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Analogue of the Bruhat–Chevalley Order for Reductive Monoids

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INTRODUCTION

The purpose of this paper is to describe the Adherence Order of $B \times B$-orbits on a reductive algebraic monoid. By the results of [10] there is already a perfect analogue of the *Bruhat decomposition* for reductive monoids. Indeed, by [10, Corollary 5.8], if M is reductive with unit group G and maximal torus $T \subseteq G$ with Borel subgroup $B \supseteq T$, then the set of two sided B-orbits, $B \setminus M/B$ is canonically identified with a certain finite inverse monoid R. In fact, $R = \overline{N_G(T)}/T$, the orbit monoid of the Zariski closure of $N_G(T)$ in M. By definition, for $\sigma, \tau \in R$ we define the *Adherence Order* by

$$\sigma \leq \tau \qquad \text{if } B\sigma B \subseteq \overline{B\tau B},$$

where $\overline{B\tau B}$ denotes the Zariski closure in M of $B\tau B$.

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In the case of reductive groups, a similar order relation was first studied by Chevalley. Since then Tits [13] developed the notion of a "BN pair" or "Tits system" which provided the abstract framework for much subsequent work in group theory (including complete homogeneous spaces and Schubert varieties [2]). Furthermore, the much studied KL-polynomials are quantified in terms of the Bruhat–Chevalley order on W [3].

For reductive monoids one would like to describe the Adherence Order on $\mathcal{R}$ in terms of the Bruhat order on W along with some other necessary invariant. This other invariant turns out to be the $\mathcal{J}$-order. For $n \times n$ matrices, $A \geq B$ in the $\mathcal{J}$-order, if $\operatorname{rank}(A) \geq \operatorname{rank}(B)$. Furthermore, each element σ of $\mathcal{R}$ can be expressed uniquely in *standard form* $\sigma = xey^{-1}$, where $e \in \Lambda$ the cross section lattice, and $x, y \in W$ are appropriately restricted (see Corollary 1.5 for details). Our main result is the following:

THEOREM. *Let $\sigma = xey^{-1}$ and $\tau = sft^{-1}$ be in standard form. Then the following are equivalent*:

(a) $\sigma \leq \tau$

(b) *$e \leq f$ and there exists $w \in W(f)W_e$ such that $x \leq sw$ and $tw \leq y$.*

Here $e \leq f$ means simply that $ef = fe = e$. This is the $\mathcal{J}$-order condition mentioned above. As usual, $W(f) = \{w \in W | wf = fw\}$ and $W_e = \{w \in W | we = ew = e\}$.

In Section 2 we analyze further the Adherence Order. To do this, we introduce the *j order on* $\mathcal{R}$. Indeed, we define

$$\sigma \leq_j \tau \qquad \text{if } B\sigma B \leq \overline{B}\tau\overline{B}.$$

Clearly, $\sigma \leq_j \tau$ implies $\sigma \leq \tau$. The main result of Section 2 is essentially the following theorem.

THEOREM. *Suppose $\sigma \leq \tau$. Then there exists $\theta_0, \theta_1, \ldots, \theta_m \in \mathcal{R}$ such that $\sigma = \theta_0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_m = \tau$, and for all $i \leq m$ either*

(a) *$\theta_i \leq_j \theta_{i+1}$ or else*

(b) *θ_i and θ_{i+1} differ by a Bruhat interchange (see Definition* 2.9).

This determines the Adherence Order in terms of the two simpler relations. The relation of (a) is not encountered in group theory but is easily analysed via Theorem 2.7. The relation of (b) is described in more detail in Definition 2.9 and Theorem 2.11. (For $M_n(k)$ it implies that θ_i and θ_{i+1} have the same nonzero rows and columns.) It is similar to the situation encountered in the study of the Bruhat–Chevalley order on a Coxeter group [4, Chap. 5].

In Section 3 we apply the general results of Sections 1 and 2 to the monoid $M_n(k)$ of $n \times n$ matrices. In this example, $\mathcal{R} = \mathcal{R}_n$ can be identified with the finite monoid of zero-one matrices with at most one nonzero entry in each row or column. These are sometimes referred to as partial permutation matrices. We use the results of Section 1 or 2 to find a purely combinatorial description of the Adherence Order on R_n.

In Section 4 we discuss a natural length function on R. Here we continue our policy of determining all "monoid quantities" in terms of "Λ-quantities" and "W-quantities." For example, if $\sigma = xey^{-1} \in R$ is in standard form we define the *length of* σ by

$$l(\sigma) = l(x) + l(e) - l(y),$$

where $l(x), l(y)$ are the lengths of x and y as elements of the Coxeter group (W, S) ($l(e)$ is defined in Section 4). We show in Section 2 that length is subadditive, namely

$$l(\sigma\tau) \leq l(\sigma) + l(\tau).$$

In the final section of the paper we pursue some further properties of the Adherence Order. While the results of this section could benefit from some further refinements, we include them for future reference.

1. THE ADHERENCE ORDER

Let M be a reductive algebraic monoid with maximal torus T and Borel subgroup $B \supseteq T$. $E(\overline{T}) = \{e \in \overline{T} | e^2 = e\}$. Let $\Lambda = \{e \in E(\overline{T}) | Be \subseteq eB\}$ be a cross section lattice, and let $R = \overline{N_G(T)}/T$. Recall from [10, Corollary 5.8] that R can be canonically identified with the set of two sided B-orbits on M. For $\sigma, \tau \in R$ we define

$$\sigma \leq \tau \tag{1}$$

if $B\sigma B \subseteq \overline{B\tau B}$ (Zariski closure in M). It is easy to verify that $\leq$ determines a partial order on R. The main result of this section describes this order relation in terms of

(i) the I-order on M

(ii) the Adherence Order on W.

The I-order (on M) has been determined explicitly in much detail by the second and third named authors for a large class of reductive monoids [8]. The Adherence Order on W is the much studied Bruhat–Chevalley order.

Before stating our main result we recall some relevant background information. Let G be a reductive group with $T \subseteq B \subseteq G$ as usual. Let W be the Weyl group of T. By definition, if $x, y \in W$, then $x \leq y$ if and only if $BxB \subseteq \overline{ByB}$.

1.1. LEMMA. *$x \leq y$ if and only if $B^- yB \subseteq \overline{B^- xB}$ where B^- is the Borel subgroup opposite to B.*

Proof. Let $w \in W$ be the longest element. Then by [4, Sect. 5.9], $x \leq y$ iff $wy \leq wx$ iff $BwyB \subseteq \overline{BwxB}$ iff $wBwyB \subseteq \overline{wBwxB}$ iff $B^- yB \subseteq \overline{B^- xB}$, since $B^- = wBw$. ∎

1.2. LEMMA. *$x \leq y$ iff $xBy^{-1} \cap B^- B \neq \varnothing$.*

Proof. By [3, Corollary 1.2], $x \leq y$ iff $B^- xB \cap ByB \neq \varnothing$. But

$$\begin{aligned} B^- xB \cap ByB \neq \varnothing &\Leftrightarrow B^- xB \cap By \neq \varnothing \\ &\Leftrightarrow B^- xB \cap B^- By \neq \varnothing \\ &\Leftrightarrow B^- xBy^{-1} \cap B^- B \neq \varnothing \\ &\Leftrightarrow xBy^{-1} \cap B^- B \neq \varnothing. \quad ∎ \end{aligned}$$

1.3. LEMMA. *For all $x \in W$, $B^- xB \subseteq B^- Bx \cap xB^- B$.*

Proof. Write $xBx^{-1} = U^- UT$ where $U^- \subseteq B^-$ and $U \subseteq B$. Then $xBx^{-1} = U^-(UT) \subseteq B^- B$. So $xB \subseteq B^- Bx$. Thus, $B^- xB \subseteq B^- Bx$. Similarly, $x^{-1}B^- x \subseteq B^- B$. Thus, $B^- xB \subseteq xB^- B$ as well. ∎

For $I \subseteq S$ let L_I be the associated Levi factor and $B_L = L_I \cap B$. Let $x \in W$. Then by [1, Proposition 2.3.3]

$$x \text{ has minimal length } l(x) \text{ in } W_I x \Leftrightarrow x^{-1} B_L x \subseteq B. \tag{2}$$

Now let M be a reductive monoid with unit group G and cross section lattice Λ. For $e \in \Lambda$ let

$$\begin{aligned} W(e) &= C_W(e) \\ W_e &= \{w \in W \mid we = e\} \lhd W(e) \\ D_e &= \{x \in W \mid x \text{ has minimal length in } xW_e\} \\ D(e) &= \{x \in W \mid l(x) \text{ is minimal in } W(e)x\} \\ L(e) &= C_G(e), \qquad H(e) = eL(e) \\ B(e) &= C_B(e) \\ B^-(e) &= C_{B^-}(e). \end{aligned}$$

If $C_W(e) = \langle s \mid s \in I \rangle$ and $x \in D_e$ then

$$\begin{aligned} Bex &= B(e)x \\ &= exx^{-1}B(e)x \\ &\subseteq exB, \qquad \text{by (2)}. \end{aligned}$$

Hence, $BexB \subseteq exB \subseteq BexB$ and so

$$BexB = exB \qquad \text{for } x \in D_e. \tag{3}$$

Now let $e, f \in \Lambda$ with $e \le f$. Then

$$W(e) = W_e[W(e) \cap W(f)]. \tag{4}$$

Clearly,

$$R = W\Lambda W = \bigcup_{e \in \Lambda} WeD(e). \tag{5}$$

Let $x \in W$, $e \in \Lambda$, $y \in D(e)$, and assume $x' \le x$. Then by Lemma 1.3 and (3),

$$\begin{aligned} Bx'eyB &\subseteq Bx'BeyB \\ &\subseteq \overline{BxB}eyB \\ &\subseteq \overline{BxBeyB} \\ &= \overline{BxeyB}. \end{aligned}$$

Thus

$$x'ey \le xey \qquad \text{if } y \in D(e) \text{ and } x' \le x. \tag{6}$$

Finally, let $y \in D(e)$ and assume $x \le y^{-1}$. Then by (6), $xey \le y^{-1}ey$. Hence

$$xey \in \overline{B} \qquad \text{if } y \in D(e) \text{ and } x \in W,\, x \le y^{-1}. \tag{7}$$

1.4. THEOREM. *Let $e, f \in \Lambda$, $x, s \in W$, $y \in D(e)$, and $t \in D(f)$. Then the following are equivalent*:

(a) $xey \le sft$

(b) *$ef = e$ and there exist $w \in W(f)W_e$, $z \in W_e$ such that $w^{-1}t \le y$ and $x \le swz$ in W.*

Proof. Let e, f, s, t, x, and y be as above and assume (b) holds for w and z as indicated. Now by (7), $t^{-1}wey \in \overline{B}$. So $sfwey = sftt^{-1}wey \in sft\overline{B} \subseteq \overline{BsftB}$. So $sfwey \le sft$. By assumption, $w = w_1w_2$ with $w_1 \in w(f)$ and $w_2 \in W_e$. Thus, $sfwey = sfw_1w_2ey = sfw_1ey = sw_1ey = sw_1w_2ey = swey$. But by (6), $xey \le swzey \le swey \le sfwey \le sft$.

Conversely, suppose $xey \le sft$. Then clearly $e \le f$ and $xey \in \overline{BsftB}$. Hence, $e \in \overline{x^{-1}BsftBy^{-1}}$. Now, for $w \in W$, let

$$A_w = x^{-1}Bs \cap B^-wB. \tag{8}$$

So $x^{-1}Bs = \bigcup_{w \in W} A_w$. Since this is a finite disjoint union of subvarieties, there exists a unique $w \in W$ such that $A_w \subseteq x^{-1}Bs$ is open and dense. So

$$e \in \overline{A_w ftBy^{-1}}.$$

Thus, $e \in e\overline{A_w ftBy^{-1}}e \subseteq \overline{eA_w ftBy^{-1}e} \subseteq \overline{eB^- wBftBy^{-1}e} \subseteq \overline{eB^- ewfBftBy^{-1}e}$. Hence, $ewf \mathrel{I} e$. So $wfw^{-1} \geq e$. But $f \geq e$ by assumption. Thus, there exists $v \in C_W(e)$ such that $vfv^{-1} = wfw^{-1}$. But then $v \in W(f) \cap W(e) \neq \varnothing$, say $w = vc$ with $c \in W(f)$. Hence, by (4), $w = vc \in W(e)W(f) \subseteq W_e[W(e) \cap W(f)]W(f) \subseteq W_e W(f)$. Conclude that

$$w = w_1 w_2 \qquad \text{for some } w_1 \in W_e, w_2 \in W(f). \tag{9}$$

Since $A_w \neq 0$, we see by Lemma 1.3 that $\varnothing \neq x^{-1}Bs \cap B^- wB \subseteq x^{-1}Bs \cap B^- Bw$. Thus $x^{-1}Bsw^{-1} \cap B^- B \neq \varnothing$. So by Lemma 1.2 and (9)

$$x \leq sw^{-1} = sw_2^{-1} w_1^{-1}. \tag{10}$$

By (10)

$$\begin{aligned} e &\in \overline{B^- w_1 w_2 BftBy^{-1}} \\ &= \overline{B^- w_1 w_2 ftBy^{-1}}, \qquad \text{by (3)} \\ &= \overline{B^- w_1 fw_2 tBy^{-1}}. \end{aligned}$$

For $u \in W$, let

$$C_u = w_2 tBy^{-1} \cap B^- uB.$$

Then

$$w_2 tBy^{-1} = \bigcup_{u \in W} C_u.$$

Since this is a finite union we see that for some $u \in W$, $C_u \subseteq w_2 tBy^{-1}$ is open and dense. Thus,

$$e \in \overline{B^- w_1 fC_u}.$$

So

$$\begin{aligned} e &\in e\overline{B^- w_1 fC_u}e \subseteq \overline{eB^- w_1 fC_u e} \subseteq \overline{eB^- - w_1 fB^- wBe} \subseteq \overline{eB^- ew_1 fB^- uBe} \\ &\subseteq \overline{eB^- efB^- uBe} = \overline{eB^- eB^- uBe} = \overline{eB^- eueBe}. \end{aligned}$$

Thus, $eue \mathrel{I} e$. Hence $u \in W(e)$. So in $H(e)$, $e \in \overline{eB^- eueBe}$. By Lemma 1.1 applied to $H(e)$, $eu \leq e$ in $W(H(e))$. Hence, $eu = e$. So $u \in W_e$. Since

$C_u \neq \varnothing$, we see from Lemma 1.3 that $\varnothing \neq w_2tBy^{-1} \cap B^-wB \subseteq w_2tBy^{-1} \cap uB^-B$. So $u^{-1}w_2tBy^{-1} \cap B^-B \neq \varnothing$. Thus, by Lemma 1.2,

$$u^{-1}w_2t \leq y. \tag{11}$$

Let $\tilde{w} = w_2^{-1}u \in W(f)W_e$, $z = u^{-1}w_1^{-1} \in W_e$. By (10) and (11)

$$x \leq sw_2^{-1}w_1^{-1} = sw_2^{-1}uu^{-1}w_1^{-1} = s\tilde{w}z, \qquad \text{and} \qquad \tilde{w}^{-1}t = u^{-1}w_2t \leq y. \quad \blacksquare$$

One can streamline the theorem somewhat as follows.

It is easy to check that each element $\sigma \in WeW$, with $e \in \Lambda$, can be written uniquely as

$$\sigma = xey^{-1}, \qquad x \in D_e,\ y \in D(e).$$

We call this the *standard form* for σ.

EXAMPLE. Let $M = M_3(k)$, and let T be the group of diagonal invertible matrices, B the group of upper triangular invertible matrices. One checks that

$$\Lambda = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

So let

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbf{R}_3.$$

Then

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (132)e(1).$$

One easily checks that this is a standard form since $D_e = W$ and $D(e) = \{1, (23), (123)\}$.

1.5. COROLLARY. *Let $\sigma = xey^{-1}$ and $\tau = sft^{-1}$ be in standard form. Then the following are equivalent*:

(a) $\sigma \leq \tau$.

(b) *$e \leq f$ and there exists $w \in W(f)W_e$ such that $x \leq sw$ and $tw \leq y$.*

Proof. This is straightforward.

Remark 1. Let $\sigma = xey^{-1}$, $\tau = sft^{-1}$ be in standard form. The characterization in Corollary 1.5(b) is "best possible" in the sense that $W(f)W_e$ cannot be replaced by $W(f)$. For example, let

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Coincidently, the same example shows that $\sigma \leq \tau \not\Rightarrow l(\sigma) - l(e) \leq l(\tau) - l(f)$. On the other hand, if $\sigma \leq \tau$ and $\sigma \in W\tau W$ then $l(\sigma) \leq l(\tau)$.

Remark 2. It is interesting to notice that $\{\sigma \in R | \sigma \leq 1\} = \{\sigma \in R | \sigma \in \overline{B}\}$ can be identified with a subset of $\{(x, y) \in W \times W | x \leq y\}$. Indeed, write $\sigma = xey^{-1}$ in standard form. Then by Lemma 2.3 below $x \leq y$. One might hope for an interesting relationship with Kazhdan–Lusztig polynomials. Indeed, the KL-polynomials are indexed by ordered pairs in W. There is already some connection between KL-polynomials and the Bruhat decomposition established by the second named author [7].

2. THE j-ORDER ON R

2.1. Definition. Let $r, s \in R$. We say $r \leq_j s$ if $BrB \subseteq \overline{B}s\overline{B}$.

One checks easily that $\leq_j$ is a partial order on R.

The major purpose of this section is to find a description of $\leq_j$ entirely within R (see Theorem 2.7 below). Along the way, we shall need to establish several other important results about $\leq$ and $\leq_j$.

2.2. Definition. Define $R^+ \subseteq R$ as $R^+ = \{r \in R | BrB \subseteq \overline{B}\}$.

One checks that this is well defined. For $M_n(k)$, R^+ is the set of upper triangular partial permutation matrices.

2.3. Lemma. *Let $r = xey^{-1}$ be in standard form. Then $r \in R^+$ if and only if $x \leq y$.*

Proof. If $r \in R^+$ then $r \leq 1$. Applying Corollary 1.5 we find that there exists $w \in W$ such that $x \leq w \leq y$. In particular, $x \leq y$. Conversely, if $x \leq y$, then plainly the criteria of Corollary 1.5(b) are satisfied for $\sigma = r$ and $\tau = 1$. ∎

It was established in [10] that if $s \in S$, the simple reflections, and $r \in R$, then

$$sBr \subseteq BrB \cup BsrB$$

and

$$rBs \subseteq BrB \cup BrsB.$$

We use this to help prove Theorems 2.4, 2.5, and 2.6 below.

2.4. THEOREM. *If $r \in R$ and $x \in W$ then $rBx \subseteq \bigcup_{y \leq x} BryB$ and $xBr \subseteq \bigcup_{y \leq x} ByrB$.*

Proof. We proceed by induction on the length of x. By the results of [13], we know that both inclusions hold if $l(x) = 1$, since $s = \{x \in W | l(x) = 1\}$. Assume the results hold if $l(x) < k$, and suppose $l(x) = k$. Then there exists $s_1, s_2, \dots, s_k \in S$ such that $x = s_1 s_2 \cdots s_k$. By induction we know that $rBx = (rBs_1 \cdots s_{k-1})s_k \subseteq (\bigcup_{y \leq x} BryB)s_k$, where $x' = s_1 \cdots s_{k-1} = xs_k$. But $BryBs_k \subseteq BryB \cup Brys_k B$ from [10, Proposition 5.3]. Thus $rBx \subseteq \bigcup_{y \leq x'} (BryB \cup Brys_k B)$. Hence $rBx \subseteq \bigcup_{y \leq x} BryB$. The other case is similar. ∎

2.5. THEOREM. *If $r, r_1 \in R$ then $rBr_1 \subseteq \bigcup_{r_2 \leq r_1} Brr_2 B$.*

Proof. Write $r_1 = xey^{-1}$ in standard form. Then by Theorem 2.4 we have that $rBr_1 = rBxey^{-1} \subseteq \bigcup_{x_1 \leq x} Brx_1 Bey^{-1}$.

Now let $L = \{x \in G | xe = ex\} = C_G(e)$, $B_L = B \cap L = C_B(e)$. Then $Be = eBe = C_B(e)e$ by [6, Theorem 6.16, Corollary 6.34]. Hence

$$Brx_1 Bey^{-1} = Brx_1 C_B(e) ey^{-1} = Brx_1 e C_B(e) y^{-1}.$$

But, $y \in D(e)$ and so $yC_B(e)y^{-1} \subseteq B$. Thus, $Brx_1 C_B(e) = Brx_1 ey^{-1} yC_B(e)y^{-1} \subseteq Brx_1 ey^{-1}B$. But notice that $x_1 \leq x$ and $y \in D(e)$, so that $r_2 = x_1 ey^{-1} \leq xey^{-1} = r_1$. Hence, $rBr_1 \subseteq \bigcup_{x_1 \leq x} Brx_1 Bey^{-1} \subseteq \bigcup_{x_1 \leq x} Brx_1 ey^{-1}B \subseteq \bigcup_{r_2 \leq r_1} Brr_2 B$. ∎

2.6. THEOREM. *If $r, r_1 \in R$ then $r_1 Br \subseteq \bigcup_{r_2 \leq r_1} Br_2 rB$.*

Proof. By [9, Theorem 8.2], there exists an involution $\tau: M \to M$ so that

$$\tau^2(x) = x \qquad \text{for all } x \in M,$$

$$\tau(xy) = \tau(y)\tau(x) \qquad \text{for all } x, y \in M,$$

$$\tau | T = id$$

and

$$\tau(B) = B^-.$$

Now let $w \in N_G(T)$ represent the longest element of W (so that $wBw^{-1} = B^-$), and define $\theta = \text{int}(w) \circ \tau$. In particular, $\theta(B) = B$ and $\theta(xy) = \theta(y)\theta(x)$ for $x, y \in R$.

If $r_1, r \in \mathsf{R}$, then by Theorem 2.5, $rBr_1 \subseteq \bigcup_{r_2 \leq r_1} Brr_2 B$. So also, $\theta(r_1)B\theta(r) \subseteq \bigcup_{r_2 \leq r} B\theta(r_2)\theta(r)B$. But $r_2 \leq r_1$ if and only if $\theta(r_2) \leq \theta(r_1)$ since $\theta(B) = B$ and θ is a homeomorphism in the Zariski topology. So our conclusion follows by applying θ directly to the inclusion $\theta(r)B\theta(r_1) \subseteq \bigcup_{r_2 \leq \theta(r_1)} B\theta(r)\theta(r_2)B$ of Theorem 2.5. ∎

2.7. THEOREM. *Let $\sigma, \theta \in \mathsf{R}$. Then $\sigma \in \mathsf{R}^+ \theta\, \mathsf{R}^+$ if and only if $B\sigma B \subseteq \overline{B}\theta\overline{B}$.*

Proof. If $\sigma \in \mathsf{R}^+ \theta\, \mathsf{R}^+$ then $\sigma = r_1 \theta r_2$ where $r_1, r_2 \in \mathsf{R}^+$. Thus,

$$\begin{aligned} B\sigma B &= Br_1\theta r_2 B \\ &\subseteq \overline{Br_1}\theta\overline{r_2 B} \\ &\subseteq \overline{B}\theta\overline{B} \qquad \text{since } \overline{Br_1}, \overline{r_2 B} \subseteq \overline{B}. \end{aligned}$$

Conversely, if $B\sigma B \subseteq \overline{B}\theta\overline{B}$ then there exist $r_1, r_2 \in \mathsf{R}^+$ such that $B\sigma B \subseteq Br_1 B\theta B r_2 B$. On the other hand, by Theorem 2.6, $r_1 B\theta \subseteq \bigcup_{r_3 \leq r_1} Br_3\theta B$, while by Theorem 2.5, $r_3\theta B r_2 \subseteq \bigcup_{r_4 \leq r_2} Br_3\theta r_4 B$. Hence $B\sigma B \subseteq Br_1 B\theta$ $Br_2 B \subseteq \bigcup_{\substack{r_3 \leq r_1 \\ r_4 \leq r_2}} Br_3\theta r_4 B$, from which it follows that $\sigma = r_3\theta r_4$ for some $r_3 \leq r_1$ and $r_4 \leq r_2$. But $r_1, r_2 \in \mathsf{R}^+$ and so $r_3, r_4 \in \mathsf{R}^+$. ∎

Theorem 2.7 above gives us one of the major ingredients in the decomposing the Adherence Order on R. It turns out that any relation $\sigma \leq \tau$ can be obtained as a chain of elementary relations $\sigma = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = \tau$, where, for each i, either $\theta_i \in \mathsf{R}^+\theta_{i+1}\mathsf{R}^+$, or else θ_{i+1} is obtained from θ_i by a "Bruhat Interchange" (see Definition 2.9 below). The next theorem will get us closer to the major result of this section.

2.8. THEOREM. *Let $\beta \in \mathrm{Ref}(W) = \bigcup_{w \in W} wSw^{-1}$ and $x, y \in W$. Assume that $x < x\beta$. Then the following are equivalent.*

(a) *$Bxey^{-1}B \subseteq \overline{B}x\beta ey^{-1}\overline{B}$ and $xey^{-1} \neq x\beta ey^{-1}$*

(b) *$\beta \notin W(e) = \{x \in W \mid xe = ex\}$.*

Proof. Assume (b). So $\beta \in \mathrm{Ref}(w) \setminus W(e)$. Then $x\beta$ can be written in reduced form as $x\beta = s_1 \cdots s_r$ where $s_i \in S$ for all i, $s_1, \ldots, s_m \in D(e)$ and $s_{m+1}, \ldots, s_r \in W(e)$. Clearly, $(x\beta)\beta = x$ and $(x\beta)\beta < x\beta$. Thus, by the strong exchange condition, there exists a unique i such that

$$x = (x\beta)\beta = s_1 \cdots s_{i-1}s_{i+1} \cdots s_r.$$

Hence $\beta = (s_r \cdots s_{i+1})s_i(s_{i+1} \cdots s_r)$. Also,

$$xe\beta x^{-1} = s_1 \cdots s_{i-1}s_{i+1} \cdots s_r e s_r \cdots s_1.$$

But now $\beta \in \mathrm{Ref}(W) \setminus W(e)$, so we must have $i \leq m$. Expanding further, we obtain

$$xe\beta x^{-1} = s_1 \cdots s_{i-1}s_{i+1} \cdots s_m e s_m \cdots s_1$$

with

$$s_1 \cdots s_{i-1}s_{i+1} \cdots s_m < s_1 \cdots s_m$$

and

$$s_1 \cdots s_m \in D(e).$$

But from Lemma 2.3 we obtain that $xe\beta x^{-1} \in R^+$. Hence,

$$\begin{aligned} Bxey^{-1}B &\subseteq \overline{B}xey^{-1}\overline{B} \\ &= \overline{B}(xe\beta x^{-1})(x\beta ey^{-1})\overline{B} \\ &\subseteq \overline{B}x\beta ey^{-1}\overline{B}. \end{aligned}$$

Finally, if $\beta \notin W(e)$ then $\beta \notin W_e$. Hence, $e \neq \beta e$ and it follows that $xey^{-1} \neq x\beta ey^{-1}$.

Conversely, assume $\beta \in W(e)$. Thus, $xe\beta y^{-1} = x\beta ey^{-1}$. Now if $xe\beta x^{-1} \in \overline{B}$ then $e\beta = e = \beta e$ and so $xey^{-1} = x\beta ey^{-1}$. Hence, either $xe\beta x^{-1} \notin \overline{B}$ or else $xey^{-1} = x\beta ey^{-1}$. If $xe\beta x^{-1} \notin \overline{B}$ then $Bxey^{-1}B \not\subseteq \overline{B}x\beta ey^{-1}\overline{B}$. (For if $Bxey^{-1}B \subseteq Bx\beta ey^{-1}\overline{B}$ then by Theorem 2.7, there exists $u, v \in R^+$ such that $ux\beta ey^{-1}v = xey^{-1}$. Replacing u and v by $uxex^{-1}$ and $yey^{-1}v$, if necessary, we can assume $u \in xex^{-1}C_W(xex^{-1})$ and $v \in yey^{-1}C_W(yey^{-1})$. But also, $u, v \in R^+$ and hence $u = xex^{-1}$ and $v = yey^{-1}$. But then $xey^{-1} = xe\beta y^{-1}$. But this implies $e = e\beta$ and so $xe\beta x^{-1} \in \overline{B}$, a contradiction.) We conclude that if $\beta \in W(e)$ then either $xey^{-1} = x\beta ey^{-1}$ or else $Bxey^{-1}B \not\subseteq \overline{B}x\beta ey^{-1}\overline{B}$. This completes the proof. ∎

2.9. Definition. Let xey^{-1} and set^{-1} be in standard form, $xey^{-1} \neq set^{-1}$. A *Bruhat Interchange* occurs between xey^{-1} and set^{-1} if there exists $\alpha \in \mathrm{Ref}(W) \cap W(e)$ such that $x \leq x\alpha$ and $x\alpha ey^{-1} = set^{-1}$.

Notice that for $xey^{-1} \neq x\alpha ey^{-1}$, $x \leq x\alpha$ if and only if $xey^{-1} \leq x\alpha ey^{-1}$.

2.10. Example. Consider, in $M_3(k)$,

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = xey^{-1}$$

$$\begin{aligned} \tau &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= x\alpha ey^{-1}. \end{aligned}$$

Here

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W(e).$$

Clearly,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x\alpha.$$

2.11. THEOREM. *Let $\sigma = xey^{-1}$ and $\theta = sft^{-1}$ be in standard form. Then $\sigma \leq \theta$ if and only if there exist $\theta_0, \theta_1, \dots, \theta_r \in R$ such that $\sigma = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_r = \theta$, and for each l, either $\theta_{l-1} \leq_j \theta_l$ or else θ_{l-1} and θ_l differ by a Bruhat Interchange.*

Proof. Obviously, the condition is sufficient. So assume $\sigma \leq \theta$. Then $Bxey^{-1}B \subseteq \overline{Bsft^{-1}B}$, and there exists $w \in W(f)W_e$ such that $x \leq sw$ and $tw \leq y$. Write $w = w_1w_2$ with $w_1 \in W(f)$ and $w_2 \in W_e$. using this fact, we obtain

$$\begin{aligned} sfwey^{-1} &= sfw_1w_2ey^{-1} \\ &= sw_1few_2y^{-1} \\ &= sw_1ew_2y^{-1} \\ &= swey^{-1}. \end{aligned}$$

Using this, we also obtain

$$\begin{aligned} Bswey^{-1}B &\subseteq \overline{B}sft^{-1}twey^{-1}\overline{B} \\ &\subseteq \overline{B}sft^{-1}\overline{B}, \qquad \text{since } twey^{-1} \in R^+. \end{aligned}$$

But also $Bxey^{-1}B \subseteq \overline{Bswey^{-1}B}$, since $x \leq sw$. In any case, we obtain

$$xey^{-1} \leq swey^{-1} \leq_j sft^{-1}.$$

So we let $\theta_r = sft^{-1}$ and $\theta_{r-1} = swey^{-1}$. Then we only need to find the chain of θ's from xey^{-1} to $swey^{-1}$. Since $x \leq sw$ we can find [4, Proposition 5.11] $\gamma_1, \dots, \gamma_k \in \operatorname{Ref}(W)$ such that

$$x < x\gamma_1 < x\gamma_1\gamma_2 < \cdots < x\gamma_1 \cdots \gamma_r = sw$$

and for every $i \in \{1, 2, \dots, r\}$ there exists $\delta_i \in W_e$ such that $x\gamma_1 \cdots \gamma_i\delta_i \in$

D_e. Since $x \in D(e)$,

$$x \leq x\gamma_1\delta_1 \leq x\gamma_1\gamma_2\delta_1 \leq \cdots < x\gamma_1\gamma_2 \cdots \gamma_r\delta_r.$$

If $x\gamma_1\gamma_2 \cdots \gamma_r\delta_r = x$ then

$$swey^{-1} = sw\delta_r^{-1}ey^{-1} = x\gamma_1 \cdots \gamma_r ey^{-1} = xey^{-1}.$$

In this case, we can just let $\theta_0 = swey^{-1}$ and $\theta_1 = sft^{-1}$ and we are done. If $x\gamma_1\gamma_2 \cdots \gamma_r \neq x$ then let

$$\begin{aligned}\theta_i &= x\gamma_1 \cdots \gamma_i\delta_i ey^{-1}\\ &= x(\gamma_1 \cdots \gamma_{i-1})\gamma_i ey^{-1}.\end{aligned}$$

If $\gamma_i \in \mathrm{Ref}(W) \setminus W(e)$ then by Theorem 2.8, $B\theta_{i-1}B \subseteq \overline{B}\theta_i\overline{B}$, while if $\gamma_i \in \mathrm{Ref}(W) \cap W(e)$ then θ_{i-1} and θ_i differ by a Bruhat Interchange. ∎

2.12. EXAMPLE. Let $M = M_4(k)$. Then

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} = \theta.$$

We can illustrate Theorem 2.11 as

$$\theta_0 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \theta_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\leq \theta_2 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \theta_3 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One could invoke Theorem 2.8 to prove $\theta_0 \leq_j \theta_1 \leq_j \theta_2$. However, we can see this directly (using Theorem 2.7) by observing

$$\theta_0 = \theta_1\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \theta_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}\theta_2.$$

Definition 2.9 easily yields that $\theta_2 \mapsto \theta_3$ is a Bruhat Interchange. (The Bruhat Interchange for $M_n(k)$ will be discussed systematically in Section 3.)

3. THE BRUHAT–CHEVALLEY ORDER ON $M_n(k)$

In this section we illustrate our theory for the reductive monoid $M_n(k)$ of $n \times n$ matrices. In this example, the symmetric inverse semigroup $\mathbf{R} = \mathbf{R}_n$ can be identified with the set of $n \times n$ zero-one monomial matrices. Our approach in this section is to identify and interpret the main results of Section 2 (especially Theorem 2.11) as they apply to $M_n(k)$. To help focus the reader, we consider the following question: Is there a purely combinatorial description for the Adherence Order on $\mathbf{R}_n$?

3.1. THEOREM. *Let E_{ij} be the $n \times n$ matrix (a_{st}) such that*

$$a_{st} = \begin{cases} 1, & (s,t) = (i,j) \\ 0, & (s,t) \neq (i,j). \end{cases}$$

Then $E_{ij} \leq_j E_{km}$ if and only if $i \leq k$ and $m \leq j$. ((i,j) is "up" and "to the right" of (k,m).)

Proof. If $E_{ij} \leq_j E_{km}$ then there exists upper triangular matrices X and Y such that $XE_{km}Y = E_{ij}$. However, this implies that $(XE_{km}Y)_{st} = 0$ if $s > k$ or $t < m$. Thus, $i \leq k$ and $m \leq j$.

Conversely, suppose $i \leq k$ and $m \leq j$. Then $X = E_{ik}$ and $Y = E_{mj}$ are both upper triangular. But one calculates $XE_{kl}Y = E_{ij}$. ∎

Remark. It is easy to check that $E_{ij} \leq_j E_{km}$ if and only if $E_{ij} \leq E_{km}$. This leads to a number of special properties in the case of $M = M_n(k)$.

We refer to E_{ij} as an *elementary matrix*.

3.2. THEOREM. *Let $A, C \in \mathbf{R}_n$ and write*

$$A = \sum_{l=1}^{s} A_l$$

$$C = \sum_{l=1}^{t} C_l,$$

where $\{A_l\}$ and $\{C_l\}$ are elementary matrices. Define $S_A = \{A_1, \ldots, A_s\}$ and $S_C = \{C_1, \ldots, C_t\}$. Then the following are equivalent:

(a) $A <_j C$

(b) *There exists an injection $\theta: S_A \to S_C$ such that $A_l \leq \theta(A_l)$ for all $l = 1, 2, \ldots, s$.*

Proof. Assume $A \leq_j C$. Notice that $s = \operatorname{rank}(A)$ and $t = \operatorname{rank}(C)$. Write

$$A = w_1 \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} w_2$$

$$C = w_3 \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} w_4,$$

where $w_1, w_2, w_3, w_4 \in S_n$, the unit group of R_n. Let

$$X = w_1 w_3^{-1}$$

$$Y = w_4^{-1} \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} w_2.$$

Then we have

$$XCY = A.$$

Furthermore, one can check that, for all $k \leq t$,

$$XC_kY \leq C_k,$$

and

$$\operatorname{rank}(XC_kY) \leq 1.$$

Hence, for each $k = 1, 2, \ldots, t$

$$\begin{aligned} &\text{either} \quad XC_kY \in S_A \\ &\text{or else} \quad XC_kY = 0. \end{aligned}$$

But for any $A_l \in S_A$ there exists C_k such that $XC_kY = A_l$ (since $XCY = A$). So define $\theta(A_l) = C_k$, where $XC_kY = A_l$.

Conversely, if there exists an injection $\theta\colon S_A \to S_C$ such that $A_l \leq_j \theta(A_l)$ for all $l = 1, 2, \ldots, s$, then we can find elementary upper triangular matrices X_l and Y_l such that $X_l\theta(A_l)Y_l = A_l$. Let

$$X = \sum_{l=1}^{s} X_l$$

and

$$Y = \sum_{l=1}^{s} Y_l.$$

Then X and Y are both upper triangular and (one checks) $XCY = A$. Thus $A \leq_j C$. ∎

3.3. EXAMPLE.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \leq_j \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C$$

since $S_A = \{E_{11}, E_{23}\}$, $S_C = \{E_{11}, E_{22}, E_{33}\}$, and θ: $S_A \to S_C$ defined by $\theta(E_{11}) = E_{11}$, $\theta(E_{23}) = E_{22}$ satisfies the criterion of Theorem 3.2.

To properly illustrate Theorem 2.11 in the situation of R_n it remains to describe concretely the notion of a Bruhat Interchange (Definition 2.9). We begin our description with the rank two case.

3.4. THEOREM. *Let $A, C \in \mathsf{R}_n$ be such that $A = E_{ij} + E_{kl}$ and $C = E_{kj} + E_{il}$. Then $A < C$ if and only if $i < k$, $j < l$ or $i > k$, $j > l$.* (*Notice that $A <$ if and only if a Bruhat Interchange has occurred* (*see Definition* 2.9).)

Proof. Suppose $i < k$ and $j < l$ (the other case is similar). If $E = E_{11} + E_{22}$ then $A = (i-1i) \cdots (12)(k-1k) \cdots (34)(23)E(23) \cdots (l-1l)(12) \cdots (j-1j)$ where (ab) is the permutation matrix which interchanges the ath row and the bth row. This is the standard form for A. One checks that

$$C = [(i-1i) \cdots (12)(k-1k) \cdots (34)(23)] \\ (12)E(23) \cdots (l-1l)(12) \cdots (j-1j).$$

But $[(i-1i) \cdots (23)] < [(i-1i) \cdots (23)](12)$. Therefore $A < C$.

Conversely, assume $i < k$ and $j > l$ or $i > k$ and $j < l$. But then the same argument, as above, shows that $C < A$. ∎

3.5. THEOREM. *Let $A = E_{i_1 j_1} + \cdots + E_{i_s j_s} \in \mathsf{R}$ (where, as usual, E_{ij} is an elementary matrix) and suppose $C = E_{i_1 j_1} + \cdots + E_{i_{k-1} j_{k-1}} + E_{i_l j_k} + E_{i_{k+1} j_{k+1}} + \cdots + E_{i_{l-1} j_{l-1}} + E_{i_k j_l} + E_{j_{l+1} j_{l+1}} + \cdots + E_{i_s j_s}$ (so C is obtained from A by interchanging two nonzero rows). Relabel the two interchanged matrices E_{ij} and E_{kl}. Then $A < C$ if and only if $i < k$, $j < l$ or $i > k$, $j > l$.*

Proof. If $i < k$ and $j < l$, then by Theorem 3.4, $(E_{ii} + E_{kk})A < (E_{ii} + E_{kk})C$. A little more calculation then shows that $A < C$.

Conversely, if $i > k$ and $j < l$, or $i < k$ and $j > l$ then the above argument shows that $C < A$. ∎

Notice that the situation above describes exactly the case where a Bruhat Interchange occurs between A and C.

Combining Definition 2.9 and Theorems 2.11 and 3.5 we obtain the following description of the Adherence Order on $\mathbf{R}_n$.

3.6. THEOREM (Theorem 2.11 for $\mathbf{R}_n$). *Let $\sigma, \tau \in \mathbf{R}_n$. Then $\sigma \leq \tau$ if and only if $\sigma = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_r = \tau$ where, for each l, either*

(a) $\gamma_{l-1} \in \mathbf{R}_n^+ \gamma_l \mathbf{R}_n^+$, *or else*

(b) γ_l *is obtained from* γ_{l-1} *via a Bruhat Interchange.*

To calculate in case (a), we use Theorem 3.2, and in case (b) we use Theorem 3.5.

Recall that for $\sigma \in \mathbf{R}_n$ we write $S_\sigma = \{E_1, \ldots, E_s\}$ if $\sigma = \sum_{i=1}^s E_i$ and each E_i is an elementary matrix.

3.7. THEOREM. *Suppose $\sigma \leq \tau$ and $l(\sigma) = l(\tau) - 1$. (So $\sigma \leq \gamma \leq \tau$ implies $\sigma = \gamma$ or $\gamma = \tau$.) Then one of the following holds:*

(a) $|S_\sigma| = |S_\tau| - 1$ *and* $S_\sigma \subseteq S_\tau$.

(b) $|S_\sigma| = |S_\tau|$ *and* $|S_\sigma \cap S_\tau| = |S_\sigma| - 1$ *with $E_\sigma < E_\tau$ where $S_\sigma \setminus (S_\sigma \cap S_\tau) = \{E_\sigma\}$ and $S_\tau \setminus (S_\sigma \cap S_\tau) = \{E_\tau\}$. Furthermore, σ and τ are either row equivalent or column equivalent.*

(c) τ *is obtained from* σ *via a Bruhat Interchange.*

Proof. Assume (c) is not the case, and $\operatorname{rank}(\sigma) < \operatorname{rank}(\tau)$. By Theorems 3.2 and 3.6 there exists an injection $\theta \colon S_\sigma \to S_\tau$ such that $\sigma_i \leq \theta(\sigma_i)$ for each $\sigma_i \in S_\sigma$. Define $\theta(\sigma) \in \mathbf{R}_n$ by $S_{\theta(\sigma)} = \theta(S_\sigma)$. Then $\sigma \leq \theta(\sigma) < \tau$. Thus $\sigma = \theta(\sigma)$. Therefore, (a) holds.

Now assume (c) is not the case, and $\operatorname{rank}(\sigma) = \operatorname{rank}(\tau)$. By Theorem 3.6 we must have $\sigma = x\tau y$ for some $x, y \in \mathbf{R}^+$. Then $\sigma = x\tau y \leq \tau y \leq \tau$. Hence, either $\sigma = \tau y$ or else $\tau y = \tau$. In the first case $\sigma = \tau y$, and in the second case $\sigma = x\tau$. Assume, without loss of generality that $\sigma = \tau y$ (σ and τ have the same nonzero rows). Now by Theorem 3.2, the map θ: $S_\sigma \to S_\tau$ has $\sigma_i \leq \theta(\sigma_i)$ for all $\sigma_i \in S_\sigma$. But by Theorem 3.1, this means that if $\sigma_i = E_{uv}$ and $\theta(\sigma_i) = E_{rs}$ then $r \geq u$ and $s \leq v$ (E_{uv} is "up" an "to the right" of E_{rs}). But σ and τ have the same nonzero rows. Since θ is row increasing it follows easily that θ is row preserving. So write

$$S_\sigma = \{E_{i1}, \ldots, E_{is}\}$$

and

$$S_\tau = \{E_{j1}, \ldots, E_{js}\},$$

where E_{ik} and E_{jk} are nonzero in the same row (so that $\theta(E_{ik}) = E_{jk}$). Now let $E_{ik} \in S_\sigma$ be the elementary matrix with the maximum column value. Define $\tau' \in R_n$ via $S_{\tau'} = (S_\tau \setminus \{E_{jk'}\}) \cup \{E_{ik}\}$ where $E_{jk'}$ is nonzero in the same row as E_{ik} (so $\theta(E_{jk}) = E_{jk'}$). Then $\tau' = \sum_{E \in S_{\tau'}} E \in R_n$ since θ is column decreasing. Also $\sigma \leq_j \tau' \leq_j \tau$ since E_{ik} has larger column value than $E_{ik'}$. If $\tau' \neq \tau$ then we are done, since we obtain $\sigma = \tau'$, and so (b) holds. If $\tau' = \tau$ (i.e., $E_{ik} = E_{jk'}$) we move on; i.e., define $\tau^{(2)}$ using the $E_{is} \in S_\sigma$ with the next largest column value, thereby substituting it for the corresponding $E_{j_{s'}} \in S_\tau$. Again one checks that $\tau^{(2)} \in R_n$, and $\sigma \leq \tau^{(2)} \leq \tau$. If $\tau^{(2)} = \tau$ then define $\tau^{(3)}$, and so on. Eventually, one obtains $\tau^{(r)} \neq \tau$. But then $\sigma = \tau^{(r)}$ since $\sigma \leq \tau^{(r)} < \tau$ and $l(\tau) = l(\sigma) + 1$. So τ is obtained from σ by "moving" one elementary matrix of σ "to the left." So (b) holds. ∎

Theorem 3.7 allows us to give a combinatorial description of the Adherence Order on R_n. First we represent the elements of R_n by sequences of nonnegative integers. Let $\sigma \in R_n$. We associate to σ a sequence $(\epsilon_1, \cdots, \epsilon_n)$ where, for all i, $1 \leq i \leq n$, ϵ_i is defined as

$$\epsilon_i = \begin{cases} 0, & \text{if } \sigma \text{ is zero in the } i\text{th column} \\ k, & \text{if } [\sigma]_{ki} = 1. \end{cases}$$

EXAMPLE.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \leftrightarrow (3042).$$

Notice that a sequence $(\epsilon_1, \dots, \epsilon_n)$ can occur this way for some $r \in R_n$ if and only if $0 \leq \epsilon_i \leq n$ for all i; and whenever $\epsilon_i = \epsilon_j$, either $i = j$ or else $\epsilon_i = \epsilon_j = 0$. For convenience we simply write $\sigma = (\epsilon_1, \dots, \epsilon_n) \in R_n$. We can now interpret Theorem 3.7 combinatorially.

3.8. THEOREM. *The Adherence Order on R_n is the smallest partial order generated by declaring*

$$\sigma = (\delta_1, \dots, \delta_n) < \tau = (\epsilon_1, \dots, \epsilon_n)$$

if either

(a) $\delta_j = \epsilon_j$ *for* $j \neq i$ *and* $\delta_i < \epsilon_i$ *(σ is obtained from τ by replacing some ϵ_i by a smaller number)*

(b) $\delta_k = \epsilon_k$ *if* $k \neq i$ *or* j, $i < j$, $\delta_i = \epsilon_i$, $\epsilon_i = \delta_j$ *and* $\epsilon_i > \epsilon_j$ *(σ is obtained from τ by interchanging ϵ_i and ϵ_j ($i < j$) where $\epsilon_i > \epsilon_j$).*

Proof. We simply interpret each case of Theorem 3.7.

Case 3.7(a). This says that σ is obtained from τ by setting some nonzero entry in τ (as a matrix) to zero. This is included in Theorem 3.8(a).

Case 3.7(b). First assume that σ and τ have the same nonzero rows. So, as we saw in the proof of Theorem 3.7, the matrix σ is obtained from the matrix τ by moving a "one" to the right. This is included in Theorem 3.8(b). If σ and τ have the same columns then σ is obtained from τ by moving a "one" upward. This is included in Theorem 3.8(a).

Case 3.7(c). This is easily seen as the special case of Theorem 3.8(b) where both ϵ_i and ϵ_j are nonzero. ∎

3.9. EXAMPLE. Let $r = (21403)$ and $s = (35201)$ in R_5. Then $r < s$, since

$$\begin{aligned} (21403) &< (31402), && \text{by Theorem 3.8(b)} \\ &< (34102), && \text{by Theorem 3.8(b)} \\ &< (35102), && \text{by Theorem 3.8(a)} \\ &< (35201), && \text{by Theorem 3.8(b)}. \end{aligned}$$

4. SUBADDITIVITY OF LENGTH

In this section we consider a natural length on R, extending the much studied length function on W, the Weyl group. In keeping with our point of view in this paper, we shall define this length function in terms of the length function on W and a notion of length for the elements of Λ. This appears to be a natural generalization of length function considered by Solomon [12] for $M = M_n(\mathbb{F}_q)$, where R can be identified with the symmetric inverse semigroup on n letters. There is also a topological/geometric definition for this length function. Indeed, one can define

$$l(x) = \dim(BxB) - \dim(B\nu B),$$

where $\nu \in WxW$ is the unique element such that $B\nu = \nu B$. This is discussed in some detail by the third named author in [11]. Our main result in this section proves that the length function is subadditive.

Let $W = \langle S \rangle$ be the Weyl group, where $S \subseteq W$ is the set of simple involutions relative to B and $T \subseteq B$. Define

$$\mathrm{Ref}(W) = \bigcup_{x \in W} xSx^{-1}.$$

For $I \subseteq S$ we have

$$\mathrm{Ref}(W_I) = W_I \cap \mathrm{Ref}(W).$$

Let

$$l(I) = |\mathrm{Ref}(W) \setminus \mathrm{Ref}(W_I)|$$

$$D(I) = \{x \in W \mid x \text{ has minimal length in } xW_I\}.$$

Thus,

$$l(I) = \max\{l(x) \mid x \in D_I\}.$$

For $I, J \subseteq S$ let

$$l(I, J) = |\mathrm{Ref}(W_I) \setminus W_J|.$$

So

$$l(I) = l(S, I)$$

and

$$\begin{aligned} l(I, J) &= l(I, I \cap J) \\ &= \max\{l(x) \mid x \in W_I \cap D_J\}. \end{aligned} \tag{1}$$

Now let R be as in section one with unit group W and cross section lattice Λ. For $e \in \Lambda$, recall that

$$W(e) = \{x \in W \mid xe = ex\}$$

$$W_e = \{x \in W \mid xe = e\} \triangleleft W(e).$$

Notice that there exists a unique subgroup $\tilde{W}(e)$ of W such that $\tilde{W}(e)$ is generated by a subset of S and

$$W(e) = W_e \times \tilde{W}(e) \tag{2}$$

$$D(e) = D_I, \qquad \text{where } W(e) = \langle I \rangle,$$

$$D_e = D_J, \qquad \text{where } W_e = \langle J \rangle,$$

and

$$l(e) = l(I).$$

If $e, f \in \Lambda$, let

$$l(f, e) = l(J, I), \qquad \text{where } W(f) = \langle J \rangle \text{ and } W(e) = \langle I \rangle.$$

If $e, f \in \Lambda$, $x \in W$, then

$$\begin{aligned} exf \in WfW \qquad &\text{iff } e \geq f \text{ and } x \in W(e)W(f) \\ fxe \in WfW \qquad &\text{iff } e \geq f \text{ and } x \in W(f)W(e). \end{aligned} \tag{3}$$

If $e, f \in \Lambda$ then

$$e \geq f \Rightarrow \tilde{W}(f) \subseteq W(e). \tag{4}$$

Thus, by (2) and (4),

$$e \geq f \Rightarrow W(f) = W(e, f)W_f, \tag{5}$$

where $W(e, f) = W(e) \cap W(f)$. Recall from the end of Section 1 that if $\sigma \in WeW$, $e \in \Lambda$, then

$$\sigma = xey^{-1} \qquad \text{for some } x \in D_e \text{ and } y \in D(e).$$

Furthermore, x and y are unique (by an easy counting argument).

4.1. DEFINITION. Let $\sigma = xey^{-1}$ be in standard form. Define

$$l(\sigma) = l(x) + l(e) - l(y),$$

where $l(x), l(y)$ are the lengths of x and y as elements of W, and $l(e)$ is as above.

It is easy to see that $l(z\sigma) \leq l(z) + l(\sigma)$ for any $z \in W$. However, much more is true. The main result of this section (Theorem 4.5 below) says that for any $\sigma, \tau \in R$, $l(\sigma\tau) \leq l(\sigma) + l(\tau)$. For example, let

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that

$$\sigma\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = xey^{-1},$$

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = uev^{-1}$$

and

$$\sigma\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = rfs^{-1}$$

determine the standard form of σ, τ, and $\sigma\tau$. Thus, by Definition 4.1

$$l(\sigma) = l(x) + l(e) - l(y) = 3 + 2 - 2 = 3$$
$$l(\tau) = l(u) + l(e) - l(v) = 1 + 2 - 0 = 3$$
$$l(\sigma\tau) = l(r) + l(f) - l(s) = 2 + 2 - 0 = 4.$$

So $l(\sigma\tau) \leq l(\sigma) + l(\tau)$ as required.

4.2. LEMMA. *Let $e, f, h \in \Lambda$ be such that $eWf \cap WhW \neq \emptyset$. Then*

$$l(h) \leq l(e) + l(f) - l(h, e) - l(h, t).$$

Proof. Let $x \in W$ be such that $exf = ahb^{-1}$ is in standard form. Then $eah \in WhW$. So by (3) and (5) above $a \in W(e)W(h) = W(e)W_h$. Since $a \in D_h$, we see that $a \in W(e)$. Also, since $hb^{-1}f \in WhW$, we see by (3) that $b^{-1} \in W(h)W(f)$. Finally, since $b \in D(h)$, we see that $b \in W(f)$. We conclude that

$$ea^{-1}xbf = a^{-1}exfb = h.$$

Thus, without loss of generality we may assume that $exf = h$. So we obtain $hxh = x$ and, by (3), $x = W(h)$. Since $x^{-1}exf$ is an idempotent we obtain that $x^{-1}exf = h$. Thus, $W(x^{-1}ex) \cap W(f) \subseteq W(h)$, and it follows that

$$x^{-1}\operatorname{Ref}(W(e))x \cap \operatorname{Ref}(W(f)) \subseteq \operatorname{Ref}(W(h)).$$

Thus, in $\operatorname{Ref}(W)$,

$$\operatorname{Ref}(W(h))^c \subseteq x^{-1}\operatorname{Ref}(W(e))^c x \cup \operatorname{Ref}(W(f))^c$$

and so

$$\operatorname{Ref}(W(h))^c \subseteq \left[x^{-1}\operatorname{Ref}(W(e))^c x \setminus \operatorname{Ref}(W(h)) \cap x^{-1}\operatorname{Ref}(W(e))^c x\right]$$
$$\cup\left[\operatorname{Ref}(W(f))^c \setminus \operatorname{Ref}(W(h)) \cap \operatorname{Ref}(W(f))^c\right].$$

But

$$\left|\operatorname{Ref}(W(h))^c\right| = l(h)$$
$$\left|x^{-1}\operatorname{Ref}(W(e))^c x\right| = \left|\operatorname{Ref}(W(e))^c\right| = l(e)$$

and

$$\left|\operatorname{Ref}(W(h)) \setminus x^{-1}\operatorname{Ref}(W(e))x\right| = \left|x^{-1}[\operatorname{Ref}(W(h)) \setminus \operatorname{Ref}(W(e))]x\right|$$
$$= l(h, e)$$

since $x \in W(h)$. Furthermore,

$$\left|\operatorname{Ref}(W(f))^c\right| = l(f)$$
$$\left|\operatorname{Ref}(W(h)) \setminus \operatorname{Ref}(W(f))\right| = l(h, f).$$

Thus, we conclude that $l(h) \leq [l(e) - l(h, e)] + [l(f) - l(h, f)]$. ∎

4.3. COROLLARY. *Let $e, f, h \in \Lambda$ and $x \in D(e)$ be such that $ex^{-1}f = ahb^{-1}$ is in standard form. Then*

$$l(a) + l(h) - l(b) \leq l(e) + l(f) - l(x) - l(h, f).$$

Proof. Now $a \in D_h$, $b \in D(h)$, and $eah, hb^{-1}f \in WhW$. Thus (3) and (5), $a \in W(e)$ and $b \in W(f)$. Hence $ea^{-1}x^{-1}bf = a^{-1}ex^{-1}fb = h$. So $ha^{-1}x^{-1}bh = hea^{-1}x^{-1}bfh = h$. But then by (3), $a^{-1}x^{-1}b \in W(h)$, and so $a^{-1}x^{-1}bh = h$. Hence, $a^{-1}xb \in W_h$. Let $\alpha = a^{-1}x^{-1}b \in W_h$. So

$$a^{-1}x^{-1} = \alpha b^{-1}. \tag{6}$$

Since $a \in W(e)$, $x \in D(e)$, $\alpha \in W_h$, and $b \in D(h)$,

$$l(a) + l(x) = l(a^{-1}x^{-1}) = l(\alpha b^{-1}) = l(\alpha) + l(b). \tag{7}$$

But $\alpha = \alpha_1\alpha_2$ with $\alpha_1 \in W_h \cap W(e)$, $\alpha_2 \in W_h \cap D(e)^{-1}$, and $l(\alpha) = l(\alpha_1) + l(\alpha_2)$. So by (6), $\alpha^{-1}a^{-1}x^{-1} = \alpha_2 b^{-1}$. Then

$$l(\alpha_1^{-1}a) = l(\alpha_1) + l(a) \qquad \text{since } \alpha_1 \in W_h, a \in D_h$$

and

$$\begin{aligned} l\big(\alpha_1^{-1}a^{-1}x^{-1} &= l\big(\alpha_1^{-1}a^{-1}\big) + l(x) \\ &= l(\alpha_1) + l(a) + l(x) \quad \text{since } \alpha_1^{-1}a^{-1} \in W(e) \text{ and} \\ &\qquad\qquad\qquad\qquad\qquad\quad x \in D(e). \end{aligned}$$

Furthermore,

$$l\big(\alpha_2 b^{-1}\big) = l(\alpha_2) + l(b) \qquad \text{since } b \in D(h).$$

Thus,

$$l(\alpha_1) + l(a) + l(x) = l(\alpha_2) + l(b).$$

But $l(\alpha) = l(\alpha_1) + l(\alpha_2)$, so by (7),

$$l(a) + l(x) = l(\alpha_1) + l(\alpha_2) + l(b).$$

Conclude that $l(\alpha_1) = 0$, and so $\alpha = \alpha_2 \in W_h \cap D(e) \subseteq W(h) \cap D(e)$. By (1), $l(\alpha) \leq l(h, e)$. Hence, $l(a) + l(x) \leq l(h, e) + l(b)$. Combining this with the lemma we obtain $l(a) + l(h) - l(b) \leq l(e) + l(f) - l(x) - l(h, f)$. ∎

4.4. COROLLARY. *Let* $e, f, h \in \Lambda$, $x \in D(e)$, $t \in D(f)$. *Then* $l(ex^{-1}ft^{-1}) \leq l(ex^{-1}) + l(ft^{-1})$.

Proof. Let $ex^{-1}f = ahb^{-1}$, $h \in \Lambda$, be in standard form. Then as in Corollary 4.3,

$$a \in D_h \cap W(e),\, b \in D(h) \cap W(f) \tag{8}$$

and

$$l(a) + l(h) - l(b) \leq l(e) + l(f) - l(x) - l(h, f). \tag{9}$$

Let

$$b^{-1}t^{-1} = \alpha v^{-1}, \qquad \alpha \in W(f),\, v \in D(h). \tag{10}$$

By (8) and (10)

$$l(b) + l(t) = l(b^{-1}t^{-1}) = l(\alpha v^{-1}) = l(\alpha) + l(v). \tag{11}$$

But $\alpha = \alpha_1\alpha_2$ with $\alpha_1 \cap (f) \cap W(h)$ and $\alpha_2 \in D(f)^{-1} \cap W(h)$. So $l(\alpha) = l(\alpha_1) + l(\alpha_2)$. By (10), $\alpha_1^{-1}b^{-1}t^{-1} = \alpha_2 v^{-1}$ and (8), $l(\alpha_1) + l(b) = l(\alpha_1^{-1}b^{-1})$. Since $\alpha_1^{-1}b^{-1} \in W(h)$ and $t \in D(h)$, we obtain

$$\begin{aligned} l(\alpha_1) + l(b) + l(t) &= l\big(\alpha_1^{-1}b^{-1}\big) + l(t) \\ &= l\big(\alpha_1^{-1}b^{-1}t^{-1}\big) \\ &= l\big(\alpha_2 v^{-1}\big) \\ &= l(\alpha_2) + l(v). \end{aligned}$$

But by (11), $l(b) + l(t) = l(\alpha_1) + l(\alpha_2) + l(v)$. Hence $l(\alpha_1) = 0$ and so $\alpha = \alpha_2 \in D(f)^{-1} \cap W(h)$. By (1)

$$l(\alpha) \leq l(h, f). \tag{12}$$

Since $f \geq h$ it follows from (5) that

$$\alpha \in W(h) \cap D(f) \subseteq W_h \cap D(f).$$

Hence, by (10)

$$\begin{aligned} ex^{-1}ft^{-1} &= ahb^{-1}t^{-1} \\ &= ah\alpha v^{-1} \\ &= ahv^{-1}. \end{aligned}$$

Finally, by (9), (11), and (12)

$$\begin{aligned} l(ex^{-1}ft^{-1}) &= l(a) + l(h) - l(b) - l(t) + l(\alpha) \\ &\leq l(e) + l(f) - l(x) - l(h, f) - l(t) + l(\alpha), \\ &\qquad \text{by Corollary 4.3} \\ &\leq l(e) + l(f) - l(x) - l(t) \\ &= l(ex^{-1}) + l(ft^{-1}). \quad \blacksquare \end{aligned}$$

4.5. THEOREM. *Let* $\sigma_1, \sigma_2 \in R$. *Then*

$$l(\sigma_1\sigma_2) \leq l(\sigma_1) + l(\sigma_2).$$

Proof. Write $\sigma_1 = xey^{-1}$ and $\sigma_2 = vft^{-1}$ in standard form with $e, f \in \Lambda$. Let $y^{-1}v = wy_1^{-1}$ where $w \in W(e)$ and $y_1 \in D(e)$. Since $y \in D(e)$ and $w \in W(e)$ we obtain that $w^{-1}y^{-1} = y_1^{-1}v^{-1}$ and $l(w^{-1}y^{-1}) = l(w) + l(y)$. Hence,

$$l(w) + l(y) = l(y^{-1}v^{-1}) \leq l(y_1) + l(v). \tag{13}$$

Finally,

$$\begin{aligned} &l(\sigma_1\sigma_2) + l(xey^{-1}vft^{-1}) \\ &\quad = l(xewy_1^{-1}ft^{-1}) \\ &\quad = l(xwey_1^{-1}ft^{-1}) \\ &\quad \leq l(xw) + l(ey_1^{-1}ft^{-1}) \\ &\quad \leq l(xw) + l(e) - l(y_1) + l(f) - l(t), \qquad \text{by Corollary 4.4} \\ &\quad \leq l(x) + l(w) + l(e) - l(y_1) + l(f) - l(t) \\ &\quad \leq l(x) + l(v) - l(y) + l(e) + l(f) - l(t), \qquad \text{by (13)} \\ &\quad = l(\sigma_1) + l(\sigma_2), \qquad \text{by definition.} \quad \blacksquare \end{aligned}$$

Remark. The question of subadditivity of length was raised by L. Solomon. It turns out that the issue of integral structure constants for monoid Hecke algebras hinges on Theorem 4.5. See the proof of [12, Theorem 4.12] for an illustration of this issue in the special case $M = M_n(\mathbb{F}_q)$.

5. RELATED RESULTS

In this section we complete our current study by establishing a number of supplementary results concerning the Adherence Order. We focus on the relationship between the Adherence Order and the $\mathcal{H}$-relation on $\mathcal{R}$. It is clear to the authors that there are an intriguing number of unchartered possibilities that could be pursued here.

Let M be a reductive monoid with maximal torus T. If $\sigma, \tau \in \mathcal{R}$ we say that σ and τ are $\mathcal{H}$-related (in $\mathcal{R}$) if $\sigma\mathcal{R} = \tau\mathcal{R}$ and $\mathcal{R}\sigma = \mathcal{R}\tau$. We write $\sigma\mathcal{H}\tau$. Similarly, elements x and y of M can be $\mathcal{H}$-related. We denote the $\mathcal{H}$-class in M of $x \in M$ by H_x and the $\mathcal{H}$-class of $x \in \mathcal{R}$ in $\mathcal{R}$ by $\mathcal{H}_x$. So $H_x = \{y \in M | yM = xM \text{ and } My = Mx\}$. Two elements x and y of M are $\mathcal{I}$-related $(x\,\mathcal{I}\,y)$ is $MxM = MyM$. We write $x \geq_{\mathcal{I}} y$ if $MyM \subseteq MxM$. For $n \times n$ matrices, $x \geq_{\mathcal{I}} y$ in the $\mathcal{I}$-order if $\operatorname{rank}(x) \geq \operatorname{rank}(y)$.

Let $e, f \in E(\overline{T})$.

5.1. LEMMA. *Suppose $x \in eMf \cap eG$. Then there exists $e' \in E(\overline{T})$ such that $e' \in Cl_W(e)$, $e' \leq f$, and $xe' \in eG$.*

Proof. Write $x = eg$ and let $b_1, b_2 \in B$ be such that $g = b_1wb_2$. Here B is a Borel subgroup containing T such that $B \subseteq \{h \in G | eh = ehe\}$. Thus $eb_1 = eb_1e = ce$ for some $c \in C_B(e)$. Hence, $eg = cewb_2 = cwew^{-1}b_2 = ge'b$, where $e' = w^{-1}ew$ and $b = b_2$. So $xe' = ege' = ye'be' \in Ge'$ since $e'be' \in Be'$. But also $eg = x = xf$. So if $ege' \in Ge'$ then $f' \geq e'$. Otherwise, $f \not\geq e'$ and so $fe' < e'$ which implies that $xfe' < e'$ in the $\mathcal{I}$-order on M. But this is absurd since $xfe' = ege' \in Ge'$. ∎

5.2. COROLLARY. *Suppose $eBf \cap eG \neq \emptyset$. Then there exists $e' \in E(\overline{T})$ such that $e'\,\mathcal{I}\,e$, $e' \leq f$, and $eBe' \cap eG \neq \emptyset$.*

Proof. Let $x \in eBf \cap eG$ and choose e' as in Lemma 5.1. Then $xe' \in eBe' \cap eG$ and the conclusion follows. ∎

For $s \in \mathcal{R}$ let H_s denote the $\mathcal{H}$-class of s in M.

5.3. COROLLARY. *Let $r, s \in \mathcal{R}$ and let $e, f \in E(\overline{T})$ be such that $e\mathcal{R} = s\mathcal{R}$ and $\mathcal{R}s = \mathcal{R}f$. Suppose $eBrBf \cap H_s \neq \emptyset$. Then there exist $e', f' \in E(\overline{T})$ with $e'\,\mathcal{I}\,f'\,\mathcal{I}\,s$ such that $eBe'rf'Bf \cap H_s \neq \emptyset$. In particular, $e'r = rf'\,\mathcal{I}\,s$ and if we write $r = e_0\sigma = \sigma f_0$ with $\sigma \in W$ then $e' \leq e_0$ and $f' \leq f_0$.*

Proof. Since $eBrBf \cap H_s \neq \varnothing$ it follows easily that $eBe_0 \cap eG \neq \varnothing$ and $f_0 Bf \cap Gf \neq \varnothing$. So by Corollary 5.2 and its left-right dual we obtain $e' \leq e_0$ a nd $f' \leq f_0$ such that $eBe' \cap eG \neq \varnothing$ and $f'Bf \cap Gf \neq \varnothing$.

If we let ${}_{e_1}H_{e_2}$ denote the $\mathcal{H}$-class of $e_1\sigma$ where $\sigma^{-1}e_1\sigma = e_2$ then it is easy to see that ${}_{e_1}H_{e_2 e_2}H_{e_3} \subseteq {}_{e_1}H_{e_3}$. But we have $\varnothing \neq eBe'r \cap {}_eH_{f'}$ (since $e'r = rf'$) and $\varnothing \neq f'Bf \cap_{f'} H_f$. Thus, recalling that $H_s = {}_eH_f$, we obtain $eBe'rf'Bf \cap H_s \neq \varnothing$. ∎

We recall now some results from [11]. Define the set of *order preserving elements* of $\mathcal{R}$ by

$$\mathcal{O} = \{ r \in \mathcal{R} \mid rBr^* \subseteq Brr^* \}.$$

Then by [11, Sect. 2]

$$\mathcal{O} \subseteq \mathcal{R} \text{ is an inverse subsemigroup},$$

$$E(\mathcal{R}) \subseteq \mathcal{O},$$

and

$$|\mathcal{O} \cap H| = 1 \qquad \text{for each } \mathcal{H}\text{-class of } \mathcal{R}.$$

For each $s \in \mathcal{R}$ we can write uniquely

$$s = s_+ s_0 s_-,$$

where $s_+, s_- \in \mathcal{O}$, $s_+\mathcal{R} = s\mathcal{R}$, $\mathcal{R}s_- = \mathcal{R}s$, $s_0 \,\mathcal{H}\, \nu$, and $\nu \,\mathcal{J}\, s$ is the unique such element that satisfies $B\nu = \nu B$.

To illustrate these notions, consider our familiar example of $\mathcal{R}_n$. One can easily calculate that, in this example,

$$\mathcal{O} = \mathcal{O}_n = \left\{ \sigma \in \mathcal{R}_n \middle| \text{if } \sigma_{ij}, \sigma_{kl} \neq 0 \text{ and } i < k, \text{ then } j < l \right\};$$

i.e., for nonzero entries, the column value is an increasing function of the row value, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{O}_4, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \notin \mathcal{O}_4.$$

It is interesting to notice that for each $\sigma \in \mathcal{R}_n$, there exists a unique $\bar{\sigma} \in \mathcal{O}_n$ so that σ be obtained form $\bar{\sigma}$ via a sequence of Bruhat interchanges. $\mathcal{O}_n$ is an inverse semigroup because it is closed under matrix multiplication and transpose.

Another easy calculation yields that

$$\{\nu \in R_n \mid B\nu = \nu B\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \dots, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right\}.$$

The decomposition

$$s = s_+ s_0 s_- \qquad (\text{for any } s \in R_n)$$

can be thought of as follows:

s_+ indicates the nonzero rows of s.

s_- indicates the nonzero columns of s.

s_0 indicates the nonzero columns as a function of the nonzero rows.

One should think of s_0 as the "order reversing" part of s. For example, $s_0 = \nu$ if and only if $s \in O$. As an illustration, consider the example

$$\begin{aligned} s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= s_+ s_0 s_-. \end{aligned}$$

As promised, s_+ says "rows one and three are involved in s," s_- says "columns one and two are involved in s." s_0 represents the function

$$R_1 \mapsto C_2$$

$$R_3 \mapsto C_1.$$

See [11, Sect. 3] for more details concerning O and the $+0-$ decomposition.

5.4. DEFINITION. For $s \in R$ let $\bar{s} \in O$ denote the unique element of $O \cap H_s$. So if $s = s_+ s_0 s_-$ as above then $\bar{s} = s_+ \nu s_-$.

For the elements of R_n, it is easy to find $\bar{s}$ from s without finding s_+, s_0, and s_-. $\bar{s}$ is the unique element of O_n with the same nonzero rows and columns as s. For example, if

$$s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{then } \bar{s} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.5. COROLLARY. *The following are equivalent for $r, s \in \mathsf{R}$ with $e\mathsf{R} = s\mathsf{R}$ and $\mathsf{R}s = \mathsf{R}f$.*

(a) $eBrBf \cap H_s \neq \varnothing$

(b) $r \geq \bar{s}$

(c) $\bar{r} \geq \bar{s}$.

Proof. By [11, Corollary 4.4], $H_s \subseteq \bigcup_{t\mathsf{H}s} BtB = BH_sB$, yet by continuity of multiplication $eBrBf \subseteq \overline{BrB}$ since $e, f \in \overline{T}$ and $TBrBT \subseteq BrB$. Hence, $eBrBf \cap H_s \neq \varnothing$ iff there exists $t\mathsf{H}s$ such that $r \geq t$. On the other hand $\bar{s} \in \mathsf{H}_s$ is the unique smallest element in H_s. Thus, (a) and (b) are equivalent. But also $r \geq \bar{r}$. So (c) implies (b). So we may assume $r \geq \bar{s}$. By Corollary 5.3 there exist $e', f' \in \mathsf{R}$, I-equivalent to s, such that $r \geq e'rf'$ and $e'rf' \geq \bar{s}$. Thus,

$$\bar{r} \geq e'\bar{r}f' = \overline{e'rf'}. \tag{1}$$

The equality here results from the fact that $e'\bar{r}f'\mathsf{H}\overline{e'rf'}$ and $e'\bar{r}f', \overline{e'rf'} \in \theta$. But also $e'rf'\,\mathsf{I}\,s$ (and $e'rf' \geq \bar{s}$). So by [11, Proposition 3.11], $e'rf' \geq \bar{s}$ implies that $(e'rf')_{\pm} \geq \bar{s}_{\pm}$. Thus

$$\overline{e'rf'} \geq \bar{s} \tag{2}$$

by [11, Proposition 3.2] since $(\overline{e'r'f'})_0 = (\bar{s}_0)$. Putting things together we obtain

$$r \geq \bar{s} \Rightarrow r \geq e'rf' \geq \bar{s} \Rightarrow \bar{r} \geq \overline{e'rf'} \qquad (\text{by } (1)).$$

But $\overline{e'rf'} \geq \bar{s}$, by (2). ∎

Remarks. (1) From the above proof we see that for $r \in \mathsf{R}$, $s \in \mathsf{O}$, and $r \geq s$ there exists $t\,\mathsf{I}\,s$ such that

$$t \geq s, \qquad \text{and} \qquad t = e'rf', \text{where } e'\mathsf{R} = t\mathsf{R} \text{ and } \mathsf{R}t = \mathsf{R}f'.$$

This should be useful in getting a better picture of the Adherence Order generally.

(2) Assume $eBrBf \cap H_s \neq \varnothing$ where $e\mathsf{R} = s\mathsf{R}$ and $\mathsf{R}s = \mathsf{R}f$. From [11, Corollary 4.4] we know that $H_s \subseteq \bigcup_{t\mathsf{H}s} BtB$. Thus, there exists a

unique $t \in H_s$ such that $eBrBf \cap BtB \subseteq \overline{eBrBf}$ is dense. It would be nice to describe this t somehow. In any case we obtain the following corollary.

5.6. COROLLARY. *If* $r \geq \bar{s}$ *then there exists a maximum* $t\mathcal{H}s$ *such that* $r \geq t$.

5.7. EXAMPLE. We can use Theorem 3.8 to illustrate Corollary 5.6. Recall from Section 3 that the elements of R_n can be thought of as sequences of nonnegative integers $r = (\epsilon_1, \dots, \epsilon_n)$. So let $n = 5$, and consider $r = (24105)$, $s = (31020) \in R_5$. Then $\bar{s} = (12030)$ and one can easily check, using Theorem 3.8, that $r \geq \bar{s}$. Again using 3.8, we see that $r \geq (23010) = t$. Furthermore, t is maximum in the Adherence Order with the properties $t\mathcal{H}s$ and $r \geq t$.

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