Introduction to Coding Theory 89-662

Final Exam, Moed Bet 2008

Exam instructions:

1. Closed book: no material is allowed
2. Answer all questions
3. Time: 2.5 hours
4. **Good luck!**

Question 1 (20 points): Prove the Gilbert-Varshamov lower bound: Let n, k and d be natural numbers such that $2 \leq d \leq n$ and $1 \leq k \leq n$. If $V_q^{n-1}(d-2) < q^{n-k}$ then there exists a linear code $[n, k]$ over F_q with distance at least d.

Question 2 (25 points): The *heaviest codeword problem* is defined as follows: Upon receiving a parity check matrix H that fully defines a *binary* linear code C, find the codeword $c \in C$ with the maximum weight (i.e., find c such that $wt(c) \geq wt(c')$ for all $c' \in C$). Give an efficient (polynomial-time) algorithm for this problem or show that it is NP-complete.

Question 3 (25 points):

1. Show that there exists no binary linear code with parameters $[2^m, 2^m - m, 3]$ for any $m \geq 2$.
2. Let C be a binary linear code with parameters $[2^m, k, 4]$ for some $m \geq 2$. Show that $k \leq 2^m - m - 1$.
3. Let δ and R be such that $R = 1 - H(\delta)$. Is it possible to construct a code with rate $R = \frac{k}{n}$ that can correct more than δn errors?

You can use any of the bounds that we learned in class (but you must state exactly what you are using and what it states).

Question 4 (30 points): A burst error of length t has the property that all errors are within distance t from each other. More formally, a vector $e \in F_2^n$ is a burst error of length t if there exist $i < j$ such that $e_1 = \cdots = e_{i-1} = 0$, $e_{j+1} = \cdots = e_n = 0$ and $j - i < t$.

Let C be a linear code $[n, k]$ over F_q such that there exists a decoder for C that corrects every burst of length t or less.

1. Show that in every nonzero codeword $c \in C$, the locations i and j of the first and last nonzero entries in c must satisfy $j - i \geq 2t$ (i.e., they must be at least $2t$ far apart).
2. Show that all of the burst errors of length t of a codeword c lie in distinct cosets of C.

3. Show that $n - k \geq 2t$.

   Hint: recall that if there are d linear dependent columns of the parity check matrix, then there exists a codeword of weight d. Combine this with item (1) of this question to conclude that no consecutive $2t$ columns can be linearly dependent. Now consider what it would mean if $n - k < 2t$.