BASIC SEMIGROUP THEORY

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We assume that the reader is more familiar with notions from the theory of groups and associative algebras than those of semigroups, so we give a brief introduction to the major tools of semigroup theory in this subsection and their relationship to representation theory in the next. For more details, see [8,20,35], where all the assertions we make in this subsection are proved.

A semigroup S is a set with an associative binary operation and a monoid M is a semigroup that has an identity element 1. If M is a monoid, then the set $U(M) = \{m \in M |$ there is $n \in M$ such that $mn = nm = 1\}$ is a group called the group of units of M. There is an evident notion of morphism between semigroups. A morphism between monoids is a semigroup morphism that also preserves the identity element. We emphasize that a semigroup morphism between monoids need not preserve the identity element.

Unlike groups and rings, semigroup (monoid) morphisms are not determined by the inverse image of a single element. This is replaced by the notion of a congruence on a semigroup. If S is a semigroup (monoid), then a congruence is an equivalence relation θ on S that is also a subsemigroup (submonoid) of $S \times S$. This condition implies easily that the product of any two equivalence classes of θ is contained in a unique equivalence class. Thus the equivalence classes form the quotient semigroup (monoid) S/θ. With this notion of quotient, the usual homomorphism theorems hold. In particular, if $f : S \twoheadrightarrow T$ is a surjective morphism of semigroups (monoids), then T is isomorphic to $S/ker(f)$, where $ker(f)$ is the congruence $ker(f) = \{(s_1, s_2) | f(s_1) = f(s_2)\}$.

Right, left and two sided ideals are defined for semigroups analogously to the definition for rings. If S is a semigroup and $s \in S$, then the principal right (left, two-sided) ideal generated by s is $R(s) = \{s\} \cup sS$,($L(s) = \{s\} \cup Ss$, $J(s) = \{s\} \cup R(s) \cup L(s) \cup SsS$). If S is a monoid, then $R(s) = sS, L(s) = Ss, J(s) = SsS$.

An element s of a semigroup S is called regular, if there is $t \in S$ such that $sts = s$. If furthermore, $tst = t$, then t is called a semigroup inverse of s. It is easy to prove that every regular element has a semigroup inverse. A semigroup S is called regular, if every element of S is a regular element. An element e of a semigroup is an idempotent if $e^2 = e$. It is a useful fact that if S is a finite semigroup then the subsemigroup generated by s, which is the set of all positive powers of s, has a unique idempotent. Notice that if $s \in S$ is a regular element with inverse t, then both st and ts are idempotents and it follows that $R(s) = R(st), L(s) = L(ts)$ and $J(s) = J(st) = J(ts)$ are generated by idempotents. In particular, if S is a regular semigroup, then every principal right, left and two-sided ideal is generated by an idempotent.

Green's preorders and equivalence relations have played an essential part of semigroup theory since their introduction in [15]. Let S be a semigroup and $s, t \in S$. Green's preorders are defined as follows.

- $s \leq_{\mathcal{J}} t$ if and only if $J(s) \subseteq J(t)$
- $s \leq_{\mathcal{R}} t$ if and only if $R(s) \subseteq R(t)$
- $s \leq_{\mathcal{L}} t$ if and only if $L(s) \subseteq L(t)$

Clearly, $\leq_{\mathcal{J}}, \leq_{\mathcal{R}}, \leq_{\mathcal{L}}$ are reflexive and transitive relations. The symmetric closures of these relations are equivalence relations given by $s \mathrel{\mathcal{J}} t$ if and only if $J(s) = J(t)$, $s \mathrel{\mathcal{R}} t$ if and only if $R(s) = R(t)$ and $s \mathrel{\mathcal{L}} t$ if and only if $L(s) = L(t)$. One also defines the relation $\mathcal{H}$ as the intersection of $\mathcal{R}$ and $\mathcal{L}$. It is known that the equivalence relations $\mathcal{L}$ and $\mathcal{R}$ commute, that is $\mathcal{L} \circ \mathcal{R}{=}\mathcal{R} \circ \mathcal{L}$ where $\circ$ is composition of binary relations. Thus $\mathcal{D}{=}\mathcal{L} \circ \mathcal{R}{=}\mathcal{R} \circ \mathcal{L}$ is also an equivalence relation that is the join of $\mathcal{L}$ and $\mathcal{R}$ in the lattice of equivalence relations of S. In particular, $\mathcal{D}{\subseteq}\mathcal{J}$. It is known that for finite semigroups $\mathcal{D}{=}\mathcal{J}$.

For example, if $M_n(K)$ is the monoid of $n \times n$ matrices over a field K, and $s, t \in M_n(K)$, then one has $s \leq_{\mathcal{J}} t$ if and only if $rank(s) \leq rank(t)$, $s \leq_{\mathcal{L}} t$ if and only if $\text{rowspace}(s) \subseteq \text{rowspace}(t)$ and $s \leq_{\mathcal{R}} t$ if and only if $\ker(t) \subseteq \ker(s)$. Thus the Green relations can be thought of as generalizing these standard notions from linear algebra to more general semigroups.

The set of idempotents of a semigroup S is denoted by $E(S)$. If $e \in E(S)$, we write G_e for the group of units of the monoid eSe. It is known that G_e is the maximal subgroup of S whose identity element is e and also that G_e is the $\mathcal{H}$-class of e. In a finite semigroup S two idempotents $e, f \in E(S)$ are $\mathcal{J}$ equivalent if and only if they are conjugate. This means that there are elements $x, y \in S$ such that $e = xy, f = yx$. From this one can prove that if $e \mathrel{\mathcal{J}} f$ then eSe is isomorphic to fSf and G_e is isomorphic to G_f. A $\mathcal{J}$-class J of a finite semigroup S is called regular if it contains a regular element. It is known that in this case, every element of J is regular and that J is regular if and only if it contains an idempotent. Thus to each regular J class of a finite semigroup one associates a unique abstract group $G = G_e$, where $e \in J$ is an idempotent.

0.1. **Representation theory of semigroups and quasi-hereditary algebras.**

The theory of quasi-hereditary finite dimensional algebras [9] was developed more than 30 years after the representation theory of finite semigroups was first developed [8], Chapter 5. However, the theorem that states that the algebra of a finite regular semigroup in good characteristic is quasi-hereditary came quite late in the game [31]. This despite that the basic ideas, techniques and results for the semigroup algebras of finite semigroups and in particular, regular ones, presages the much later development of quasi-hereditary algebras.

Thus, for instance, the Munn-Ponizovskii description of the simple modules of the semigroup algebra of a finite semigroup [8], chapter 5, described in the 1950's are exactly via the construction of the standard modules and taking the minimal irreducible constituent in the partial order; the co-standard modules appear in the work of Rhodes and Zalcstein [36] from the 1960's and Nico [23, 24] early on showed that the algebra of a regular finite semigroup in good characteristic has finite global dimension and computed the exact bound that one would obtain from the theory of quasi-hereditary algebras. We remark that the algebra of a finite regular semigroup is stratified in the sense of [10] in arbitrary characteristic, but our proposal is entirely about the case of good characteristic, that is, when the characteristic of the field does not divide the order of any subgroup of a finite semigroup. We always assume that we are in good characteristic in this proposal.

Thus semigroup algebras of finite regular semigroups form a wide, varied and natural class of quasi-hereditary algebras and the theory of quasi-hereditary algebras provides the language, results and tools to understand these algebras. The connection between algebras of finite regular semigroups and quasi-hereditary algebras, which until the work of Putcha [31] and the recent work of Margolis and Steinberg [22] has gone unnoticed by both semigroup theorists and algebraists, forms one of the central themes of this proposal. A number of natural classes of regular semigroups and their algebras and how they appear in a number of diverse ways in a number of different fields of mathematics will be detailed in the next section.

Let S be a semigroup and K a field. The semigroup or representation algebra, KS of S over K is the K-algebra whose underlying vector space has basis the elements of S, and whose multiplication is induced by that of S. A representation of S over K is a morphism from S to the monoid $M_n(K)$ of all $n \times n$ matrices over K. Just as in the representation theory of groups, representations of S are in 1-1 correspondence with those of KS.

We briefly review the definition of quasi-hereditary algebras. We refer to [9, 11] for more details, properties and proofs concerning quasi-hereditary algebras. There are a number of equivalent definitions of this concept, but the following one is particularly useful for application to the algebras of finite regular semigroups.

A finite dimensional algebra A is quasi-hereditary if A has a chain of ideals $\{0\} = I_0 \subset I_1 \subset I_2 \ldots \subset I_t = A$ such that for each $1 \leq k \leq t$.

(1) I_k/I_{k-1} is a projective A/I_k module.
(2) $I_k^2 = I_k$
(3) $I_k \operatorname{Rad}(A) I_k \subset I_{k-1}$ where $\operatorname{Rad}(A)$ is the radical of A.

As mentioned previously, Putcha [31] proved that the semigroup algebra of a finite regular semigroup is quasi-hereditary. The proof is essentially this. If S is a finite regular semigroup and K is a field, then every ideal I of S gives an ideal KI of KS. Then a principal ideal series of S [8] lifts in this way to an ideal chain that defines the quasi-hereditary structure of KS.

The importance of quasi-hereditary algebras in representation theory is that an algebra A is quasi-hereditary if and only if its module category is a highest weight category [9]. If one uses a principal ideal series, as above, to define the quasi-hereditary structure of KS for a finite regular semigroup S, then the poset required in the definition of a highest weight category is the $\mathcal{J}$-class order, $S/\mathcal{J}$. Thus the natural parameters of semigroup theory give the natural parameters of the theory of quasi-hereditary algebras.

We now give a modern description of the simple modules of a finite semigroup algebra. The description of the simple modules for a finite semigroup are well known, see for instance [8, 14]. We follow here the presentation and ideas of [14], which is the shortest and easiest accounting.

Let $J_1, \ldots, J_n$ be the collection of $\mathcal{J}$-classes of S. Assume that we have ordered them so that $J_i \leq_{\mathcal{J}} J_\ell$ implies $i \leq \ell$. Choose idempotents $e_1, \ldots, e_n$ with $e_i \in J_i$ and let G_i be the maximal subgroup at e_i. Define

$$\begin{aligned} J_i{}^{\downarrow} &= \{s \in S \mid s <_{\mathcal{J}} e_i\} \\ J_i{}^{\not\uparrow} &= \{s \in S \mid s \not\geq_{\mathcal{J}} e_i\}. \end{aligned}$$

Both $J_i{}^{\downarrow}$ and $J_i{}^{\not\uparrow}$ are ideals of S. Notice that $J_i{}^{\downarrow} \subseteq J_i{}^{\not\uparrow}$ and $e_i J_i{}^{\downarrow} = e_i J_i{}^{\not\uparrow}$ (and dually).

First note that, $e_i(kS/kJ_i{}^{\downarrow})e_i \cong kG_i$. For each $i = 1, \ldots, n$, define functors

$$\mathrm{Ind}_i, \mathrm{Coind}_i \colon \text{mod-}kG_i \to \text{mod-}kS/kJ_i{}^{\not\uparrow} \subseteq \text{mod-}kS/kJ_i{}^{\downarrow}$$

by

$$\begin{aligned} \mathrm{Ind}_i(V) &= V \otimes_{kG_i} e_i kS/kJ_i{}^{\not\uparrow} = V \otimes_{kG_i} e_i kS/kJ_i{}^{\downarrow} \\ \mathrm{Coind}_i(V) &= \mathrm{Hom}_{kG_i}(kS/kJ_i{}^{\not\uparrow})e_i), V) = \mathrm{Hom}_{kG_i}(kS/kJ_i{}^{\downarrow})e_i, V). \end{aligned}$$

These functors are exact and are the respective left and right adjoints of the restriction functor $M \mapsto Me_i$ from mod-$kS/kJ_i{}^{\downarrow} \to$ mod-kG_i (in fact $e_i(kS/kJ_i{}^{\downarrow})$ and $(kS/kJ_i{}^{\downarrow})e_i$ are free kG_i-modules since G_i acts freely on $e_iS \cap J$ and dually). Also $\mathrm{Ind}_i(V)e_i \cong V \cong \mathrm{Coind}_i(V)e_i$. The functor Ind_i preserves projectivity and the functor Coind_i preserves injectivity as functors to mod-$kS/kJ_i{}^{\downarrow}$ (but not in general to mod-kS). Both functors preserve indecomposability.

If V is a simple kG_i-module, then it is known that $\mathrm{Ind}_i(V)$ has a unique maximal submodule $\mathrm{rad}(\mathrm{Ind}_i(V))$, which is in fact the largest submodule annihilated by e_i (or equivalently is the submodule of elements annihilated by J_i). The quotient $\widetilde{V} = \mathrm{Ind}_i(V)/\mathrm{rad}(\mathrm{Ind}_i(V))$ is then a simple kS-module and can be characterized as the unique simple kS-module M such that:

(1) e_i is $\leq_{\mathcal{J}}$-minimal with $Me_i \neq 0$;
(2) $Me_i \cong V$ as kG_i-modules.

Also one can show that $\widetilde{V}$ is the socle of $\mathrm{Coind}_i(V)$ and can be described as $\mathrm{Coind}_i(V)e_ikS$. One calls J_i the *apex* of the simple kS-module $\widetilde{V}$. It is known that every simple module for kS has an apex, i.e., is of the form $\widetilde{V}$

for a unique i and a unique simple kG_i-module V. See [14]. It is convenient to put a partial order on the simple kS-modules by setting $V \leq U$ if $V = U$ or the apex of V is strictly $\mathcal{J}$-below the apex of U. We call this the *canonical quasi-hereditary structure* on kS. One can show that kS is indeed quasi-hereditary [9, 11, 31] with respect to this partial ordering and that the modules of the form $\mathrm{Ind}_i(V)$ are the standard modules, whereas the modules $\mathrm{Coind}_i(V)$ are the co-standard modules [31].

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