Faster Deterministic Fully-Dynamic Graph Connectivity

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  - `connected`(u, v): reports whether vertices u and v are connected in G
- We refer to `insert` and `delete` as update operations and to `connected` as a query operation

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- Kapron, King, Mountjoy, 2013: $O(\mathrm{polylog}\, n)$ update and query (Monte Carlo)

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  - New result: $O(\log^2 n / \log \log n)$ update, $O(\log n / \log \log n)$ query

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  - Boolean 'and', 'or', and 'xor'

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- If (u', v') exists, add it as a tree edge; otherwise do nothing

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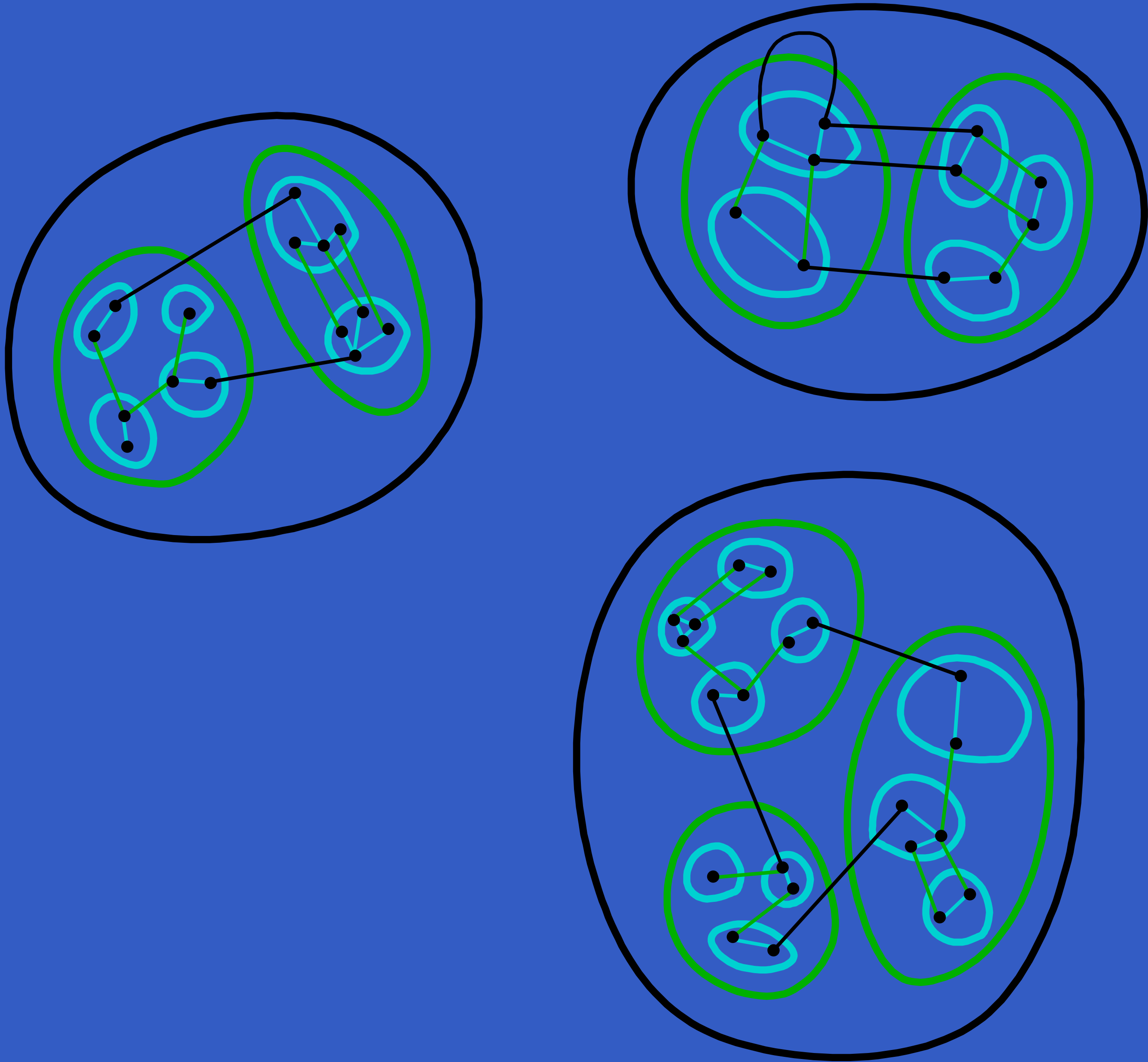
- Level 0-clusters are the connected components of G and level $\ell_{\max}$-clusters are vertices of G

Clusters



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- Roots of $\mathcal{C}$ correspond to connected components of G and leaves of $\mathcal{C}$ correspond to vertices of G
- Given $\mathcal{C}$, we can determine whether two vertices u and v are connected in G in $O(\log n)$ time

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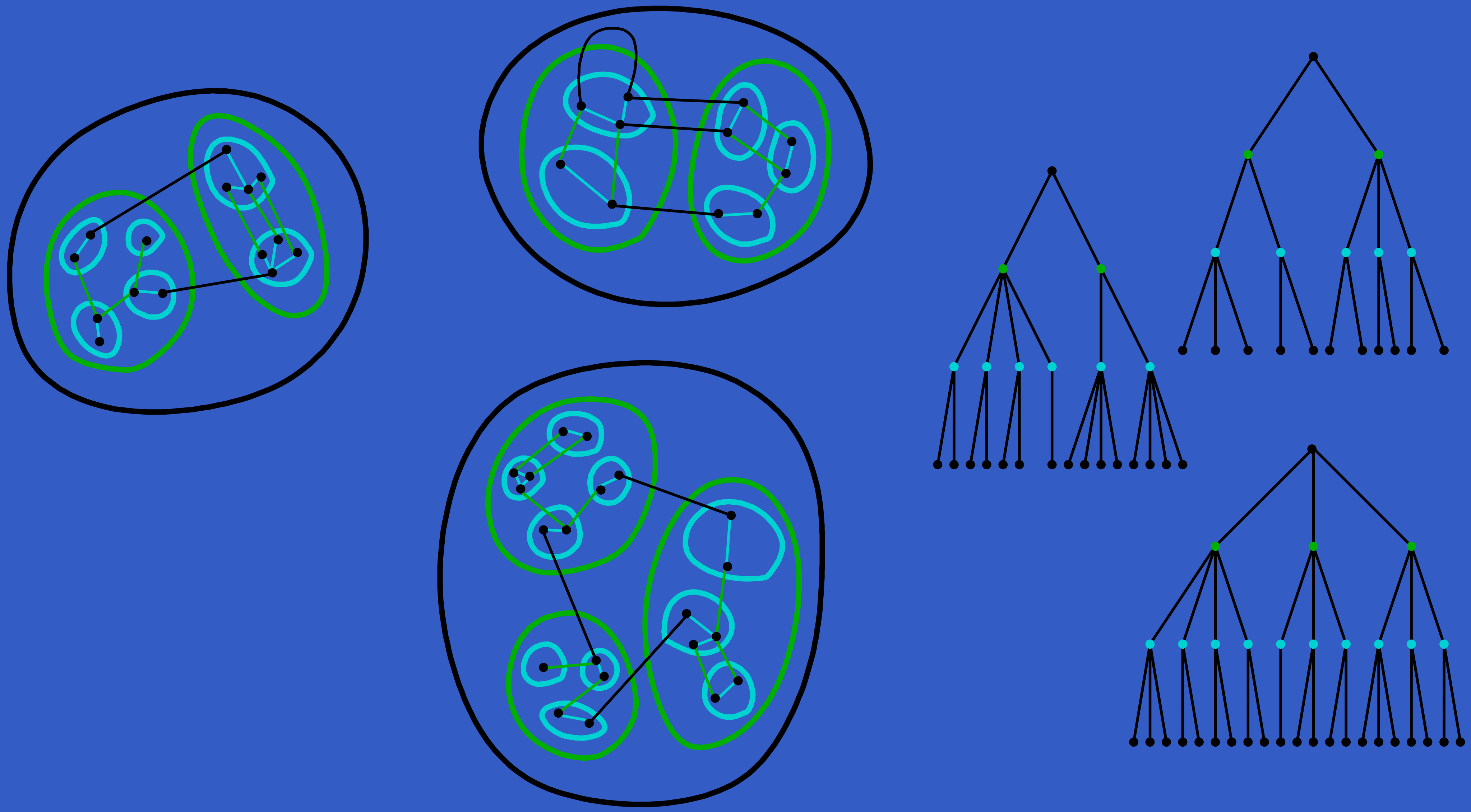

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- This corresponds to merging $C(r_u)$ and $C(r_v)$

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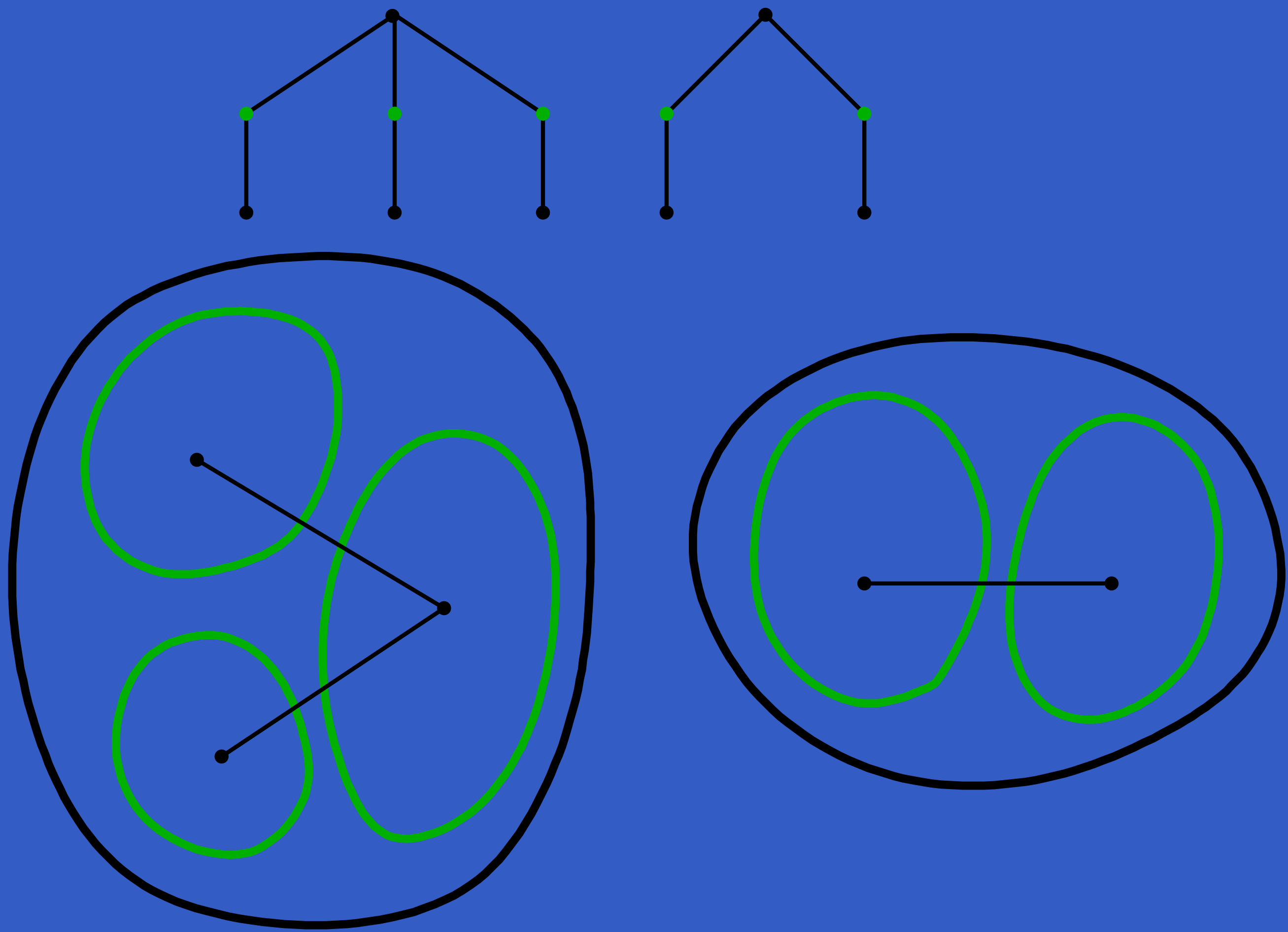

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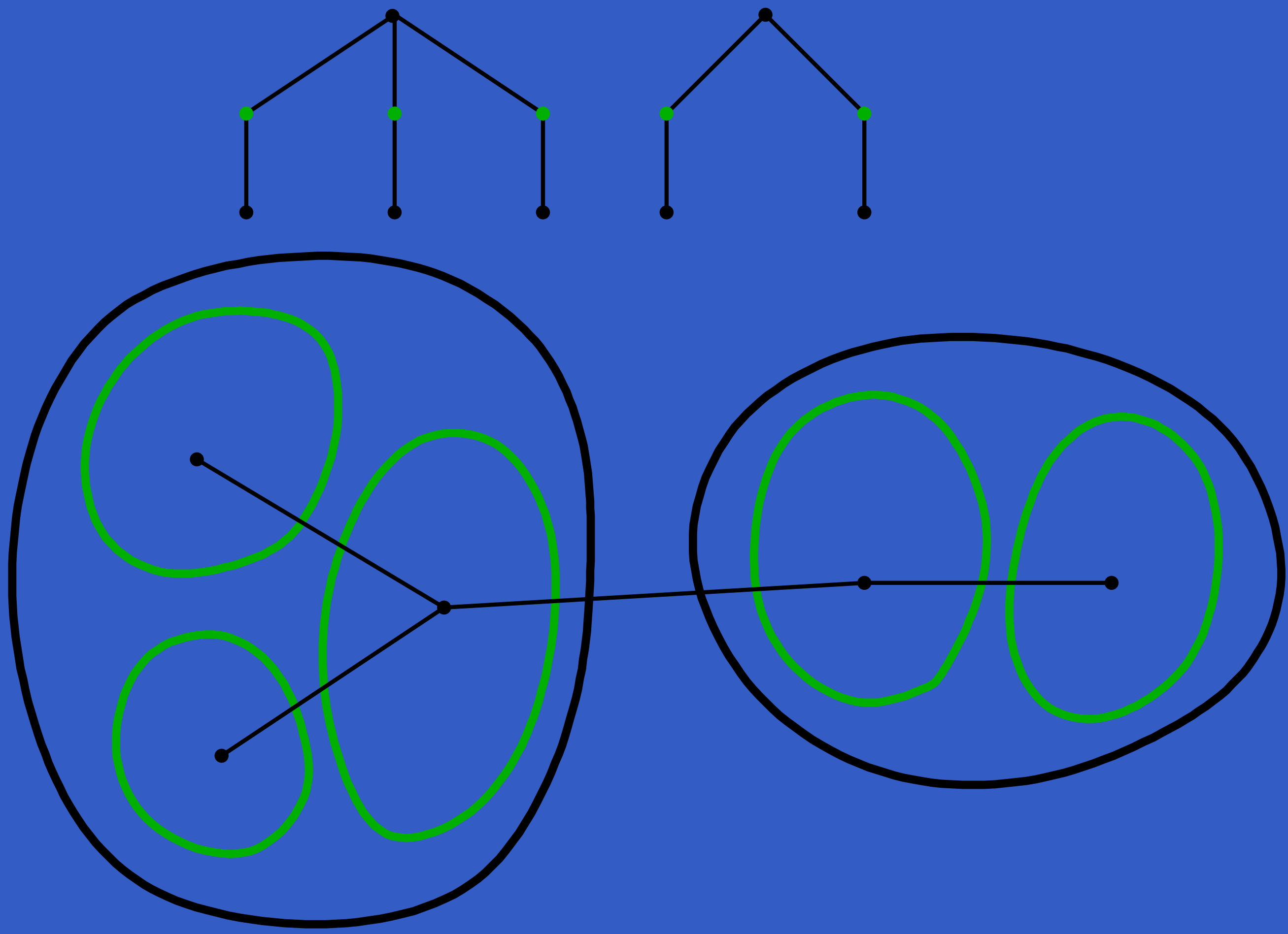

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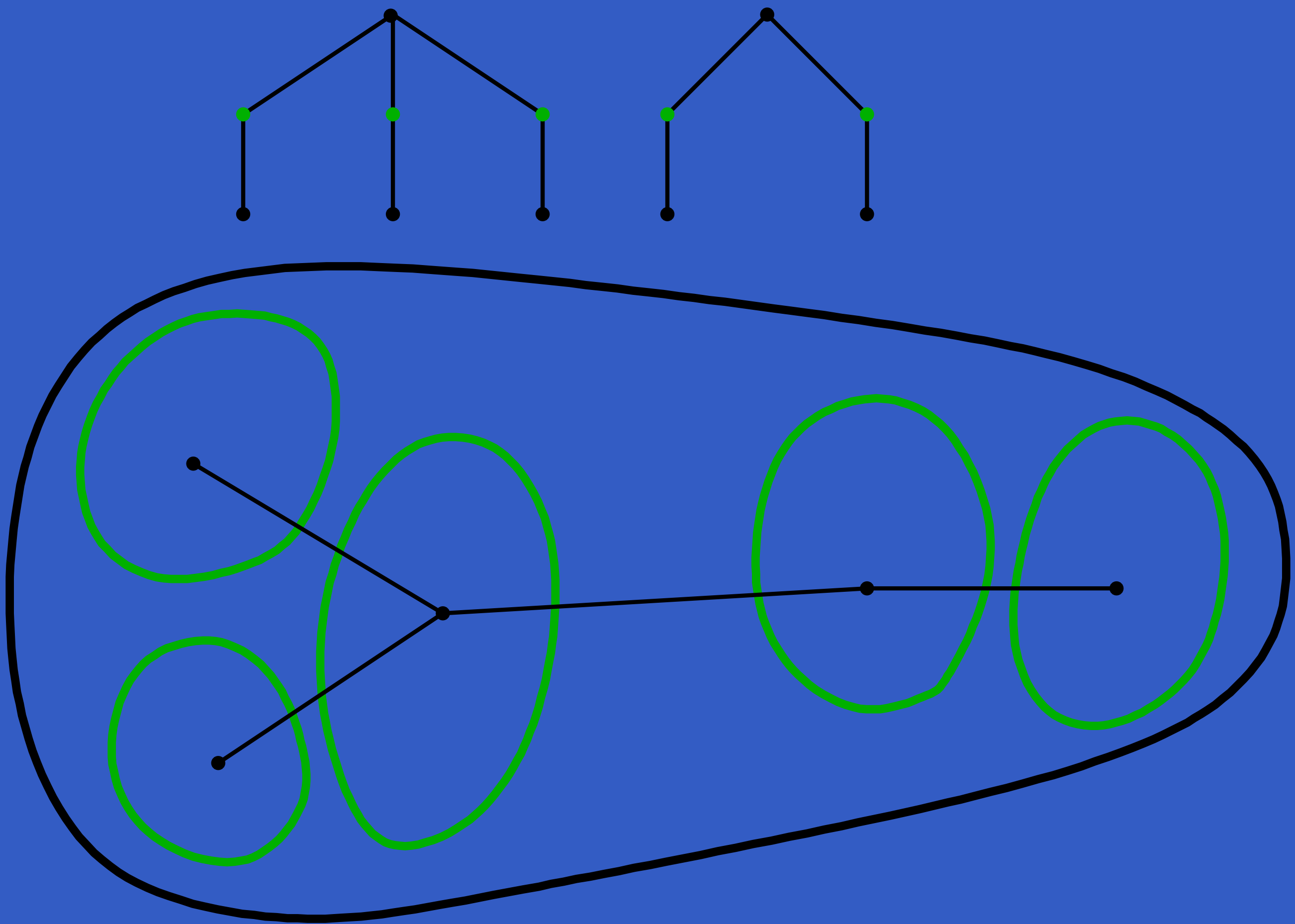

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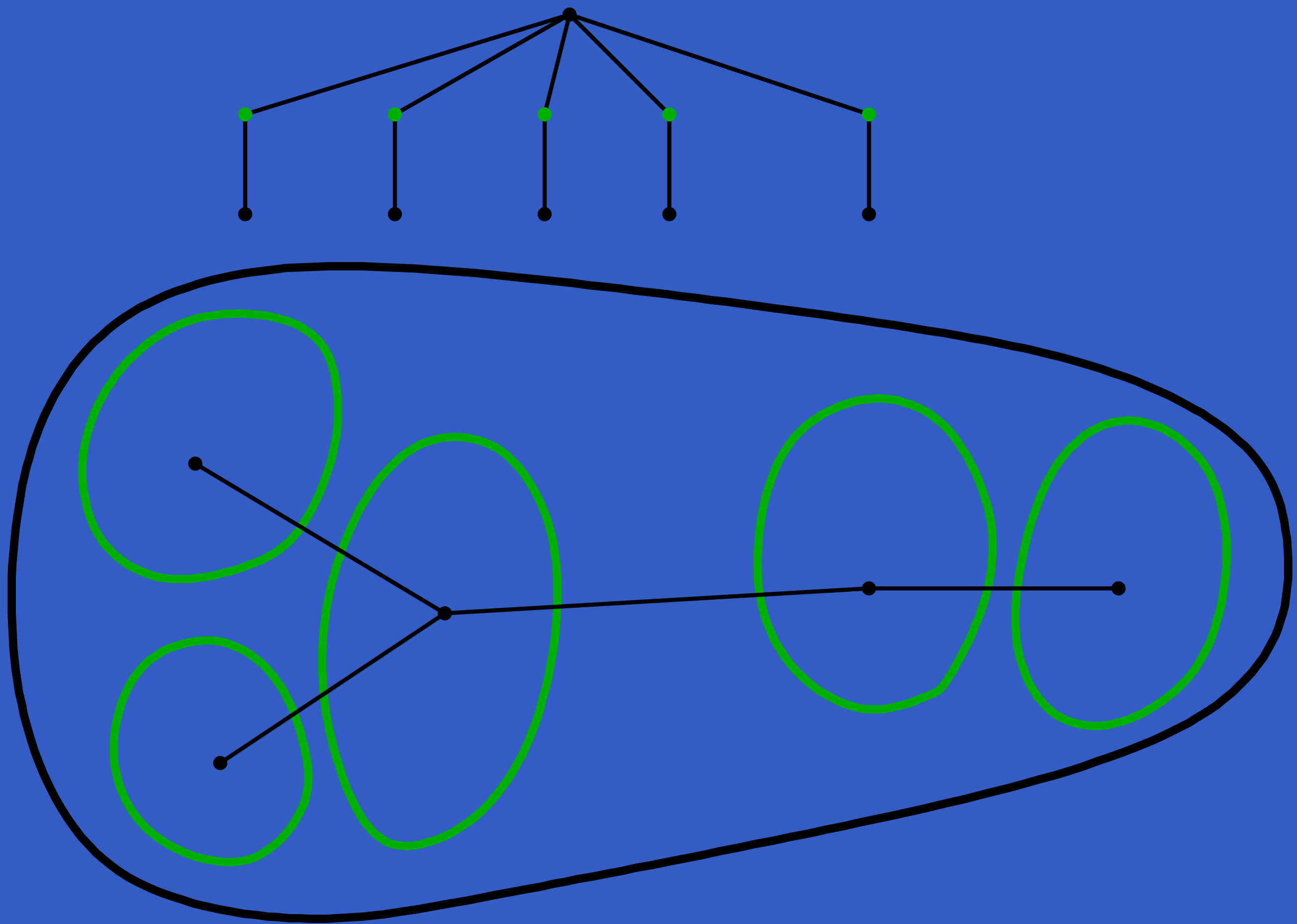

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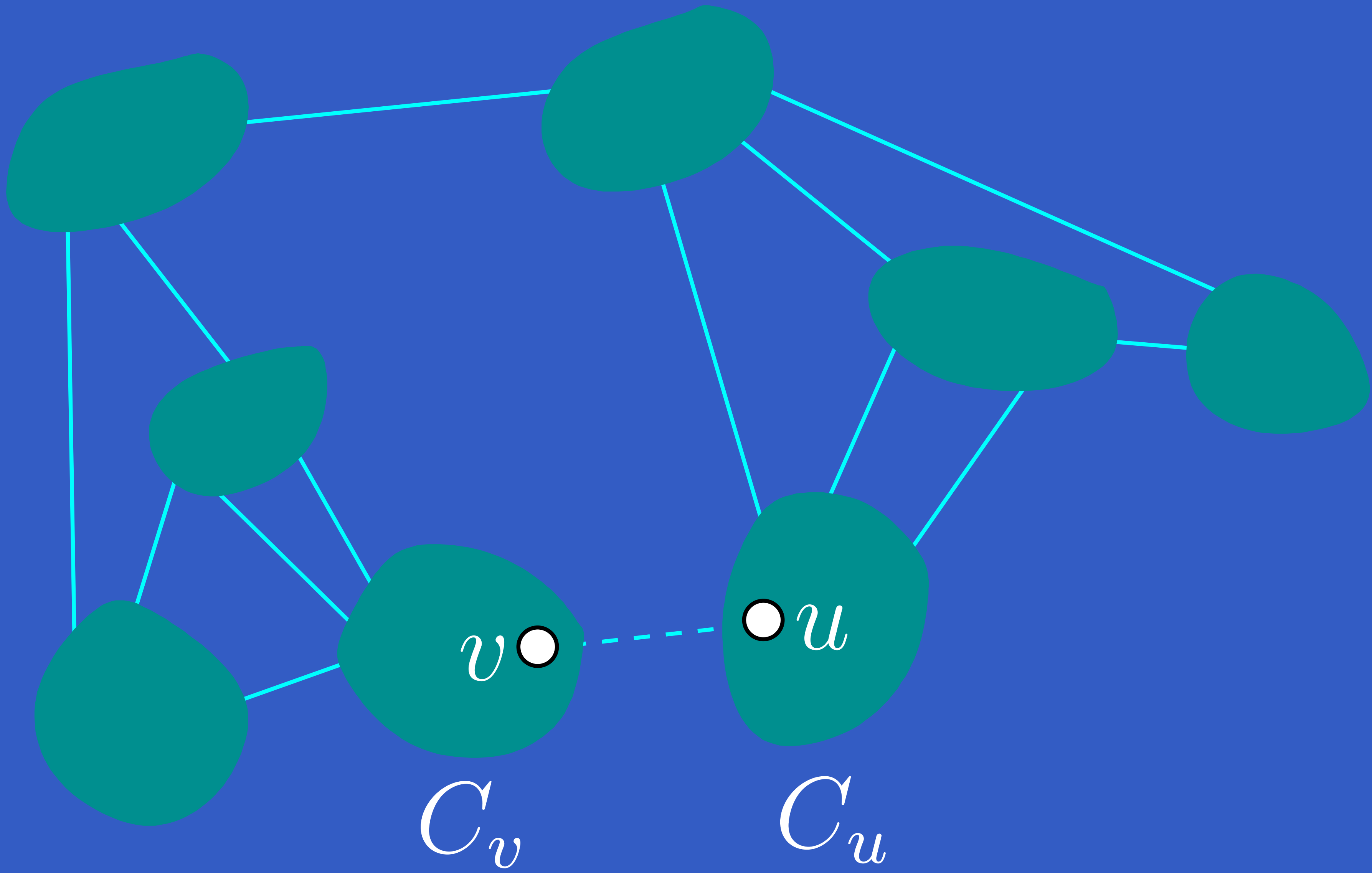

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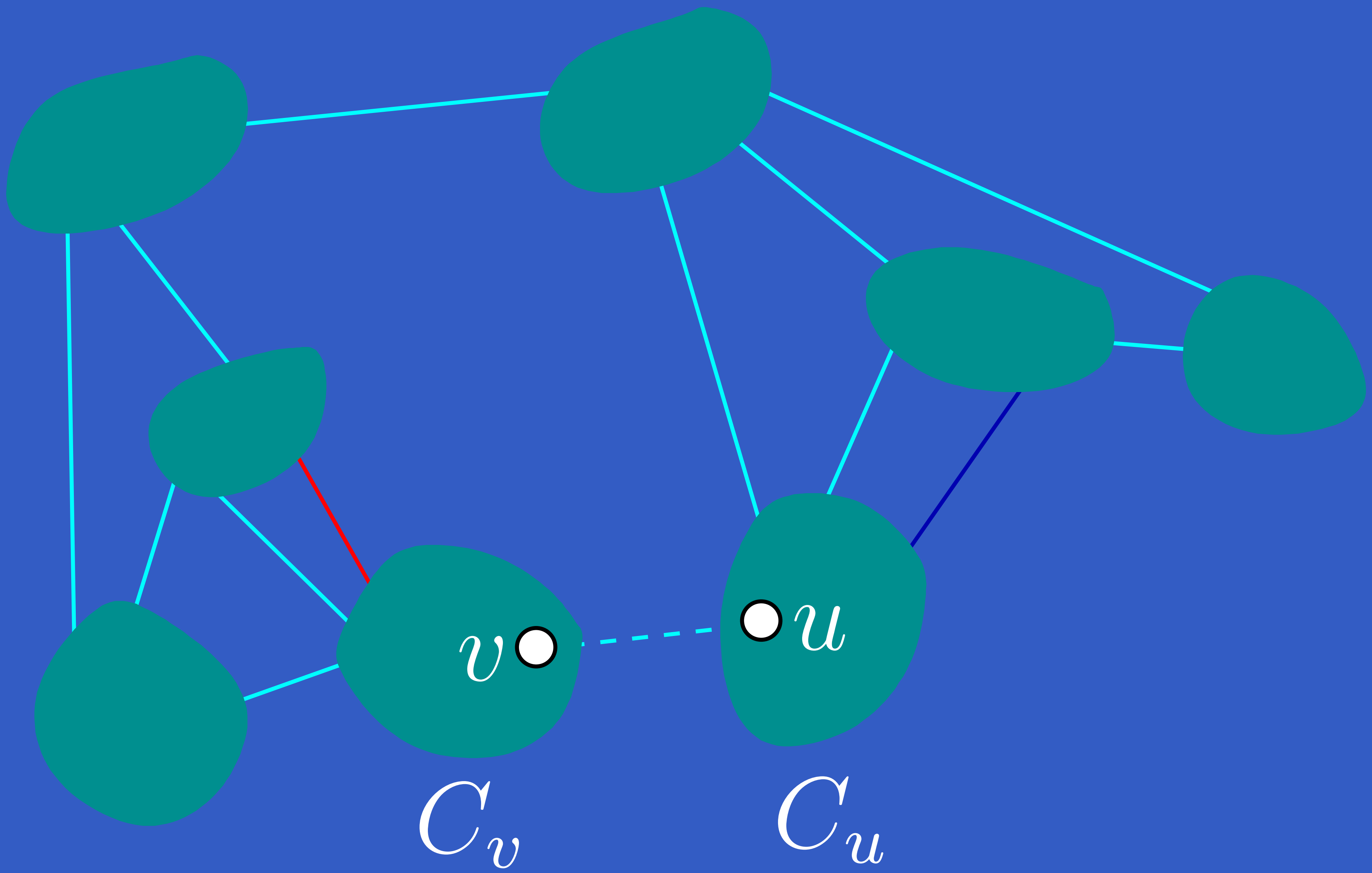

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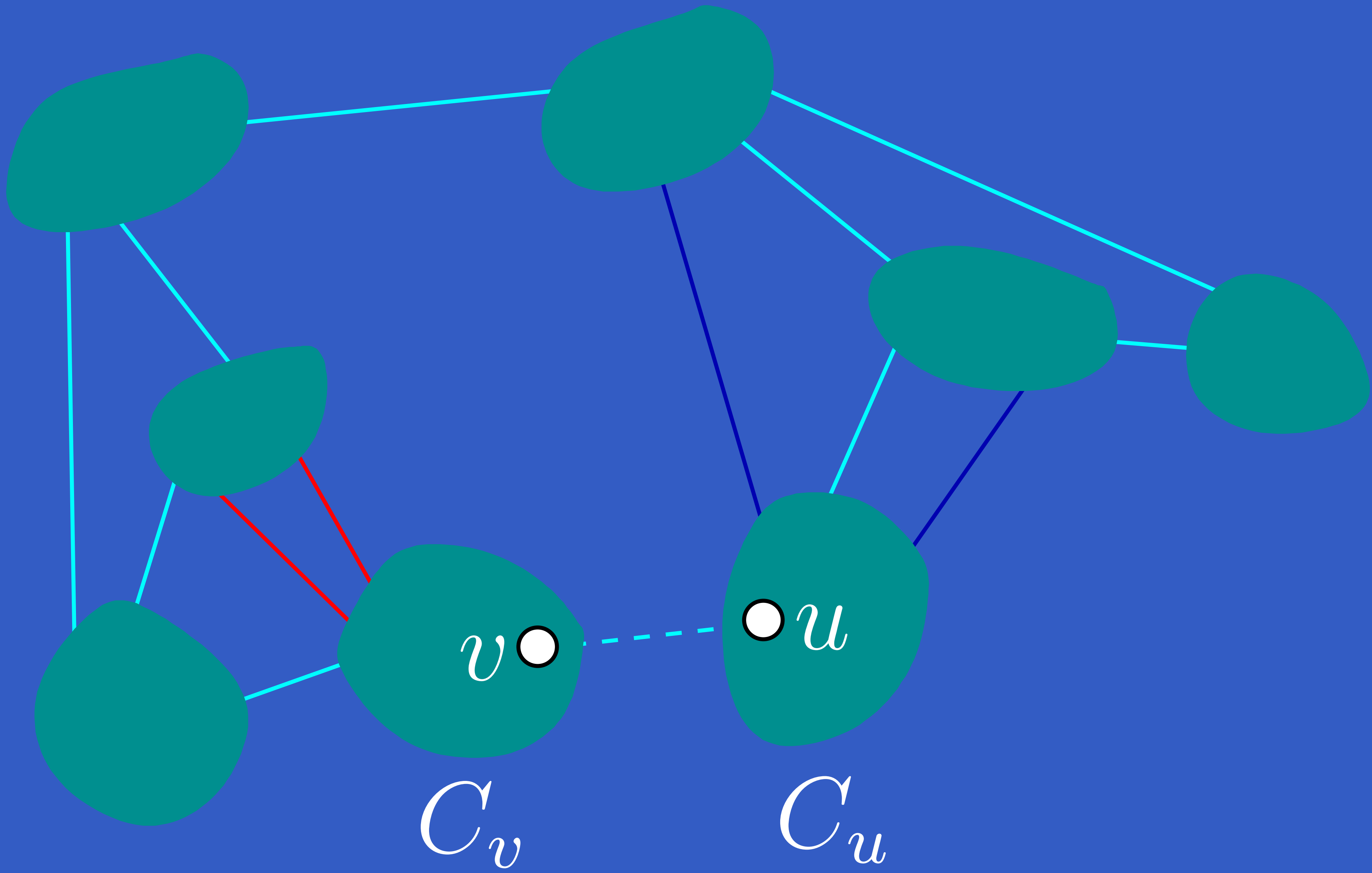

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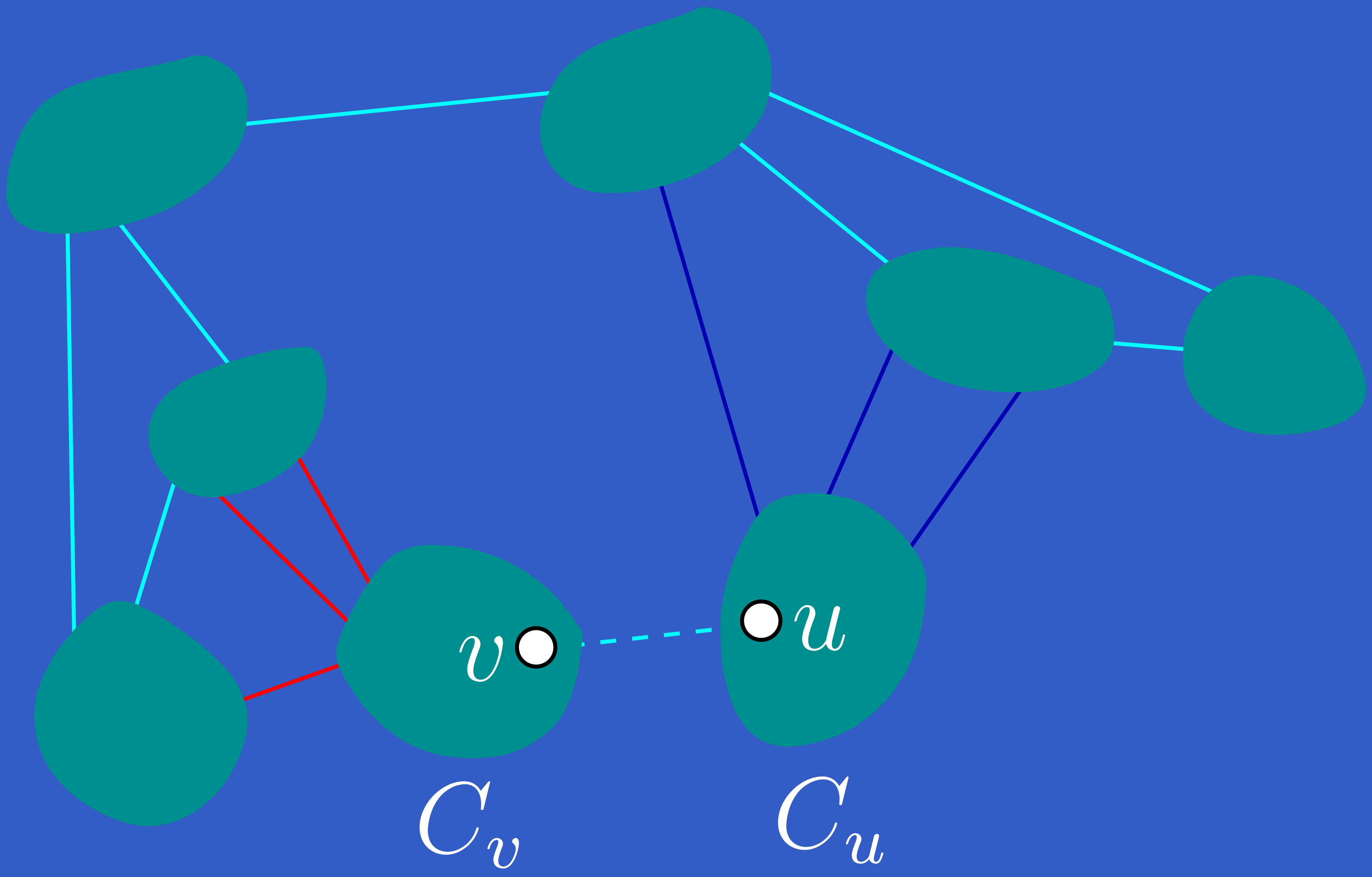

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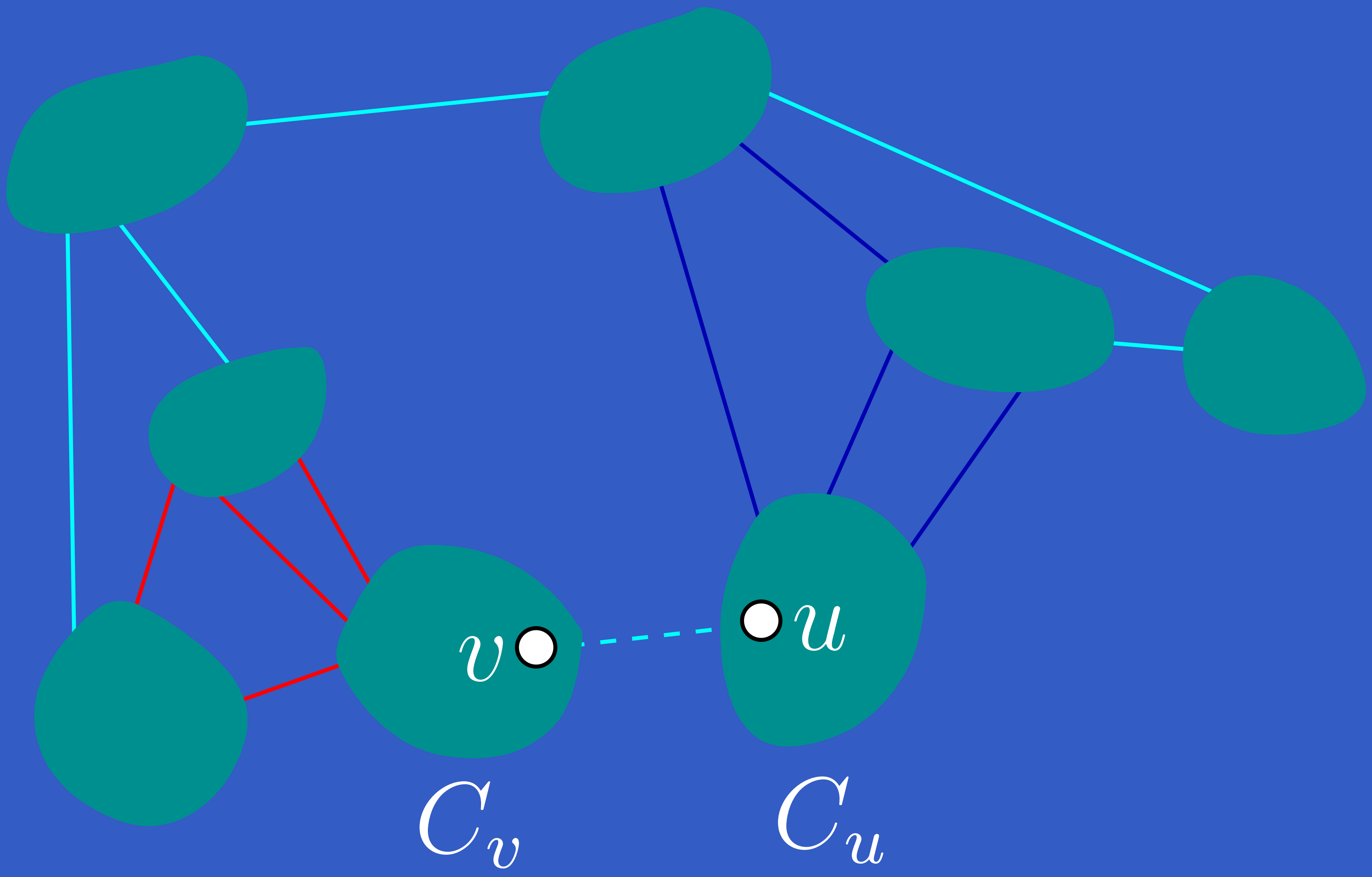

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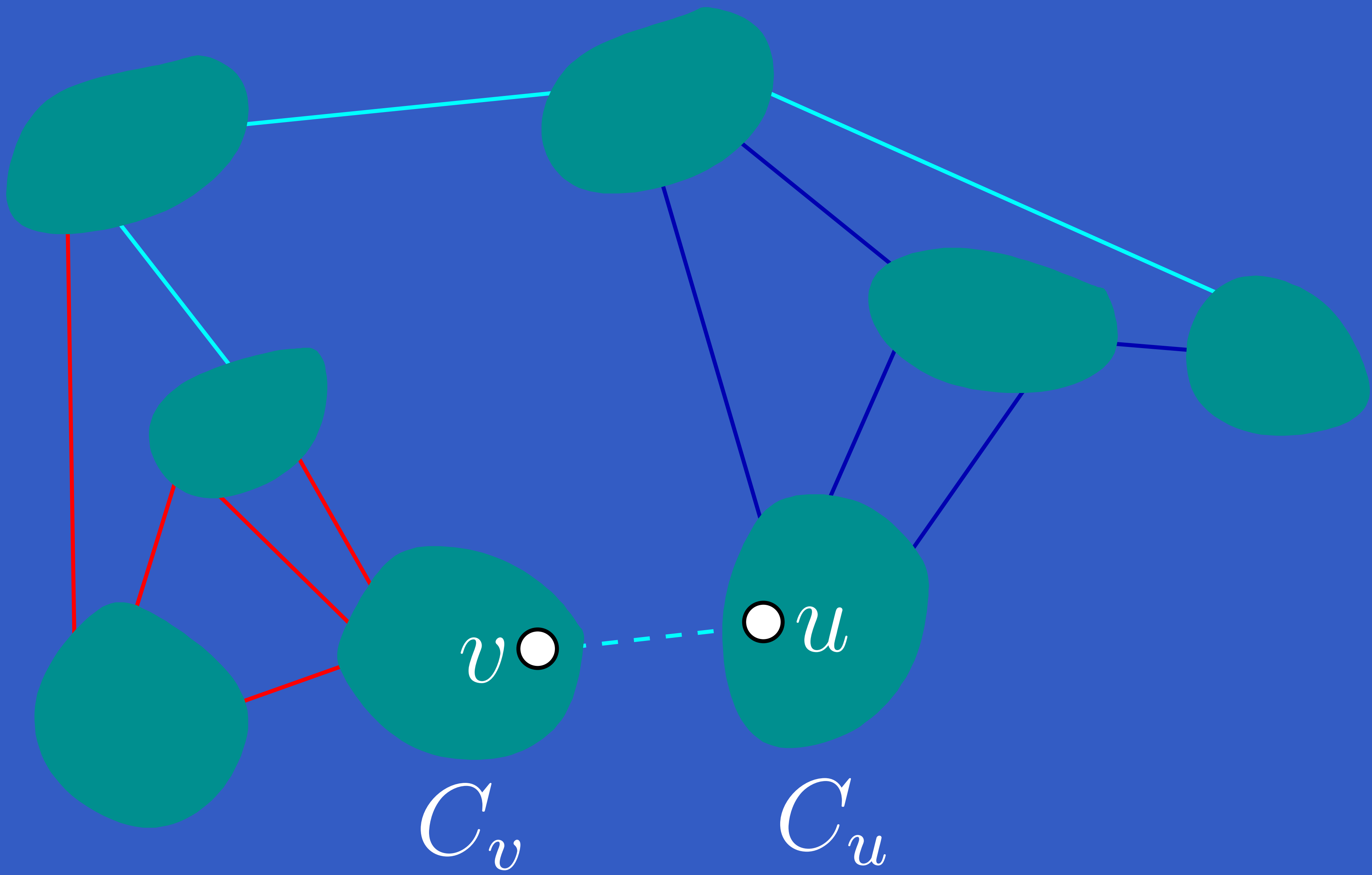

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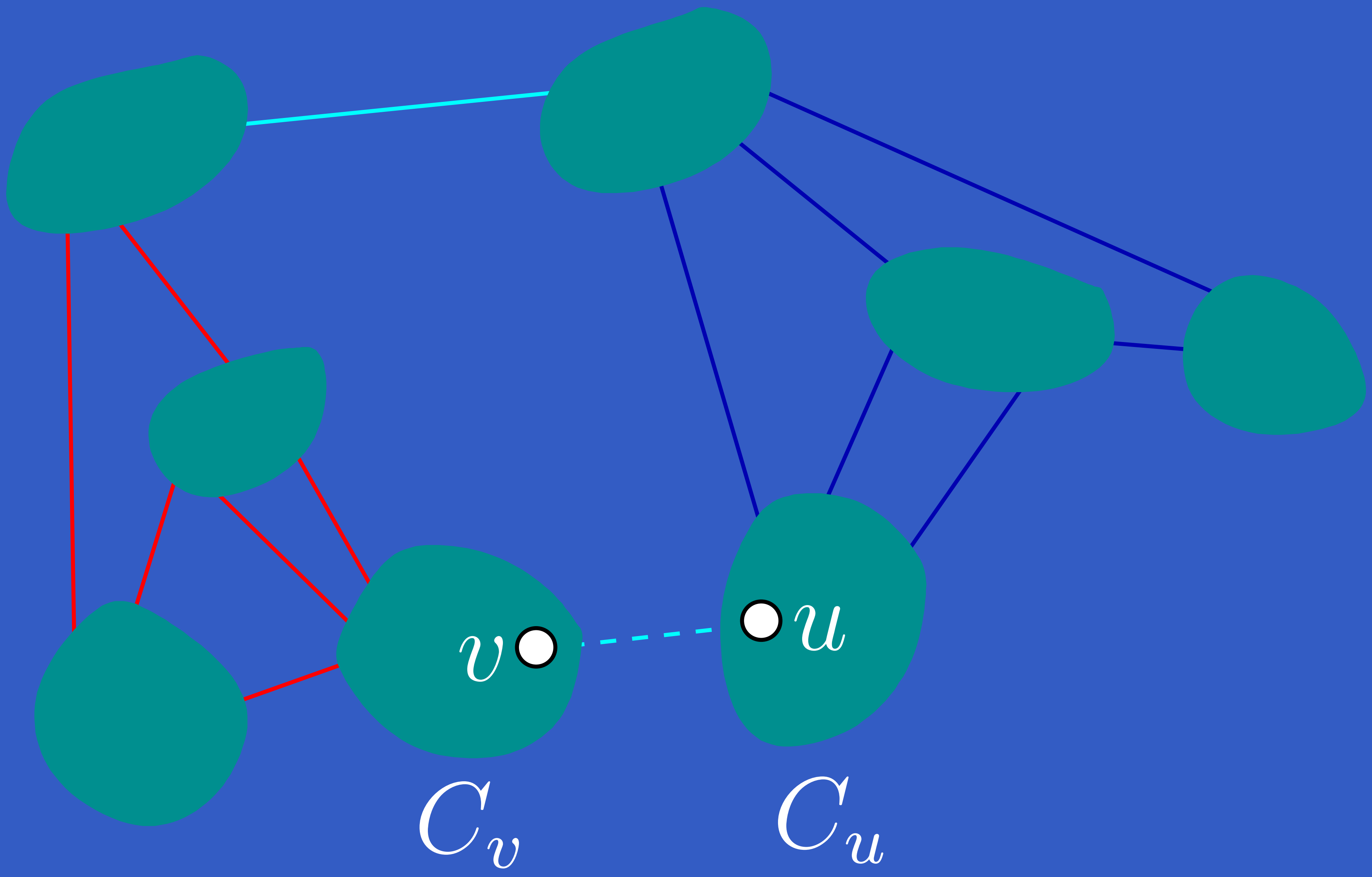

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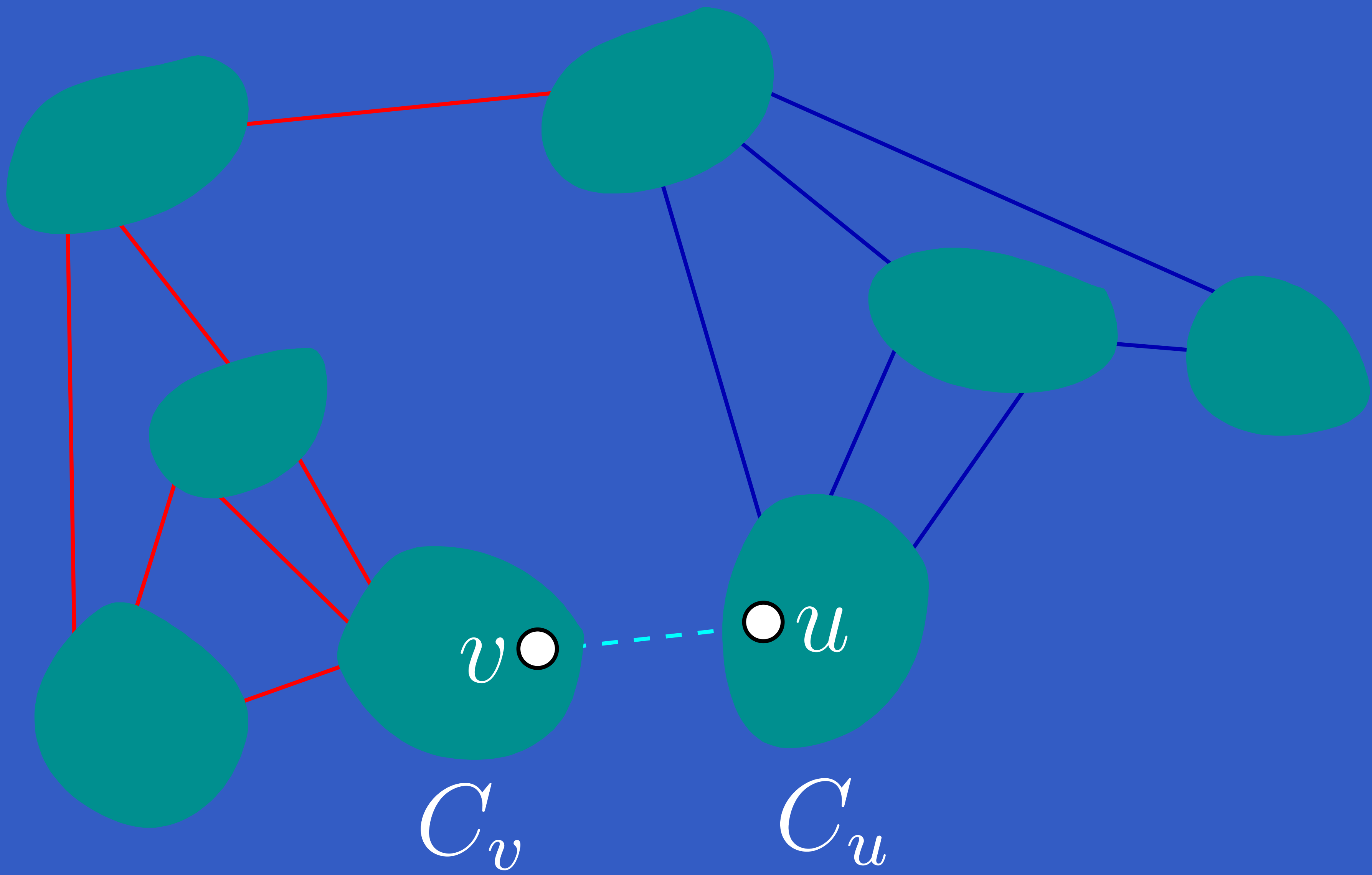

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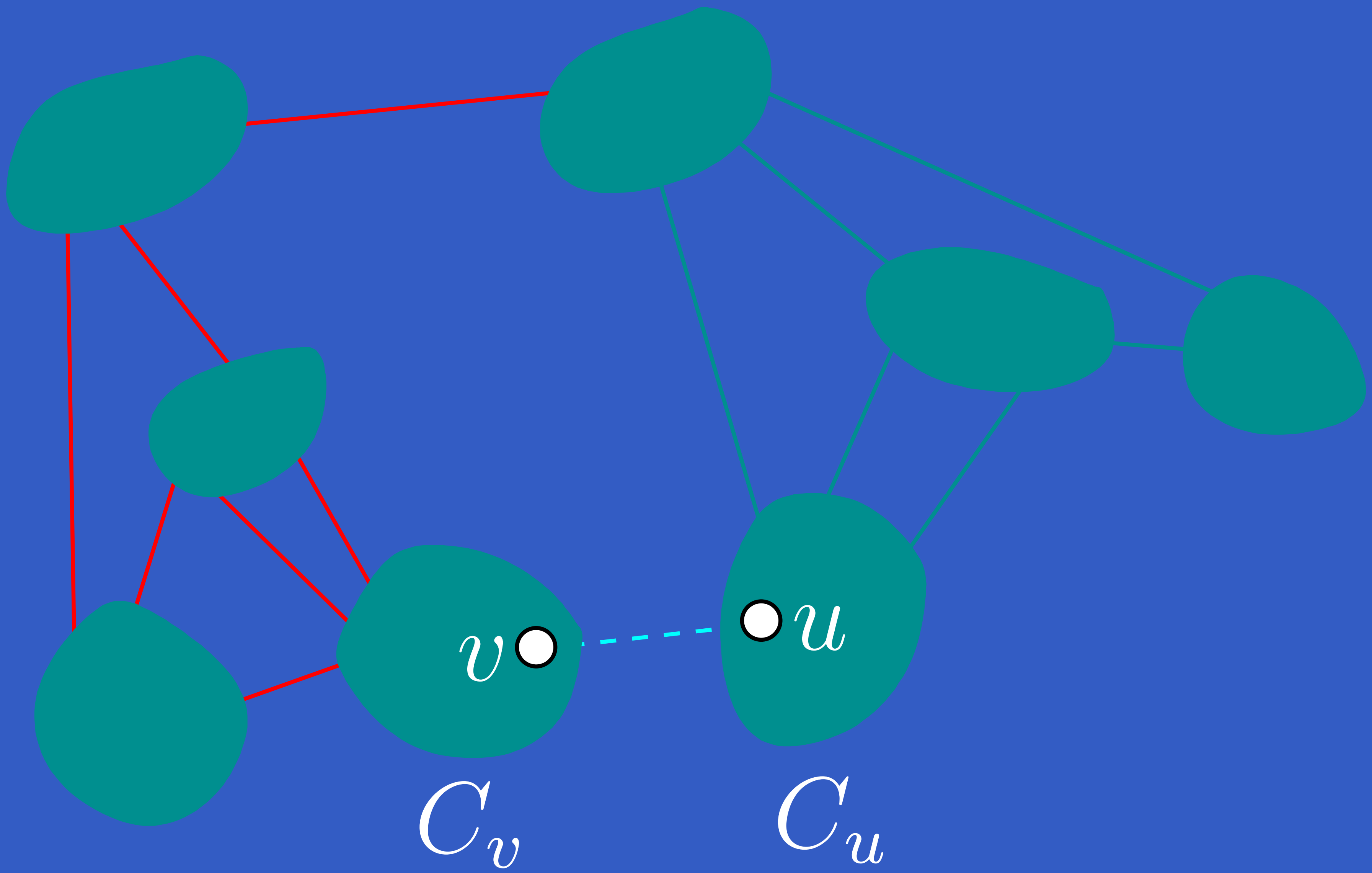

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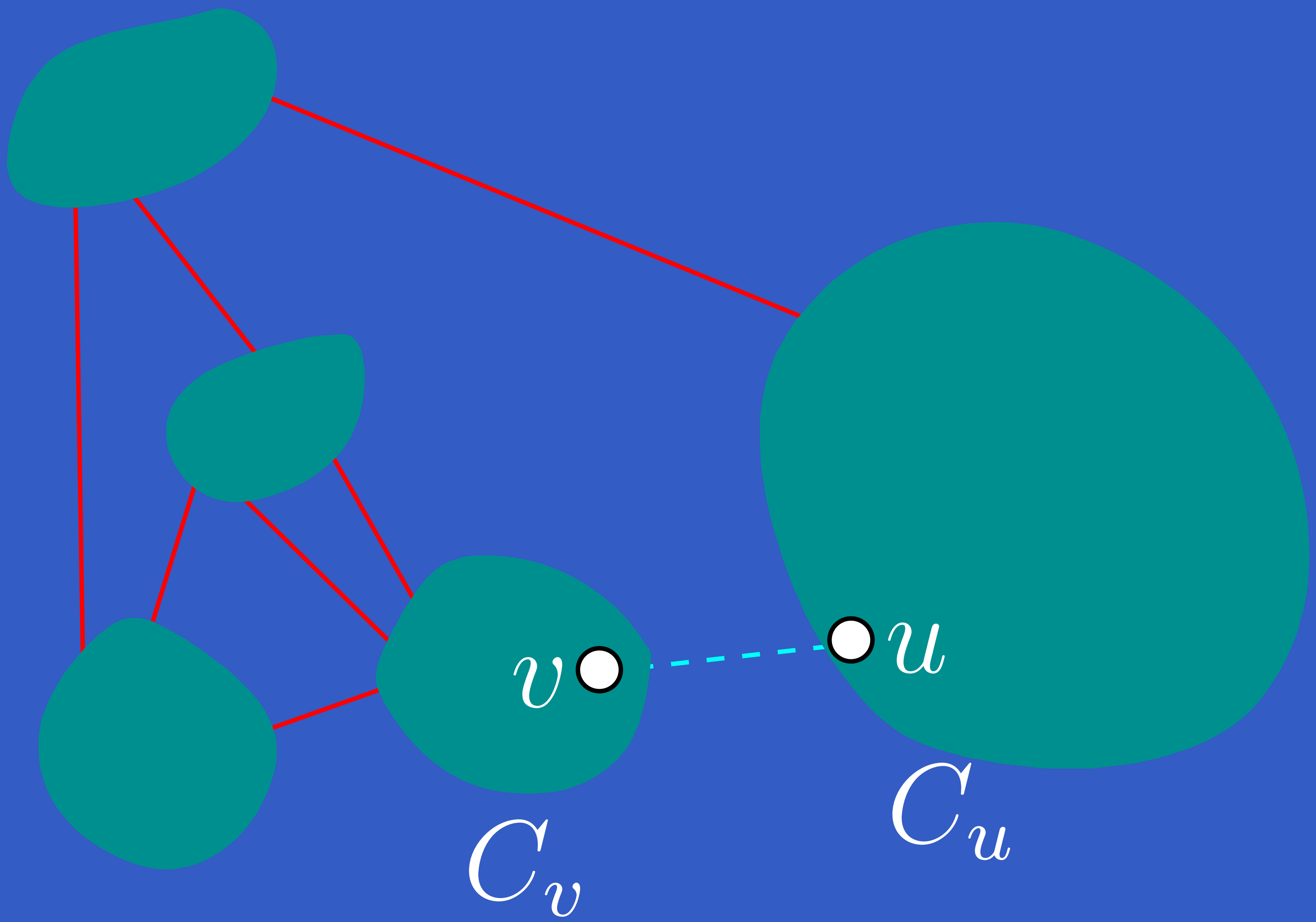

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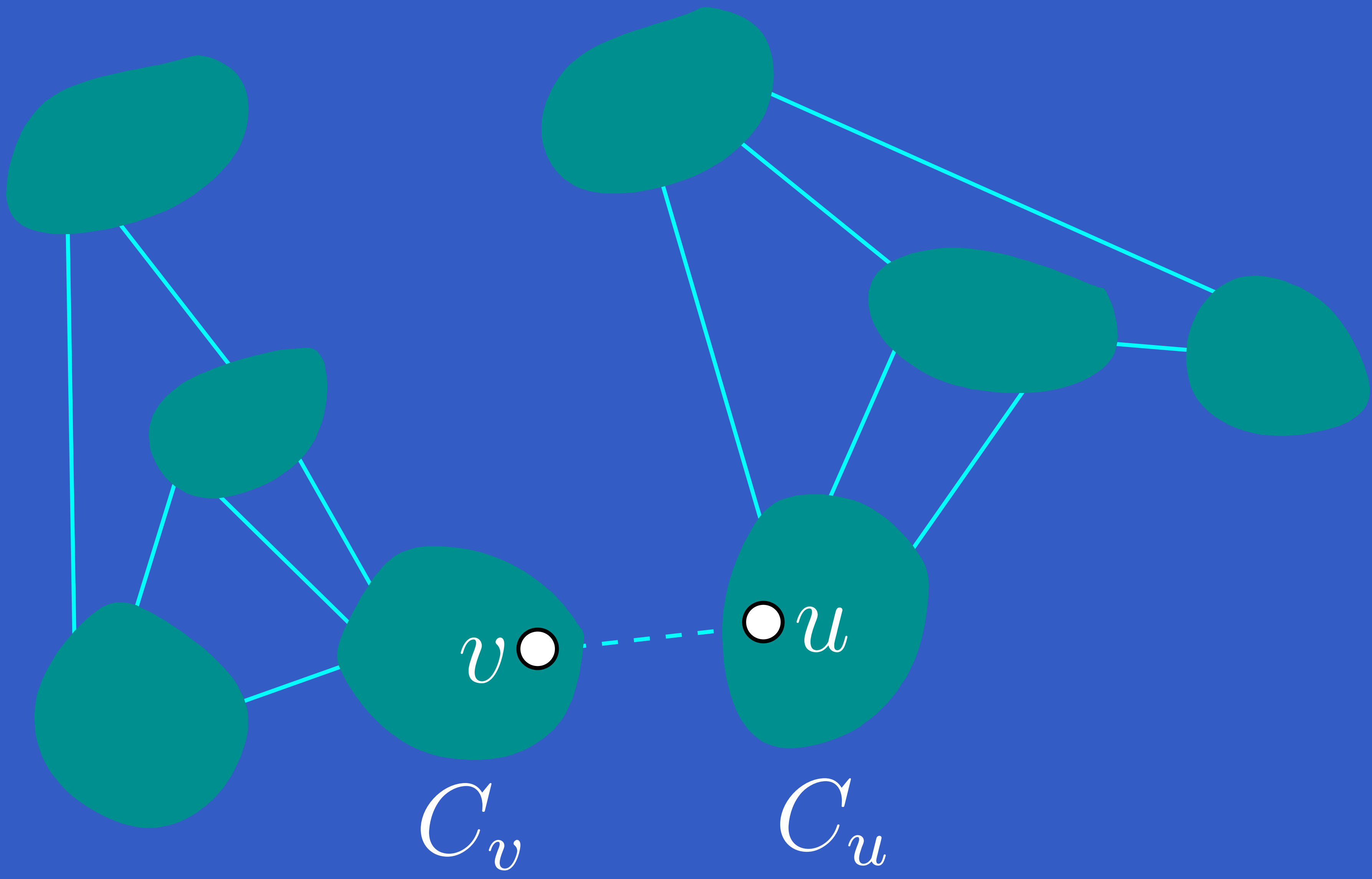

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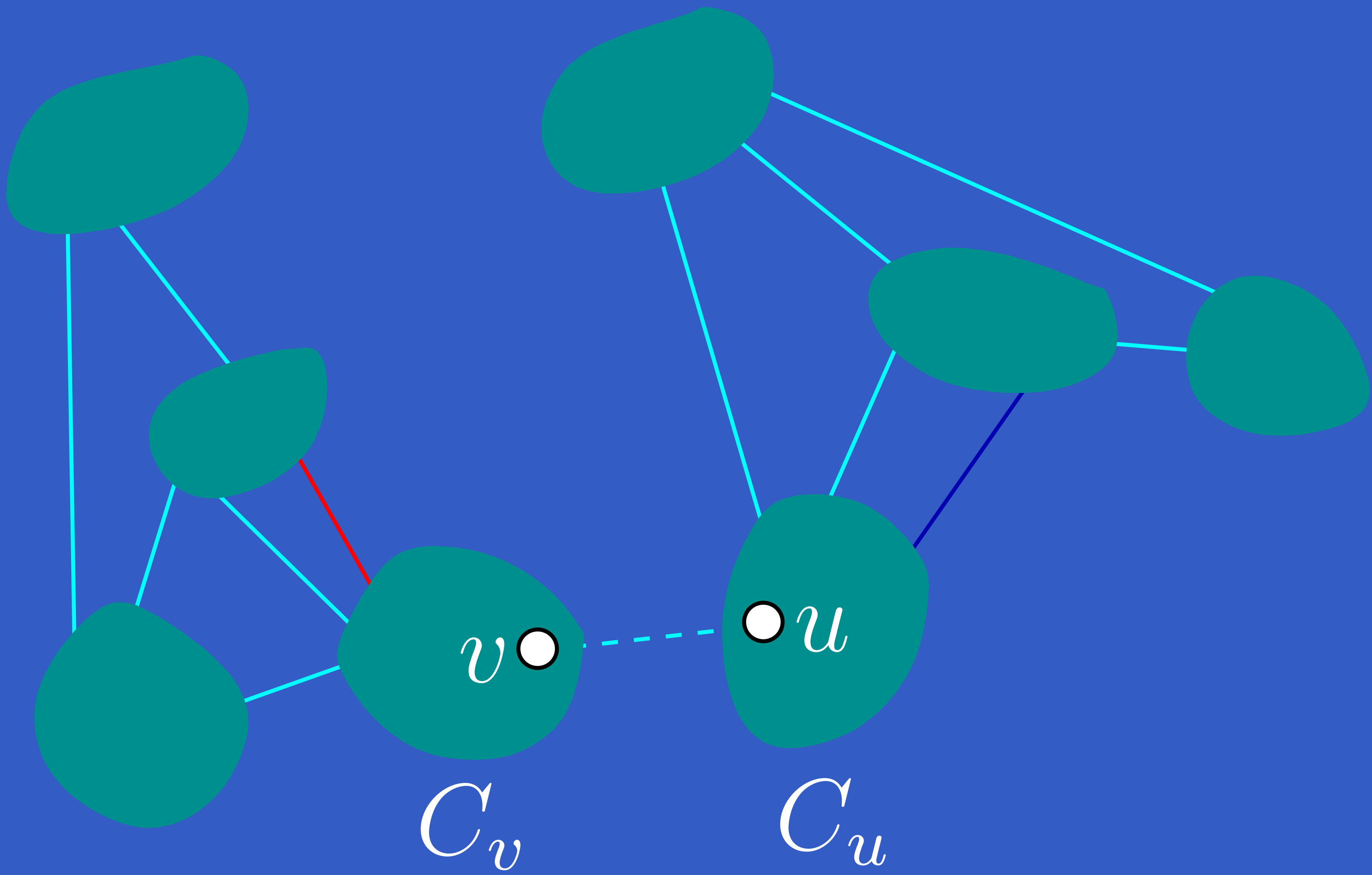

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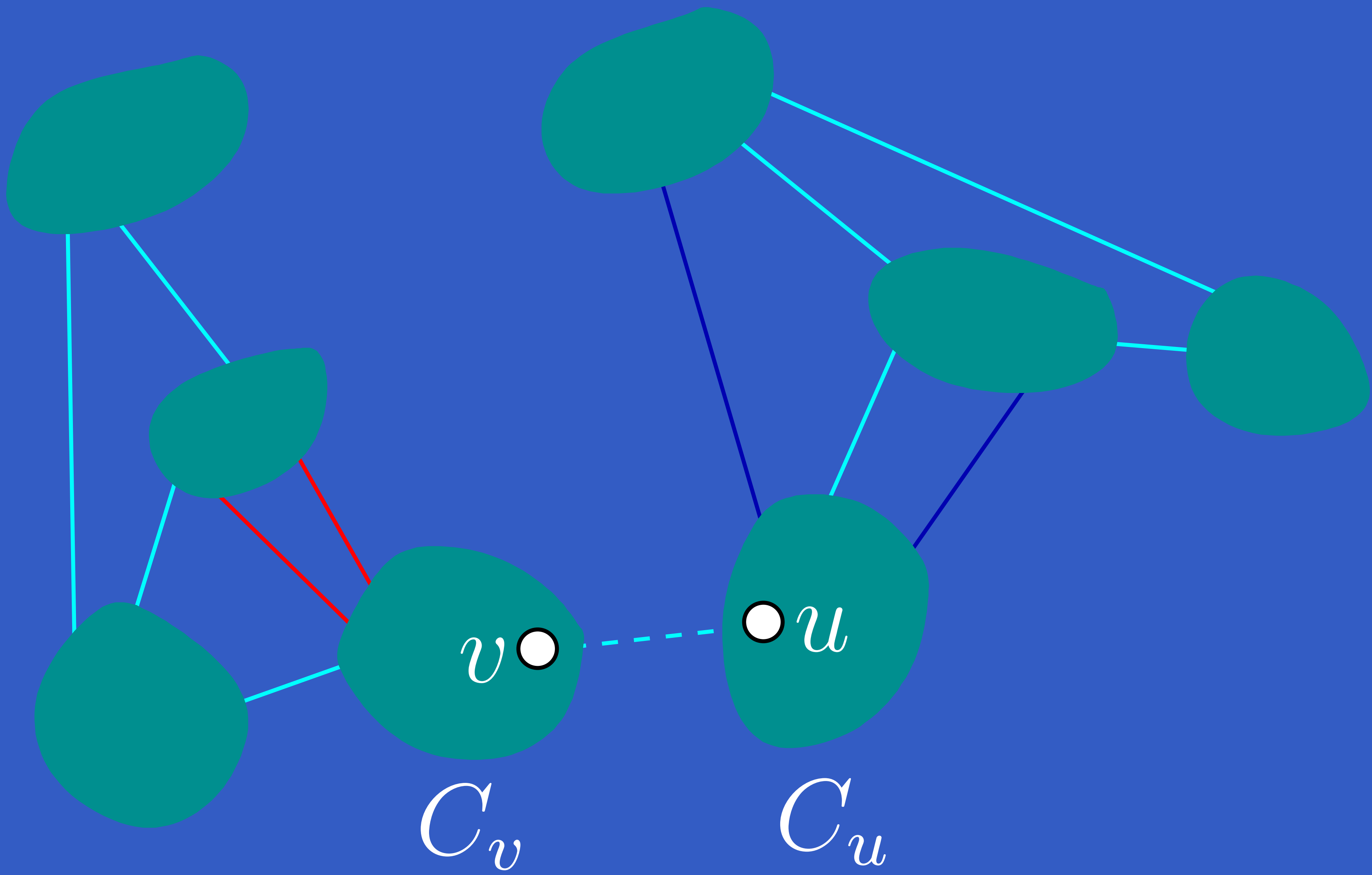

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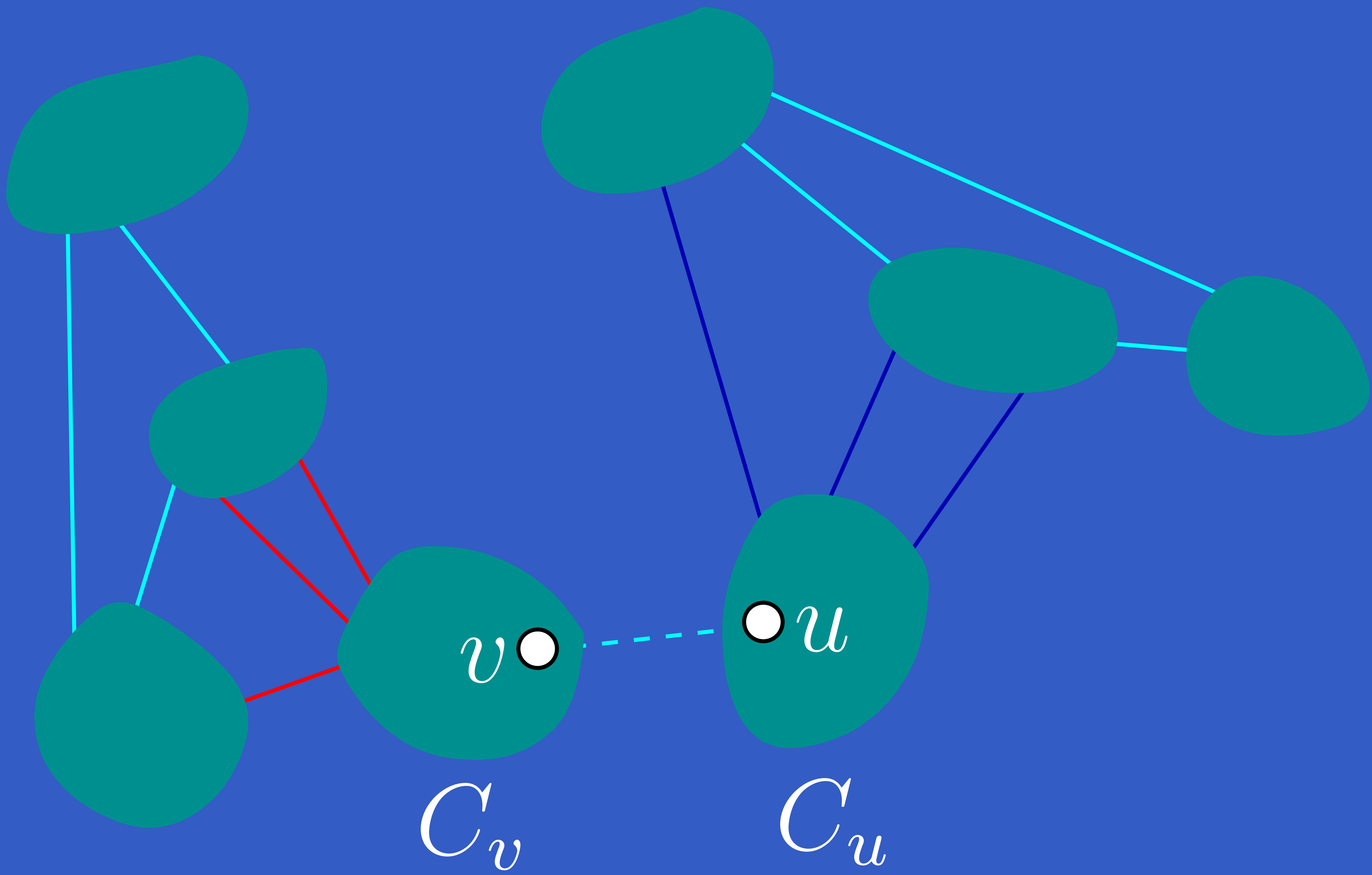

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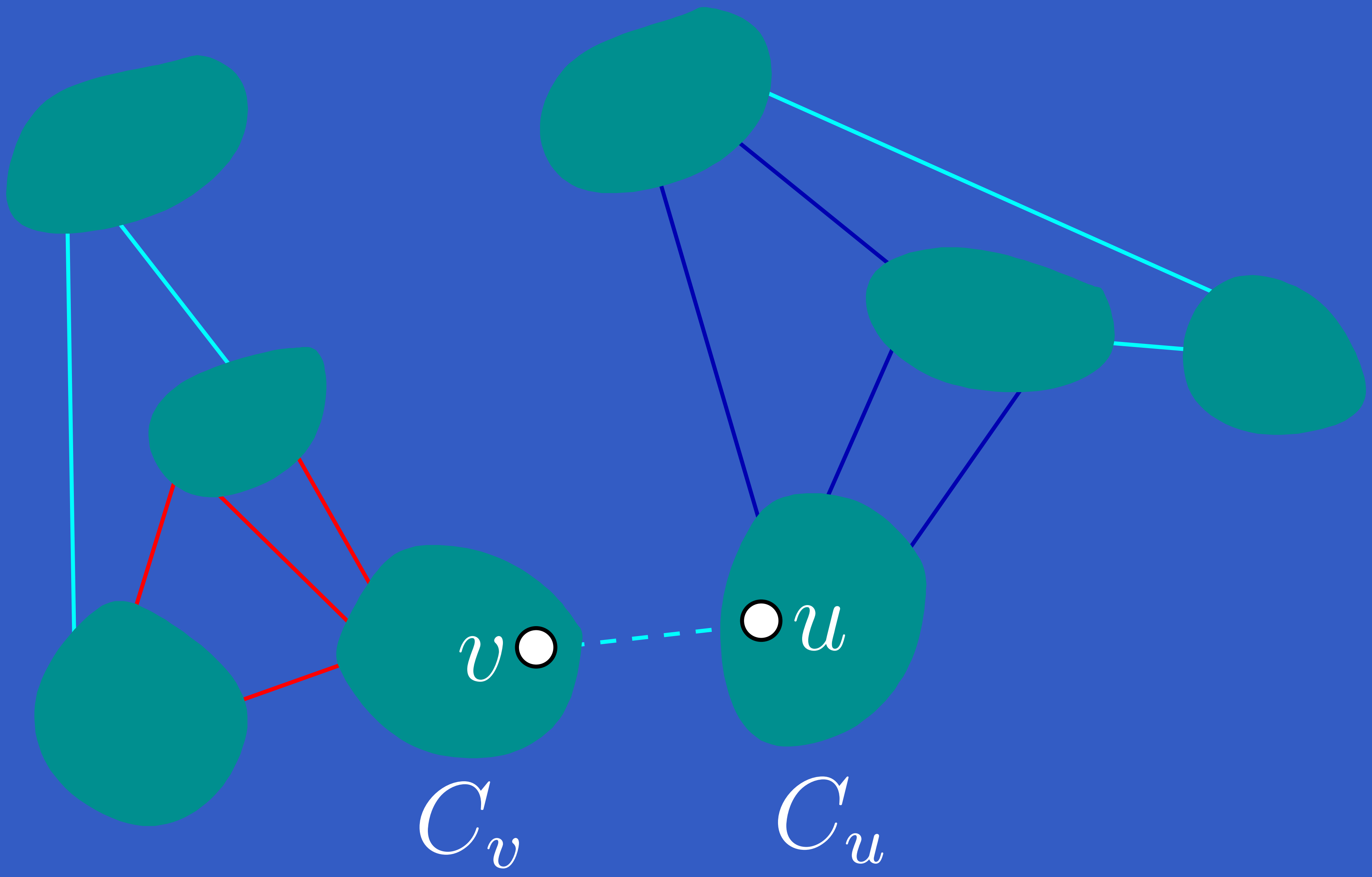

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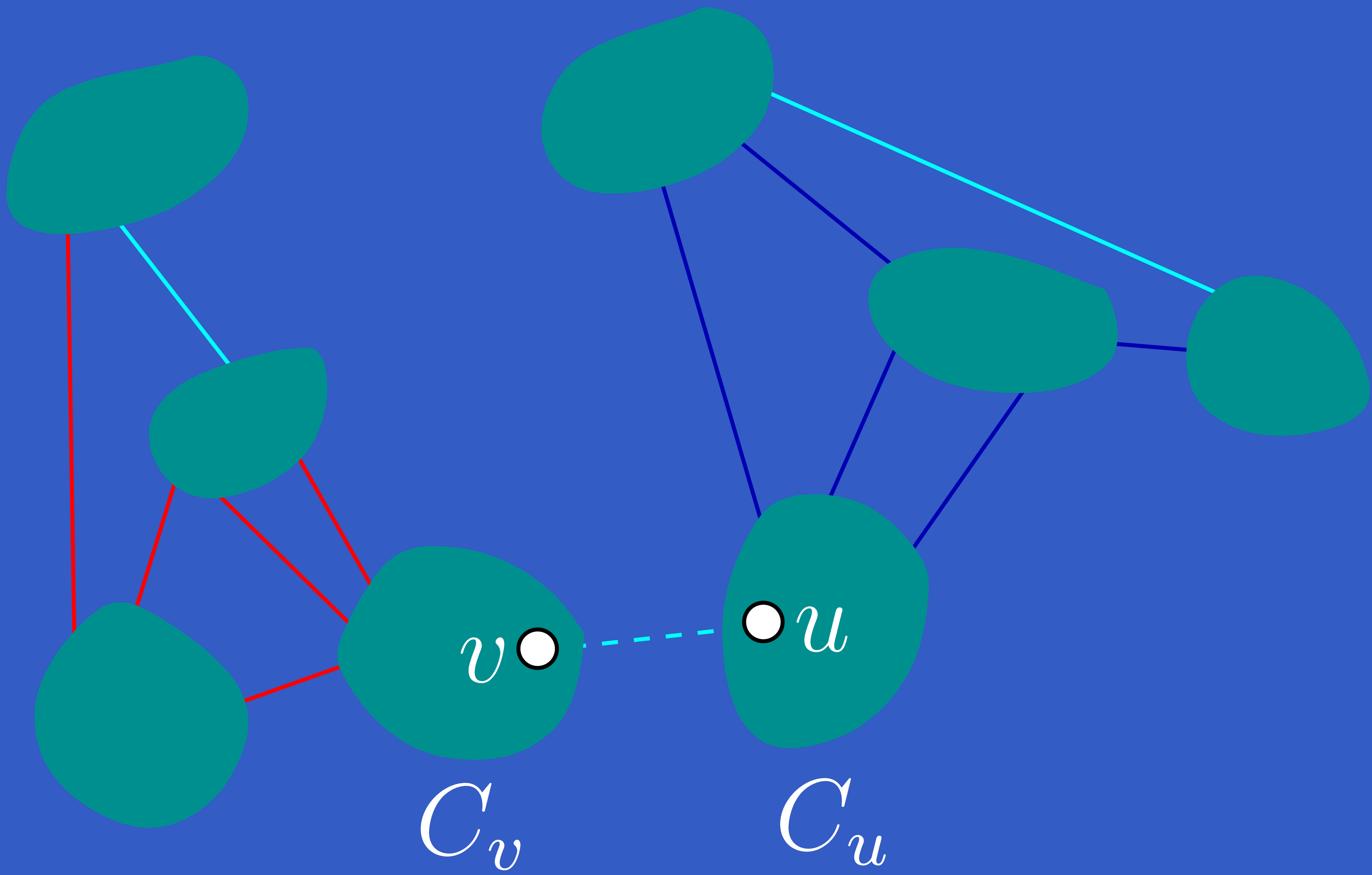

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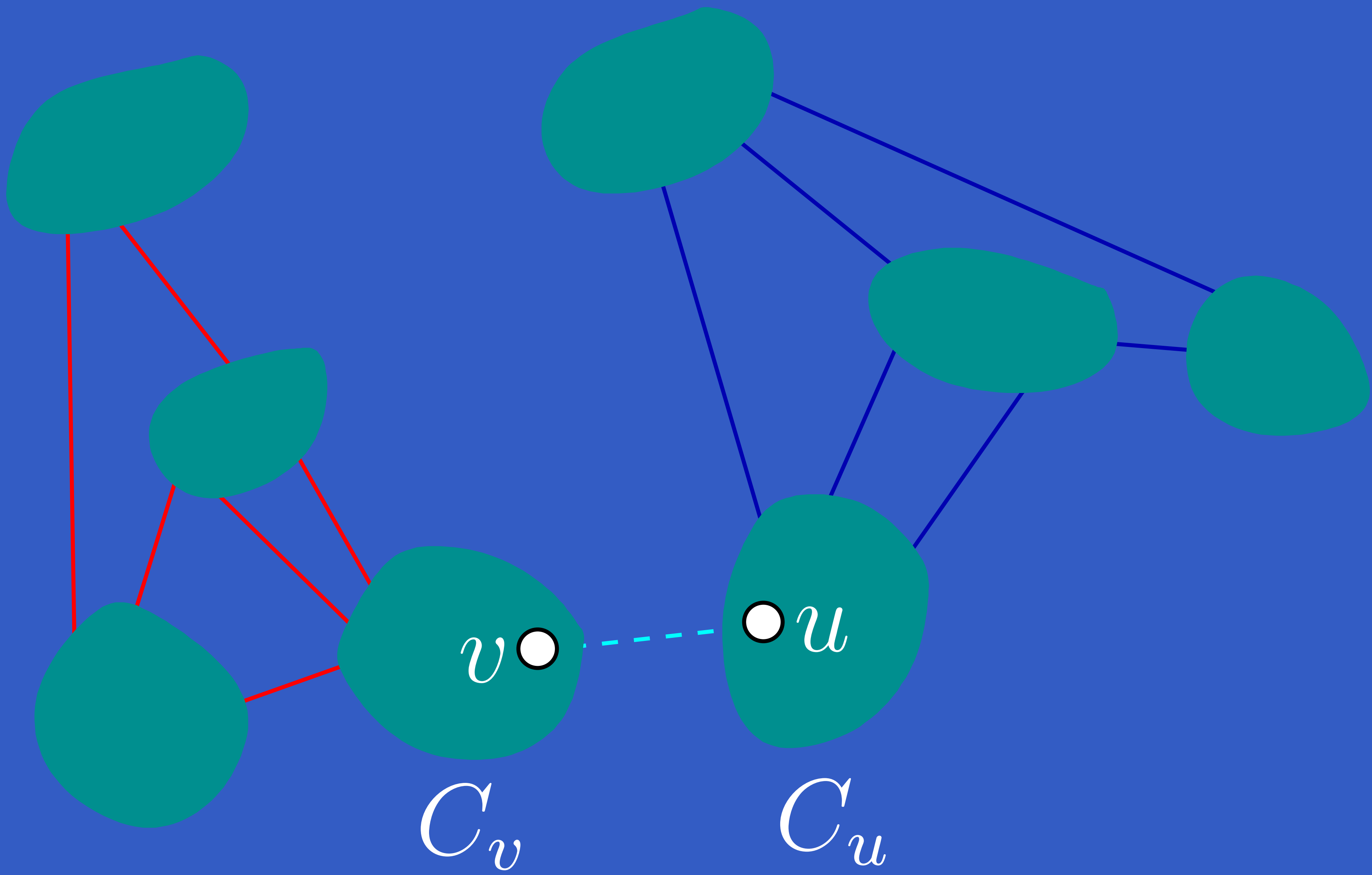

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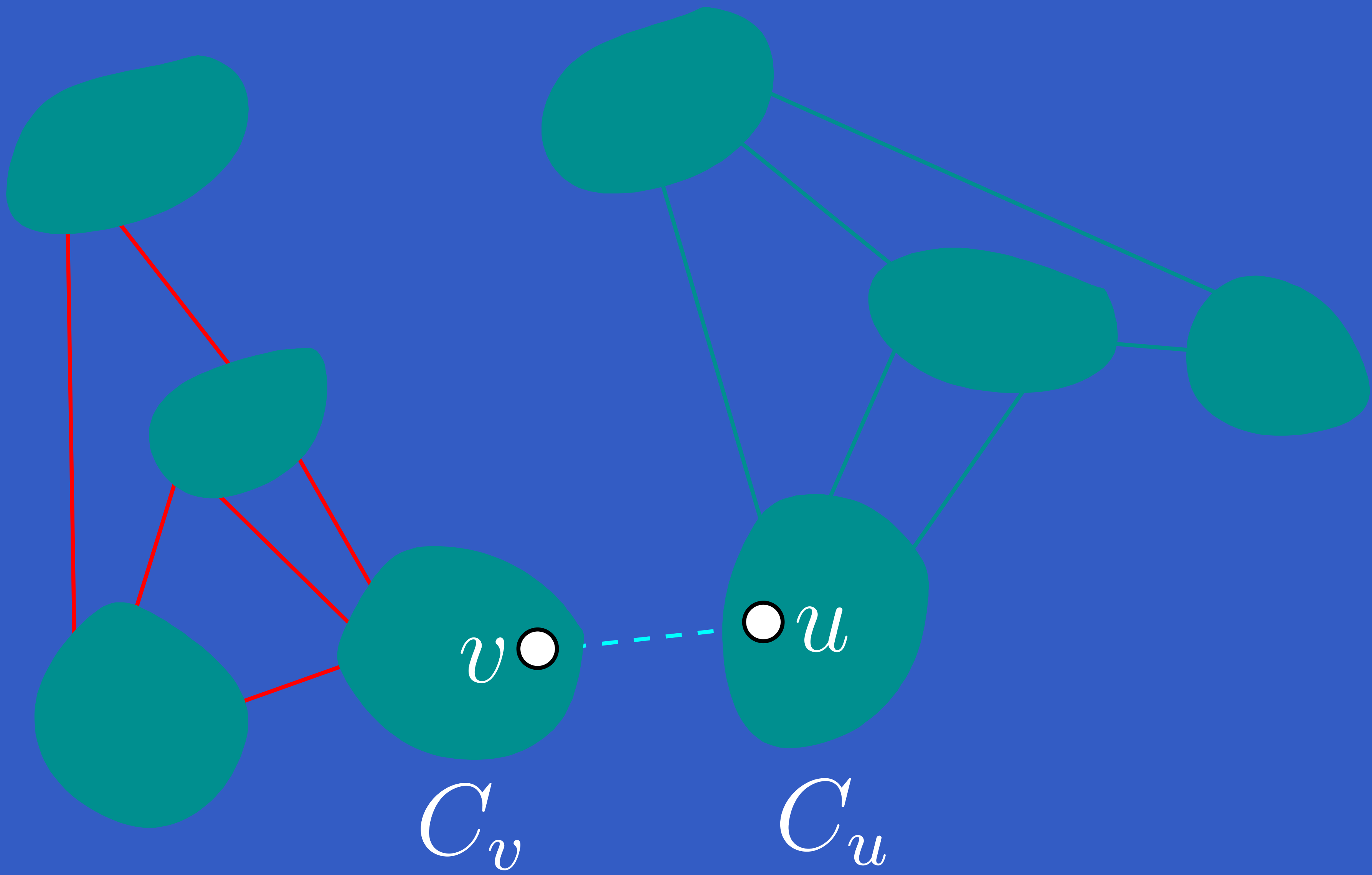

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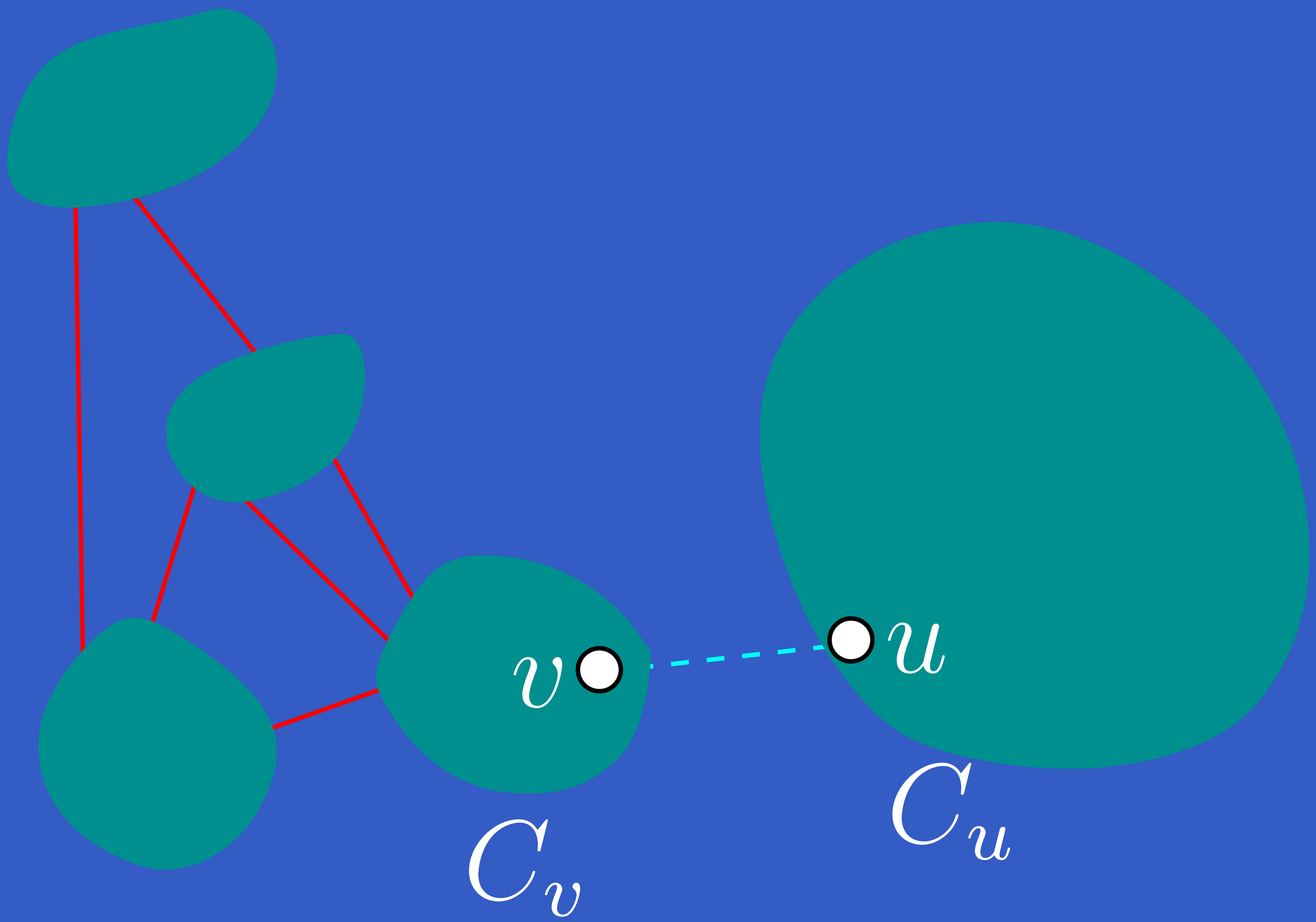
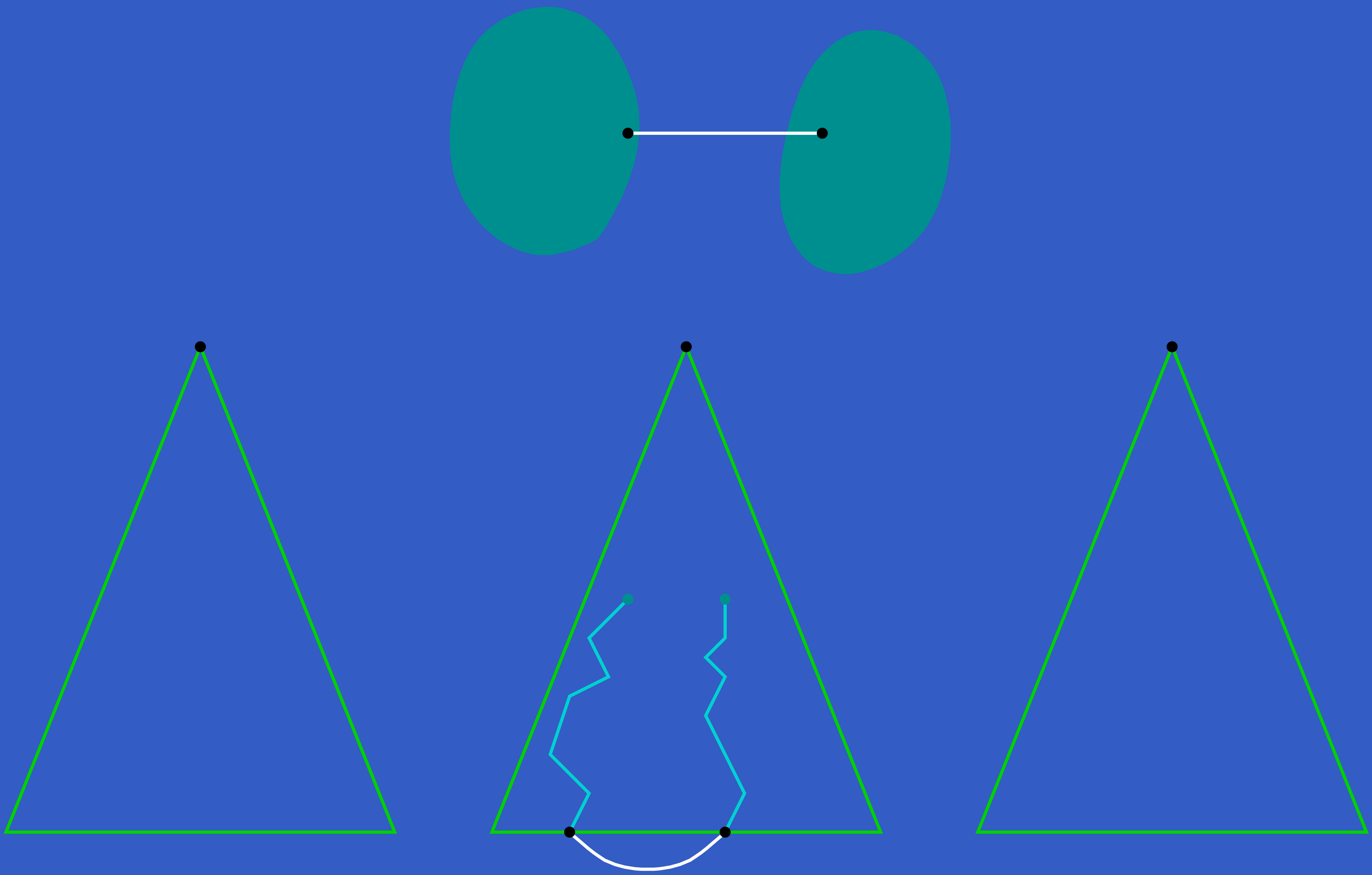


Traversing a single graph edge



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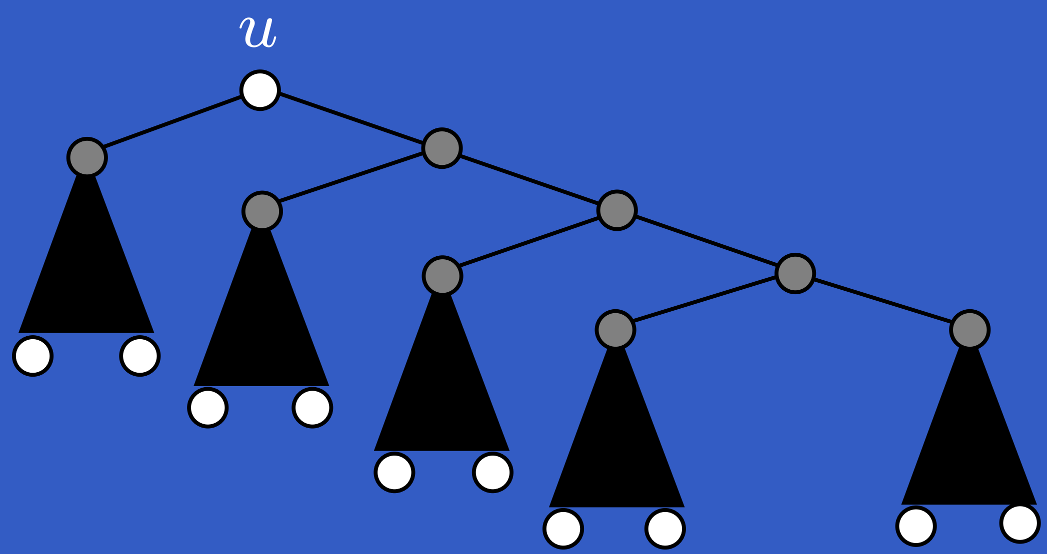

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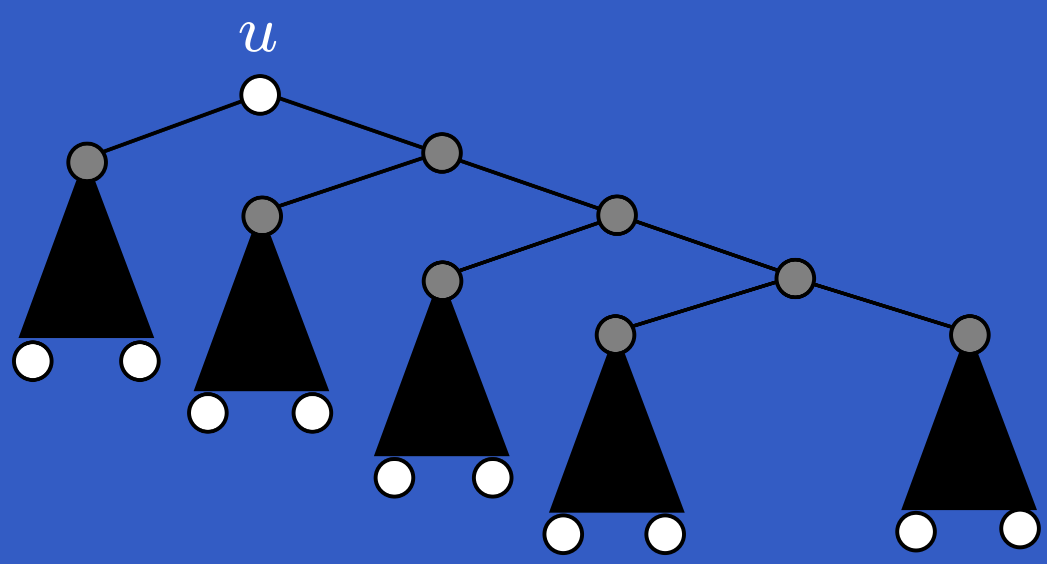

- Denote by $\mathcal{C}_L$ the resulting forest of binary trees; its height is $O(\log n)$

Searching for multigraph edges

- With each node $u \in \mathcal{C}_L$, associate a bitmap $\mathrm{edge}(u)$ where $\mathrm{edge}(u)[i] = 1$ iff a level i-edge is incident to a leaf of the subtree of $\mathcal{C}_L$ rooted at u

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- Whenever an edge is explored, we move up $\mathcal{C}_L$ to identify the new level $(i+1)$-cluster visited
- Since $\mathcal{C}_L$ has height $O(\log n)$, the search procedures run in $O(\log n)$ time per edge explored

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Performance of simple data structure

- For each edge level increase, we spend $O(\log n)$ time
- Total time charged to an edge: $O(\log^2 n)$
- $O(\log^2 n)$ amortized update time
- Query time: $O(\log n)$
- Goal: improve both bounds by a factor of $\log \log n$
- Main ideas: add shortcuts to $\mathcal{C}_L$ and use *lazy* local trees

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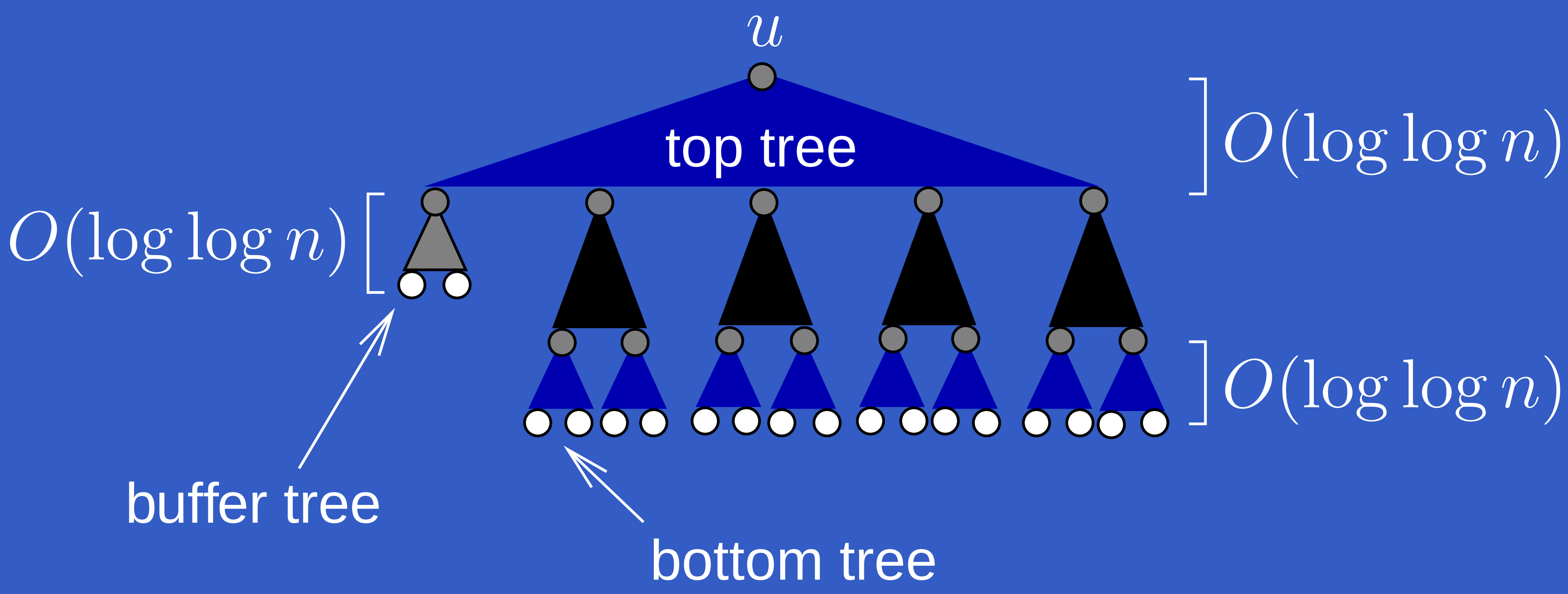

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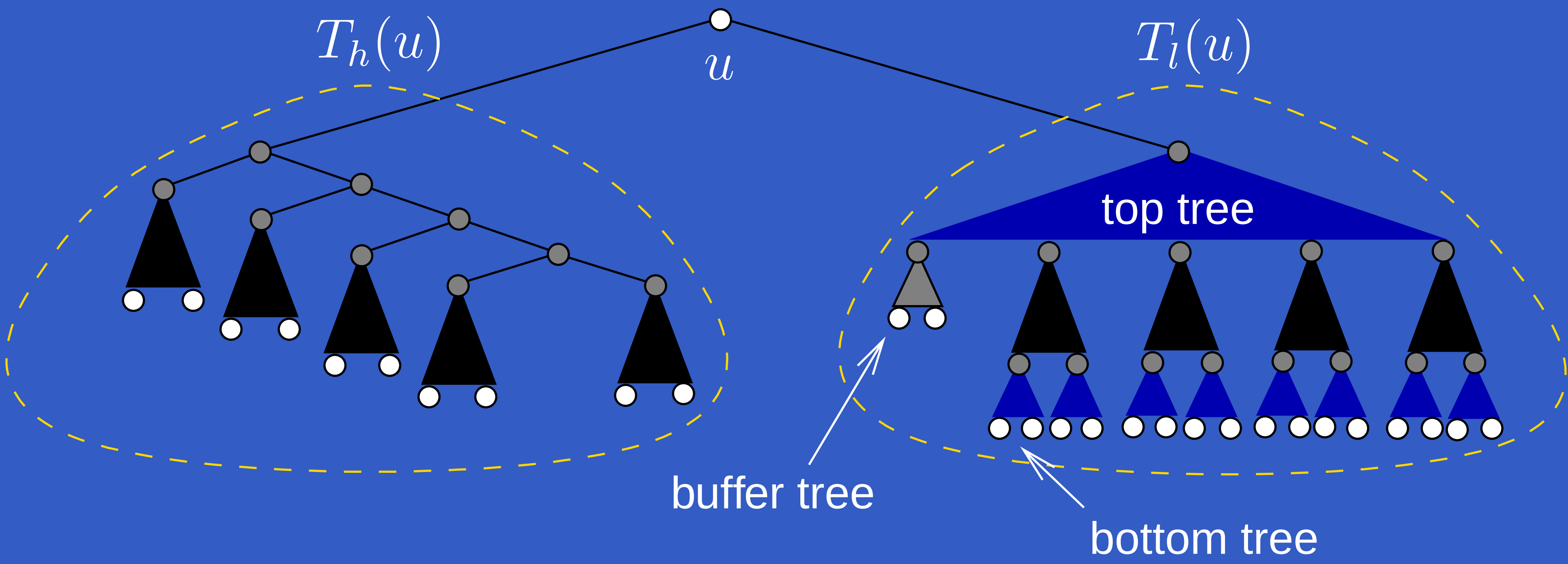

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