use the structure of the face semigroup algebra $k\mathcal{F}$ to determine properties of the descent algebra. In particular, we will construct a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ and use these to construct a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^{S_n}$. Then we briefly outline how these determine the projective indecomposable modules, the radical and the quiver of the descent algebra.

3.1. **A Complete System of Primitive Orthogonal Idempotents in $k\mathcal{F}$.** Let A be a k-algebra. An element $e \in A$ is *idempotent* if $e^2 = e$. It is a *primitive idempotent* if e is idempotent and we cannot write $e = e_1 + e_2$ where e_1 and e_2 are nonzero idempotents in A with $e_1e_2 = 0 = e_2e_1$. Equivalently, e is primitive iff Ae is an indecomposable A-module. A set of elements $\{e_i\}_{i\in I} \subset A$ is a *complete system of primitive orthogonal idempotents* for A if e_i is a primitive idempotent for every i, if $e_ie_j = 0$ for $i \neq j$ and if $\sum_i e_i = 1$.

A complete system of primitive orthogonal idempotents $\{e_i\}_{i\in I}$ provides a lot of information about the structure of the algebra A. Up to isomorphism, the indecomposable projective A-modules are given by the A-modules Ae_i for $i \in I$ and the simple modules are isomorphic to $Ae_i/\operatorname{rad}(Ae_i)$ for some $i \in I$. Moreover, if M is any A-module, then there is an A-module decomposition of M given by the idempotents: $M \cong \bigoplus_{i\in I} Me_i$.

We now turn to constructing a complete system of primitive orthogonal idempotents for $k\mathcal{F}$. For each $X \in \mathcal{L}$, fix an $x \in \mathcal{F}$ with $\operatorname{supp}(x) = X$ and define elements in $k\mathcal{F}$ recursively by the formula,

$$e_X = x - \sum_{Y>X} xe_Y. \tag{3.1}$$

Note that $e_{\hat{1}}$ is an arbitrarily chosen chamber.

Example 5. Let us construct these elements for the braid arrangement for $n = 3$. The intersection lattice $\mathcal{L}$ consists of five elements.

$$\mathcal{L} = \Big\{\{1, 2, 3\}, \{12, 3\}, \{13, 2\}, \{1, 23\}, \{123\}\Big\}.$$

For each of the elements of $\mathcal{L}$ we fix a set composition of that support.

$$(1, 2, 3), (12, 3), (13, 2), (1, 23), (123).$$

Using Equation (3.1) we construct elements e_X for $X \in \mathcal{L}$ recursively.

$$\begin{aligned}
e_{\{1,2,3\}} &= (1, 2, 3).\\
e_{\{12,3\}} &= (12, 3) - (12, 3)e_{\{1,2,3\}} = (12, 3) - (1, 2, 3).\\
e_{\{13,2\}} &= (13, 2) - (13, 2)e_{\{1,2,3\}} = (13, 2) - (1, 3, 2).\\
e_{\{1,23\}} &= (1, 23) - (1, 23)e_{\{1,2,3\}} = (1, 23) - (1, 2, 3).\\
e_{\{123\}} &= (123) - e_{\{12,3\}} - e_{\{13,2\}} - e_{\{1,23\}} - e_{\{1,2,3\}}\\
&= (123) - (12, 3) - (13, 2) - (1, 23) + (1, 3, 2) + (1, 2, 3).
\end{aligned}$$

It is easy to see that the sum of these elements is (123), the identity element of $k\mathcal{F}$.

Note that $e_{\{1,2,3\}}e_X = 0$ for all $X \neq \{1, 2, 3\}$ since the coefficients in e_X sum to zero. Next we'll compute $e_{\{13,2\}}e_{\{1,2,3\}}$.

$$e_{\{13,2\}}e_{\{1,2,3\}} = \Big((13, 2) - (1, 3, 2)\Big)(1, 2, 3) = (1, 3, 2) - (1, 3, 2) = 0.$$

It is straightforward to verify that $e_Xe_Y = 0$ if $X \neq Y$ for all $X, Y \in \mathcal{L}$.

Next we'll compute $(e_{\{13,2\}})^2$.

$$\begin{aligned}
(e_{\{13,2\}})^2 &= \Big((13, 2) - (1, 3, 2)\Big)^2\\
&= (13, 2)^2 - (13, 2)(1, 3, 2) - (1, 3, 2)(13, 2) - (1, 3, 2)^2\\
&= (13, 2) - (1, 3, 2) - (1, 3, 2) - (1, 3, 2)\\
&= e_{\{13,2\}}.
\end{aligned}$$

It is straightforward to verify $(e_X)^2 = e_X$ for all $X \in \mathcal{L}$.

To prove that the elements e_X give a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ we'll need the following lemma.

Lemma 3.1. *Let $w \in \mathcal{F}$ and $X \in \mathcal{L}$. If* $\operatorname{supp}(w) \not\leq X$*, then* $we_X = 0$.

Proof. We proceed by induction on X. This is vacuously true if $X = \hat{1}$. Suppose the result holds for all $Y \in \mathcal{L}$ with $Y > X$. Suppose $w \in \mathcal{F}$ and $W = \operatorname{supp}(w) \not\leq X$. Using the definition of e_X and the identity $wxw = wx$ (Proposition 2 (2)),

$$we_X = wx - \sum_{Y>X} wxe_Y = wx - \sum_{Y>X} wx(we_Y).$$

By induction, $we_Y = 0$ if $W \not\leq Y$. Therefore, the summation runs over Y with $W \leq Y$. But $Y > X$ and $Y \geq W$ iff $Y \geq W \vee X$, so the summation runs over Y with $Y \geq W \vee X$.

$$we_X = wx - \sum_{Y>X} wx(we_Y) = wx - \sum_{Y \geq X \vee W} wxe_Y.$$

Now let z be the element of support $X \vee W$ chosen in defining $e_{X \vee W}$. So $e_{X\vee W} = z - \sum_{Y > X \vee W} ze_Y$. Note that $ze_{X \vee W} = e_{X\vee W}$ since $z = z^2$. Therefore, $z = \sum_{Y \geq X \vee W} ze_Y$. Since $\operatorname{supp}(wx) = W \vee X = \operatorname{supp}(z)$, it follows from Proposition 2 (5) that $wx = wxz$. Combining the last two statements,

$$we_X = wx - \sum_{Y \geq X \vee W} wxe_Y = wx \left(z - \sum_{Y \geq X \vee W} ze_Y \right) = 0. \qquad \square$$

Theorem 3.2. *The elements $\{e_X\}_{X \in \mathcal{L}}$ form a complete system of primitive orthogonal idempotents in $k\mathcal{F}$.*

Proof. Complete. $1 = \bigcap_{H \in \mathcal{A}} H$ is the only element of support $\hat{0}$. Hence, $e_{\hat{0}} = 1 - \sum_{Y > \hat{0}} e_Y$. Therefore,

$$\sum_X e_X = e_{\hat{0}} + \sum_{X \neq \hat{0}} e_X = \left(1 - \sum_{X > \hat{0}} e_X \right) + \sum_{X \neq \hat{0}} e_X = 1.$$

Idempotent. Since e_Y is a linear combination of elements of support at least Y, $e_Y z = e_Y$ for any z with $\operatorname{supp}(z) \leq Y$ (Proposition 2 (5)). Using the definition of e_X, the facts $e_X = xe_X$ and $e_Y = e_Y y$, and

Lemma 3.1,

$$e_X^2 = \left(x - \sum_{Y>X} xe_Y\right) e_X = xe_X - \sum_{Y>X} xe_Y(ye_X) = xe_X = e_X.$$

Orthogonal. We show that for every $X \in \mathcal{L}$, $e_X e_Y = 0$ for $Y \neq X$. If $X = \hat{1}$, then $e_X e_Y = e_X x e_Y = 0$ for every $Y \neq X$ by Lemma 3.1 since $X = \hat{1}$ implies $X \not\leq Y$. Now suppose the result holds for $Z > X$. That is, $e_Z e_Y = 0$ for all $Y \neq Z$. If $X \not\leq Y$, then $e_X e_Y = 0$ by Lemma 3.1. If $X < Y$, then $e_X e_Y = xe_Y - \sum_{Z>X} x(e_Z e_Y) = xe_Y - xe_Y^2 = 0$.

Primitive. Let $E_X = \sum_{Y \geq X} \mu(X,Y)Y$ for all $X \in \mathcal{L}$. Then the above arguments show that the elements E_X are orthogonal idempotents in $k\mathcal{L}$ summing to 1. The number of these elements is the number of elements of $\mathcal{L}$, so the elements E_X form a basis of $k\mathcal{L}$. Moreover, $(k\mathcal{L})E_X = \mathrm{span}_k(E_X) \cong k$, which is an indecomposable $k\mathcal{L}$-module. So these elements form a complete system of primitive orthognal idempotents in $k\mathcal{L}$.

We now prove that the elements e_X lift the primitive idempotents E_X for all $X \in \mathcal{L}$. Indeed, if $X = \hat{1}$, then $\mathrm{supp}(e_{\hat{1}}) = \hat{1} = E_{\hat{1}}$. Suppose the result holds for $Y > X$. Then $\mathrm{supp}(e_X) = \mathrm{supp}(x - \sum_{Y>X} xe_Y) = X - \sum_{Y>X}(X \vee E_Y)$. Since E_Y is a linear combination of elements $Z \geq Y$, it follows that $X \vee E_Y = E_Y$ if $Y > X$. Therefore, $\mathrm{supp}(e_X) = X - \sum_{Y>X} E_Y$. The Möbius inversion formula applied to $E_X = \sum_{Y \geq X} \mu(X,Y)Y$ gives $X = \sum_{Y \geq X} E_X$. Hence, $\mathrm{supp}(e_X) = X - \sum_{Y>X} E_Y = E_Y$.

To see that this is sufficient, suppose E is a primitive idempotent in $k\mathcal{L}$ and that e is an idempotent lifting E. Suppose $e = e_1 + e_2$ with e_i orthogonal idempotents. Then $E = \mathrm{supp}(e) = \mathrm{supp}(e_1) + \mathrm{supp}(e_2)$. Since E is primitive and $\mathrm{supp}(e_1)$ and $\mathrm{supp}(e_2)$ are orthogonal idempotents, $\mathrm{supp}(e_1) = 0$ or $\mathrm{supp}(e_2) = 0$. Say $\mathrm{supp}(e_1) = 0$. Then e_1 is in the kernel of supp. This kernel is nilpotent so $e_1^n = 0$ for some $n \geq 0$. Hence $e_1 = e_1^n = 0$. Therefore, e is a primitive idempotent. $\square$

Remark 3.3. We can replace $x \in \mathcal{F}$ in (3.1) with any linear combination $\tilde{x} = \sum_{\operatorname{supp}(x)=X} \lambda_x x$ of elements of support X whose coefficients λ_x sum to 1. The proofs still hold since the element $\tilde{x}$ is idempotent and satisfies $\operatorname{supp}(\tilde{x}) = X$ and $\tilde{x}y = \tilde{x}$ if $\operatorname{supp}(y) \leq X$. We will use this observation in the next section to construct a complete system of primitive orthogonal idempotents in for the descent algebra.

Remark 3.4. It can be shown that the above generalizes to give a complete system of primitive orthogonal idempotents in the semigroup algebra of a left regular band. A left regular band is a semigroup S satisfying $x^2 = x$ and $xyx = xy$ for all $x, y \in S$. It follows from this definition that there exists a lattice L and a surjection $\operatorname{supp} : S \to L$ such that $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$, and $xy = x$ iff $\operatorname{supp}(y) \leq \operatorname{supp}(x)$ for all $x, y \in S$. These are precisely the properties of $\mathcal{F}$ that we used to prove the above theorem.

Corollary 3.5. *The set $\{xe_{\operatorname{supp}(x)} \mid x \in \mathcal{F}\}$ is a basis of $k\mathcal{F}$ of primitive idempotents.*

Proof. Let $y \in \mathcal{F}$. Then by Theorem 3.2 and Lemma 3.1,

$$y = y1 = y \sum_X e_X = \sum_{X \geq \operatorname{supp}(y)} ye_X = \sum_{X \geq \operatorname{supp}(y)} (yx)e_X.$$

Since $\operatorname{supp}(yx) = \operatorname{supp}(y) \vee \operatorname{supp}(x) = X$, the face y is a linear combination of the elements of the form $xe_{\operatorname{supp}(x)}$. So these elements span $k\mathcal{F}$. Since the number of these elements is the cardinality of $\mathcal{F}$, which is the dimension of $k\mathcal{F}$, the set forms a basis of $k\mathcal{F}$. The elements are idempotent since $(xe_X)^2 = (xe_X)(xe_X) = xe_X^2 = xe_X$ (since $xyx = xy$ for all $x, y \in \mathcal{F}$). Since xe_X also lifts the primitive idempotent $E_X = \sum_{Y \geq X} \mu(X,Y)Y \in k\mathcal{L}$, it is also a primitive idempotent (see the end of the proof of Corollary 3.2). $\square$

3.2. **Idempotents in the Descent Algebra.** In this section we will use the above to construct idempotents in the invariant subalgebra $(k\mathcal{F})^{S_n}$. In the following let $W = S_n$, let $\mathcal{F}$ denote the set of faces of the braid arrangement in $\mathbb{R}^n$, and let $\mathcal{L}$ be the lattice of set partitions of $[n]$.

For each $X \in \mathcal{L}$, let $\widehat{X}$ denote the normalized sum of all elements of support X.

$$\widehat{X} = \frac{1}{\#\{x \in \mathcal{F} : \operatorname{supp}(x) = X\}} \left(\sum_{\operatorname{supp}(x)=X} x \right).$$

Then $w(\widehat{X}) = \widehat{w(X)}$ for all $w \in W$. Then the elements $\{e_X\}_{X \in \mathcal{L}}$ in $k\mathcal{F}$ constructed using the formula $e_X = \widehat{X} - \sum_{Y>X} \widehat{X} e_Y$ is a complete system of primitive orthogonal idempotents for $k\mathcal{F}$ (see Remark 3.3).

Lemma 3.6. *For each $w \in W$ and $X \in \mathcal{L}$, we have $w(e_X) = e_{w(X)}$.*

Proof. We proceed by induction on $X \in \mathcal{L}$. If $w \in W$, and $X = \hat{1}$, then $w(e_{\hat{1}}) = w(\widehat{1}) = \widehat{1} = e_{\hat{1}} = e_{w(\hat{1})}$. Now suppose that $w(e_Y) = e_{w(Y)}$ for all $Y > X$. Then

$$\begin{aligned} w(e_X) &= w\widehat{X} - \sum_{Y>X} w\left(\widehat{X} e_Y\right) \\ &= w\widehat{X} - \sum_{Y>X} w\left(\widehat{X}\right) w(e_Y) \\ &= \widehat{wX} - \sum_{Y>X} \widehat{wX} e_{wY} \\ &= \widehat{wX} - \sum_{Y>wX} \widehat{wX} e_Y \\ &= e_{wX}. \end{aligned}$$

□

We now get a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$ by summing all the idempotents e_Y for Y in the orbit of X. If $[X]$ denotes the W-orbit of $X \in \mathcal{L}$, then let

$$\varepsilon_X = \varepsilon_{[X]} = \sum_{Y \in [X]} e_Y.$$

Since the elements e_Y are orthogonal, it follows that the elements ε_X are orthogonal idempotents, and it is not difficult to show that these are also primitive. Since $\sum_Y e_Y = 1$, it follows immediately that the elements ε_X sum to 1. Therefore, we get a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$. We summarize this in the following.

Proposition 3.7. *For each $X \in \mathcal{L}$ let $\widehat{X}$ denote a linear combination of elements of support X whose coefficients sum to 1. Suppose that $w(\widehat{X}) = \widehat{w(X)}$ for all $w \in W$ and $X \in \mathcal{L}$. Define e_X for $X \in \mathcal{L}$ recursively by $e_X = \widehat{X} - \sum_{Y>X} \widehat{X} e_Y$. Then the elements $\sum_{Y \in [X]} e_Y$, one for each orbit $[X] \in \mathcal{L}/W$, form a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$.*

Using these idempotents we also obtain a basis of $(k\mathcal{F})^W$ by idempotents, similar to Corollary 3.5.

Corollary 3.8. *The elements*

$$\sum_{w\in W} w\Big(x e_{\mathrm{supp}(x)}\Big) = \left(\sum_{w\in W} w(x)\right) \varepsilon_{[\mathrm{supp}(x)]},$$

where $x \in \mathcal{F}$, is a basis of $(k\mathcal{F})^W$ of primitive idempotents.

Exercise 6. This exercise will construct a complete system of primitive orthogonal idempotents in the descent algebra using a construction analogous to that used to construct the elements e_X in $k\mathcal{F}$.

Let (W, S) denote a finite Coxeter system. Let L denote the poset of subgroups $W_J = \langle J \rangle$ generated by subsets J of S. Show that this poset is a lattice isomorphic to the lattice of subsets of S.

Define an equivalence class on these subgroups as follows.

$$W_J \sim W_K \text{ iff there exists } w \in W \text{ such that } wW_Jw^{-1} = W_K.$$

Show that the partial order in L induces a partial order on the equivalence classes as follows. If $[W_J]$ denotes the equivalence class containing W_J, then

$$[W_J] \leq [W_K] \text{ iff } W_J \text{ is conjugate to a subgroup of } W_K.$$

For each equivalence class $[W_J]$ fix an element x_J (the elements used to define the descent algebra of W). Define elements $\varepsilon_{[W_J]}$ as recursively using the formula

$$\varepsilon_{[W_J]} = x_J - \sum_{[W_K]>[W_J]} x_J \varepsilon_{[W_K]}.$$

Show that the elements $\varepsilon_{[W_J]}$ form a complete system of primitive orthogonal idempotents in the descent algebra.

Exercise 7. Generalize the result in the previous exercise. Find conditions on an k-algebra A so that the above gives a complete system of primitive orthogonal idempotents for A. Let me know if you do.

3.3. **A Sampling of Other Results.** Here is a sampling of the some of the results about the face semigroup algebra of a hyperplane arrangement and the descent algebras.

Theorem 3.9. *There are nice descriptions of the simple modules and the indecomposable projective modules of $k\mathcal{F}$. We can determine the quiver with relations of $k\mathcal{F}$. The quiver is the directed graph obtained from the Hasse diagram of the intersection lattice $\mathcal{L}$ by orienting all edges away from the top vertex (corresponding to $\mathbb{R}^n$). There is one relation for each interval of length two, obtained by summing the paths of length two in that interval.*

Corollary 3.10. *The algebra $k\mathcal{F}$ depends only on $\mathcal{L}$. That is, starting from $\mathcal{L}$ there is a construction that will recover $k\mathcal{F}$. (Note that there are arrangements with isomorphic intersection lattices, but non-isomorphic face semigroups, so this is saying something nontrivial.)*

Corollary 3.11. *$k\mathcal{F}$ is a* Koszul algebra*. Its* Koszul dual *is the incidence algebra of the lattice obtained from $\mathcal{L}$ by reversing the order.*

Theorem 3.12. *Using the above results it is possible to determine information about the simple modules, the indecomposable projective modules, the radical and the quiver of $(k\mathcal{F})^W$, hence of the descent algebra of W.*

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