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Essays on Coxeter groups

Bruhat closures

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This essay is concerned with the Bruhat-Chevalley ordering in a Coxeter system.

My sources have been §§5.8–5.11 of [Humphreys:1990] and [Dixmier:1974].

Contents

1. Strong exchange

Suppose (W, S) to be a Coxeter system, C the open positive chamber in a realization of (W, S), so that elements of S correspond to reflections in the walls of C. The **Tits cone** $\mathcal{C}$ of the realization is the interior of the union of the W-transforms of $\overline{C}$. If H a half space bounded by a wall of C and containing C, then $\mathcal{C} \cap H$ is called a **simple geometric root** of the system, and the W-transforms of these are the **geometric roots**. A (geometric) root is positive if it contains C. For w in W and $\lambda > 0$, $w\lambda < 0$ if and only if C and $w^{-1}C$ are on opposite sides of the boundary of λ. Let Δ be the set of positive roots corresponding to the walls of C, Σ the set of all roots, and Σ^+ the set of positive roots.

For every root λ, let s_λ be the reflection in the boundary of λ. If $\lambda = w\alpha_s$ with α_s in Δ, then $s_\lambda = ws_\alpha w^{-1}$.

If r is a root reflection, so is wrw^{-1}. If $w = urv$ then uv is what we get by deleting r. But $uv = uru^{-1} \cdot urv = uru^{-1}w$. Hence:

[deletion] **Proposition 1.1.** *If $w = s_1 \ldots s_n$ is an expression for w as product of elements in S, then*

$$u = (s_1 \ldots s_{i-1}) \cdot (s_{i+1} \ldots s_n) = (s_1 \ldots s_{i-1}) \cdot s_i \cdot (s_{i-1} \ldots s_1) \cdot (s_1 \ldots s_n)$$

is of the form rw where r is a reflection in W.

Conversely:

[strong-exchange] **Proposition 1.2.** *Let w be in W, $r = r_\lambda$ a root reflection with $\lambda > 0$. Then $\ell(rw) < \ell(w)$ if and only if $w^{-1}\lambda < 0$, and if $w = s_1 \ldots s_n$ then*

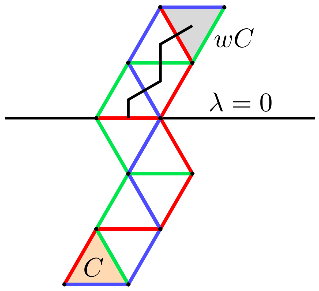
$$rw = s_1 \ldots s_{i-1} \cdot s_{i+1} \ldots s_n$$

for some intermediate s_i. If the expression for w is reduced, then s_i is unique.

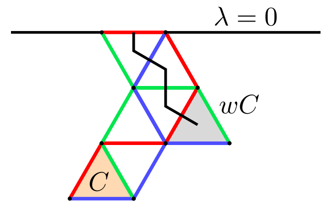
Proof. Suppose the gallery $C, s_1C, \ldots, wC$ crosses the hyperplane $\lambda = 0$ in a wall labeled s_i. Then

$$rw = s_1 \ldots \widehat{s}_i \ldots s_n$$

for the usual geometric reasons, and $\ell(rw) < \ell(w)$.



$w = s_2 s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_1$



$s_\lambda w = s_2 s_1 s_3 s_2 s_3 \widehat{s}_1 s_3 s_2 s_1$

If we start with a reduced expression, the gallery crosses $\lambda = 0$ exactly once, guaranteeing uniqueness. If $w^{-1}\lambda > 0$, then $w^{-1}r^{-1}\lambda < 0$, so we can apply this argument to rwC. ∎

If r is not in S, the reduced expression $s_1 \ldots \widehat{s}_i \ldots s_n$ may collapse further, as it does in the diagrams above.

Set $x \Leftarrow y$ if $\ell(x) < \ell(y)$ and $xr = y$ for some r in R, and define $x \le y$ to mean we can reach y from x by 0 or more such reflections. Since $wr = wrw^{-1} \cdot w$, it doesn't matter whether we use left or right multiplications by reflections in this definition. This order is called the **strong Bruhat order**. I define the **strong Bruhat graph** to be that with elements of W as nodes and oriented edges $x \Leftarrow y$. The **closure** of y is the set of all $x \le y$, and if $x \le y$ the **interval** $[x, y]$ is the set of w with $x \le w \le y$.

Of course $x \le y$ if and only if $x^{-1} \le y^{-1}$.

The following is one of the two main results about the strong Bruhat order:

[subexpressions] **Proposition 1.3.** *If $y = s_1 \ldots s_n$ then $x \le y$ if and only if x can be expressed as a subexpression of this one.*

Proof. if $x = ry < y$ then by strong exchange x can be expressed as a subexpression. This gives one half the Proposition.

On the other hand, suppose we have the reduced expression

$$y = s_1 \ldots s_n = us_i v\,.$$

Then $u\widehat{s}_i v = us_i u^{-1} \cdot us_i v$. ∎

It follows immediately from this that the Bruhat ordering of W_T is the same as it is on W_T as a subset of W_S.

It follows from this result that the set of Coxeter group elements represented by a subexpression of a given reduced expression do not depend on the particular reduced expression. But this can be seen in some sense more directly. The following is a special case of a result of [Tits:1968].

[braids] **Lemma 1.4.** *If*

$$w = s_1 \ldots s_n = t_1 \ldots t_n$$

are two reduced expressions for w then one may be obtained from the other by a sequence of braid relations.

Proof. The proof is by induction on n. The cases $n = 1$ or 2 are trivial. So assume $n > 1$, and that

$$ss_1 \ldots s_n = tt_1 \ldots t_n$$

are reduced. If $s = t$ we can cancel the common left factor and apply induction. Otherwise suppose $s \neq t$. Let

$$x = s_1 \dots s_n, \quad y = t_1 \dots t_n .$$

Then $sw < w$ so $w\alpha_s < 0$, and $tw < w$ so $w\alpha_t < 0$. Thus $w\lambda < 0$ for all positive roots in the span of α_s and α_t, so w may be expressed as

$$w = sw_{s,t}x = tw_{t,s}x$$

where $sw_{s,t} = tw_{t,s} = w_\ell$ is the longest element in the Weyl group generated by s and t, with $\ell(w) = \ell(w_\ell) + \ell(x)$. Since $sw_{s,t}x = ss_1 \dots s_n$, we may cancel s and by induction obtain $w_{s,t}x$ from $s_1 \dots s_n$ by a sequence of braid relations. Similarly for $w_{t,s}x$ and $t_1 \dots t_n$. But then we can also obtain $sw_{s,t}x$ from $tw_{t,s}x$ by a single braid relation, so the Lemma is proved. ∎.

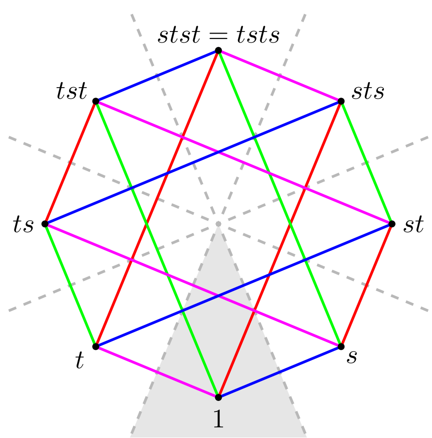
[reduced-independent] **Corollary 1.5.** *The elements represented by subexpressions of a given reduced expression does not depend on the particular reduced expression.*

Proof. By the Proposition, it suffices to prove that it is true for two reduced expressions related by a braid relation. But this is immediate. ∎

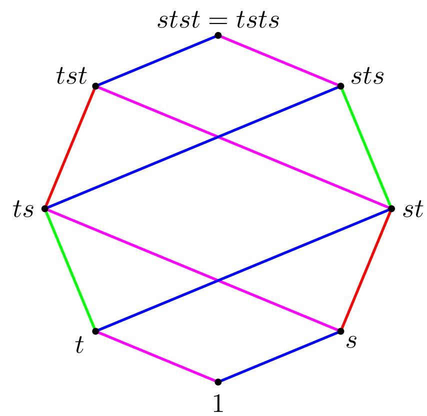
2. Structure of the graph

We'll now look at some examples of the strong Bruhat order.

Example. Let (W, S) be the dihedral group of order 8, with generators s, t. The following figure indicates how root reflections transform elements of W (and shows also the lines of reflection and the chambers):



There are a number of things to notice about this graph. First of all, there is some redundancy here. For example, the reflection sts takes t to $stst$, so $t \leq stst$. But this can be seen also by the chain t-ts-sts-$stst$. With the redundant links removed, the graph of the order looks like this:



All dihedral groups exhibit the same behaviour—for these groups, $x \le y$ if and only if $\ell(x) \le \ell(y)$.

Second of all, multiplication by s is an involution of the group. *How does this involution relate to the closure graph?* Very nicely. All possibilities are shown in this figure. It takes edges to edges, and in in a very simple way, which the next Proposition will explain.

Example. Now let W be the symmetric group $\mathfrak{S}_n$, S the subset of elementary transpositions interchanging i and $i+1$. A permutation is expressed by the array $\big(\sigma(i)\big)$. The reflections are the swaps of two coordinates. The definition says that $x \prec y$ if y is obtained from x by swapping x_j and x_k in the array (x_i), where $j < k$ and $x_j < x_k$. For example, $[2, 4, \boldsymbol{1}, \boldsymbol{5}, 3] \prec [2, 4, \boldsymbol{5}, \boldsymbol{1}, 3]$. [Humphreys:1990] (on p. 119) attributes to Deodhar a simple criterion. First some notation: if $(x_1, \ldots, x_m)$ is any array, let $\langle x_1, \ldots, x_m\rangle$ be the same array sorted from smallest to largest. If $x \le y$ if and only if

$$\langle x_1, \ldots, x_k\rangle \preceq \langle y_1, \ldots, y_k\rangle$$

for each k, in the sense that after sorting corresponding entries are less than or equal. This is clearly a necessary condition, and probably not too hard to construct for such x and y a chain of reflections.

[xsys] **Proposition 2.1.** *Suppose s in S, $x \Leftarrow y$. Then exactly one of the following occurs:*

(a) $sx = y$, so that s reverses the edge in the strong Bruhat graph between them;
(b) s maps the edge $x \Leftarrow y$ to the edge $sx \Leftarrow sy$.

In other words, applying s to the edge doesn't reverse the orientation of the edge, unless it just exchanges its endpoints.

Proof. Suppose $x \Leftarrow y$. The case $y = sx$ is trivial, so suppose $r \ne s$. Let $r = r_\lambda$ with $\lambda > 0$.

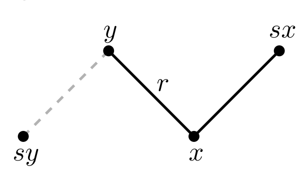
♣ [strong-exchange] Since $r_\lambda y < y$, Proposition 1.2 implies that $y^{-1}\lambda < 0$. But then

$$sx = sr_\lambda y = ss_\lambda s \cdot sy = s_{s\lambda} sy\,.$$

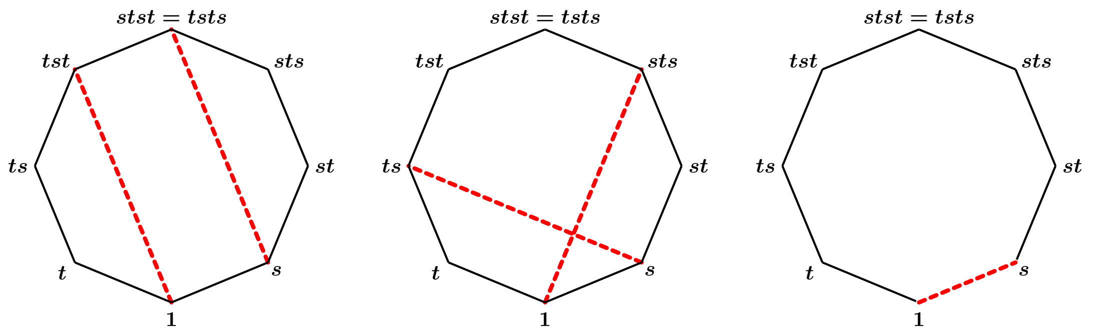
Since $r \ne s$, $s\lambda > 0$, so that $sx < sy$ if and only if $(sy)^{-1}s\lambda < 0$. But

$$(sy)^{-1}s\lambda = y^{-1}\lambda < 0\,.$$ ∎

Basically, what is forbidden is this configuration:



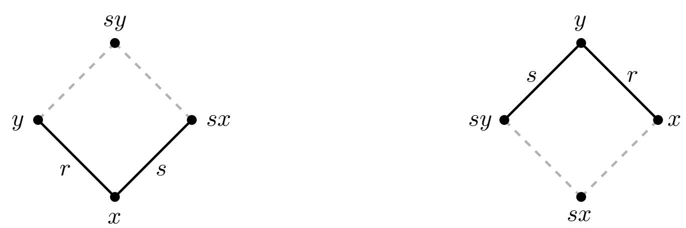
There are thus three kinds of edge-swaps: (a) an edge reverses itself; (b) $sx < y$ and $sx < sy$; or (c) $x < y$, $sx < x$, $sy > y$:



[dixmier] **Corollary 2.2.** *Suppose $x \Leftarrow y$ with $\ell(y) = \ell(x) + 1$. Then*

- *(a) if $sx > x$ then either $y = sx$ or $sx \Leftarrow sy$;*
- *(b) if $sy < x$ then either $y = sx$ or $sx \Leftarrow sy$.*

In diagrams:



Proof. This is just a restatement of what's forbidden. ▯

[rank2-diff] **Corollary 2.3.** *Suppose $x < y$, with $\ell(y) - \ell(x) = 2$, $sy < y$. Either $sx > x$ and $[x, y] = \{x, sx, sy, y\}$ or $sx < x$ and the interval $[x, y]$ is isomorphic to $[sx, sy]$.*

Proof. Since $x > y$, parity considerations require that the interval between x and y be filled with edges of length 1. If $[x, y] \neq \{x, sx, sy, y\}$ then there exists $x < z < y$ with $z \neq sx$, $z \neq sy$. In this case the Proposition implies that $sx < sz < sy$, and since $sy < y$ we must have $sx < x$. In particular $sx \notin [x, y]$.

Now there is a further dichotomy: either $sy \in [x, y]$ or not. In the second case, s is an isomorphism of $[x, y]$ with $[sx, sy]$. In the first case, the map $z \mapsto sz$, $sy \mapsto x$ is an isomorphism of $[x, y]$ with $[sx, sy]$. ▯

[s-stability] **Corollary 2.4.** *Suppose $sx < x$. Then $y \leq x$ if and only if $sy \leq x$.*

Proof. The proof is by induction on $\ell(x) - \ell(y)$. If it is 0, there is nothing to prove. Otherwise, we can find a chain

$$x_n = y \Leftarrow x_{n-1} \Leftarrow \ldots \Leftarrow x_0 = x$$

The case $n = 1$ is that of the Proposition. If $n > 1$, we have Say $y \Leftarrow x_{n-1}$ with $x_{n-1} < x$. Induction tells us $sx_{n-1} \leq x$. The Proposition says either $sy < sx_{n-1}$ or $x_{n-1} = sy$. Either way, $sy \leq x$. ▯

3. Minimal links

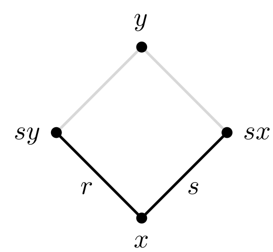
We have seen in the case of dihedral groups that the Bruhat order is generated by pairs $x = ry$ with $\ell(x) = \ell(y) - 1$. This is a general fact, and the second of the two most important results.

There is one very simple case:

[interval] **Proposition 3.1.** *Suppose $x < y$ and $\ell(x) = \ell(x) - 2$. Then there exist exactly two w with $x < w < y$.*

That is to say, the Bruhat interval $[x, y]$ in this case is very simple.

Proof. By induction on $\ell(y)$. The minimum this can be is 2, in which case $x = 1$, $y = st$, and $[x, y] = \{1, s, t, st\}$.



♣ [rank2-diff] Otherwise, choose $sy < y$. If $sx < x$, then Corollary 2.3 tells us that $[x, y]$ is isomorphic to $[sx, sy]$, and we apply induction. If $sx > x$ the same result tells us $[x, y] = \{x, sx, sy, y\}$.

Define $x \prec y$ to mean $x = ry < y$ and $\ell(y) - \ell(x) = 1$.

[dist1] **Proposition 3.2.** *If $x < y$, then there exists a chain $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$.*

This allows a very simple algorithmic description of closures. In the proof, I follow [Dixmier:1974], pp. 250–252.

Proof. We may assume that $x = ry < y$. We proceed by induction on $\ell(y) + \big(\ell(y) - \ell(x)\big)$.. If $\ell(x) = \ell(y) - 1$, there is nothing to be proven. So we may assume $\ell(y) \geq \ell(x) + 3$.

Choose s with $sy < y$. Then $sx = sry = srs \cdot sy$ and

$$\ell(sx) < \ell(x) + 1 \leq \ell(y) - 2 < \ell(y) - 1 = \ell(sy)\,.$$

So $sx < sy$. We may apply induction to get a chain from sx to sy:

$$sx = w_0 < w_1 < w_2 < \ldots < w_n = sy < w_{n+1} = y$$

with (say) $w_{i+1} = r_i w_i$. In particular, $r_n = s$.

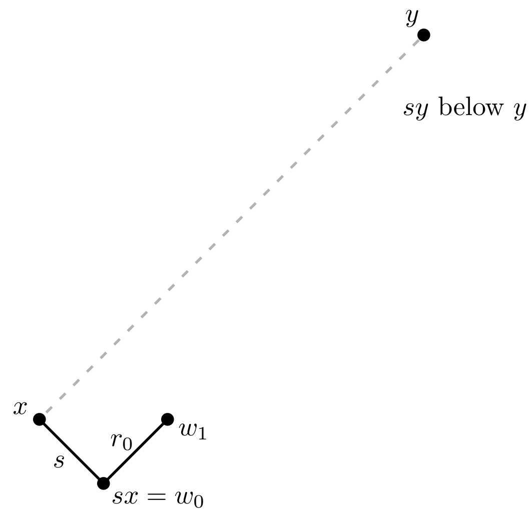
• If $x < sx$, we can just extend the chain to include x:

$$x < sx = w_0 < w_1 < w_2 < \ldots < w_n = sy < w_{n+1} = y$$

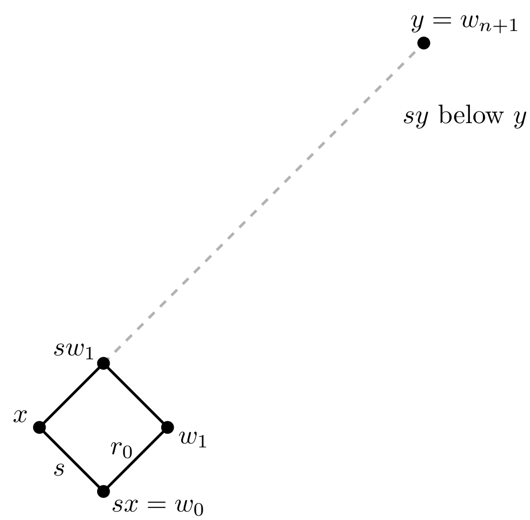
• If $x > sx$ and $w_1 = x$, the chain we want is

$$x = w_1 < w_2 < \ldots < w_n = sy < w_{n+1} = y\,.$$

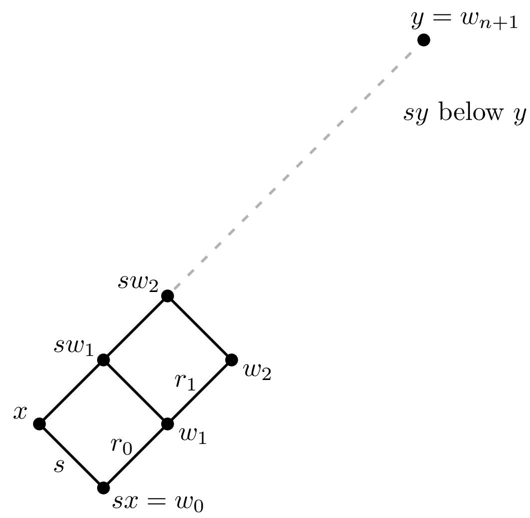
• Otherwise, $sx < x$ and $w_1 \neq x$. The situation is indicated by this diagram:



Let $t_0 = sr_0s$. Since $s \neq r_0$, we know that $sw_1 > w_1$ and that $t_0x = sw_1$, so we may fill in the diagram.



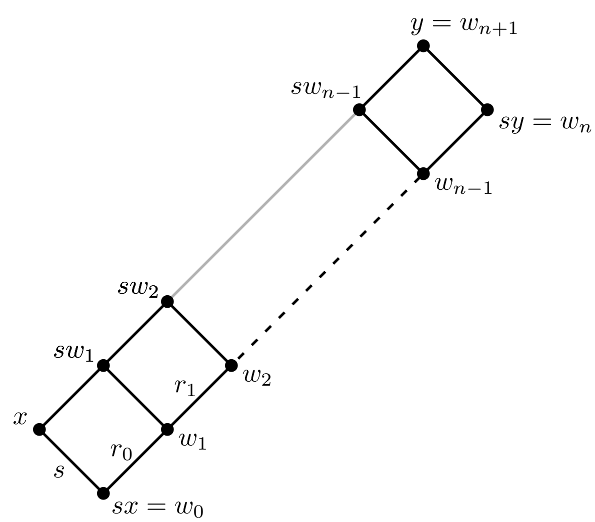
The diagram is deceptive, though, because we do not know (yet) that $sw_1 < y$. Even so, we may keep on filling in as long as $r_i \neq s$:



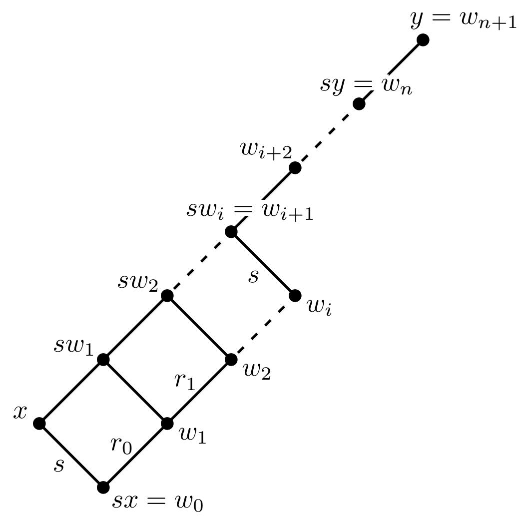
We have $r_n = s$; let i be least with $r_i = s$. So then we get a chain

$$x < sw_1 < sw_2 < \ldots < sw_i = w_{i+1} < w_{i+2} < \ldots sy < y$$

If $i = n$, the picture is this:



♣ **[dixmier]** In this case, $sw_{n-1} < y$ by Corollary 2.2. But then $x < sw_1 < sw_2 < \ldots < sw_{n-1} < y$ is the chain we want. Otherwise $i < n$, and the picture is this:



In this case, the chain is indicated in the diagram. ▯

[cl-construction] **Corollary 3.3.** *Suppose $y = sx > x$. Then the $z < y$ with $z \prec y$ are (a) x together with (b) all the sw where $w \prec x$ and $sw > w$.*

[cl-ysxs] **Corollary 3.4.** *Suppose $xs < x$, $ys < y$, $y < x$. Then $ys < xs$.*

♣ [dixmier] *Proof.* By induction. If $\ell(x) = \ell(y) + 1$ this is Corollary 2.2. Otherwise, according to the Proposition we ♣ [dixmier] may find $y < z \prec x$. Again by Corollary 2.2 we have $zs < z$, and we may apply induction. ▯

4. References

1. Jacques Dixmier, **Algèbres enveloppantes**, Gauthier-Villars, Paris, 1974.

2. James E. Humphreys, **Reflection groups and Coxeter groups**, Cambridge University Press, 1990.