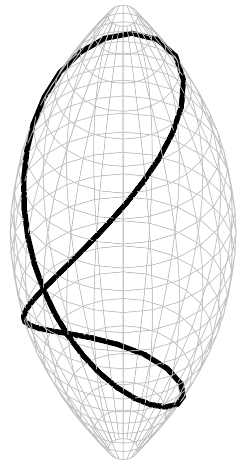

A SIXTY SECOND INTRODUCTION TO SYSTOLES

Contents

Each of the sections below should not take more than 60 seconds to read and understand (with the possible exception of the proof in Section 5). Please contact katzmik at macs.deletethis.biu.andthis.ac.il for clarifications.

1. Isoperimetric inequality

Pu's inequality can be thought of as an "opposite" isoperimetric inequality, in the following precise sense.

The classical isoperimetric inequality in the plane is a relation between two metric invariants: length L of a simple closed curve in the plane, and area A of the region bounded by the curve. Namely, every simple closed curve in the plane satisfies the inequality

$$\frac{A}{\pi} \leq \left(\frac{L}{2\pi}\right)^2.$$

This classical *isoperimetric inequality* is sharp, insofar as equality is attained precisely by round circles.

2. Pu's inequality

In the 1950's, Charles Loewner's student P. M. Pu [Pu52] proved the following theorem. Let $\mathbb{RP}^2$ be the real projective plane endowed with an arbitrary metric, *i.e.* an imbedding in some $\mathbb{R}^n$. Then

$$\left(\frac{L}{\pi}\right)^2 \leq \frac{A}{2\pi}, \tag{2.1}$$

where A is its total area and L is the least length of a non-contractible loop. This *isosystolic inequality*, or simply *systolic inequality* for short, is also sharp, to the extent that equality is attained precisely for a metric of constant Gaussian curvature, namely antipodal quotient of a round sphere. In the systolic notation where L is replaced by $\mathrm{sys}\pi_1$, Pu's inequality takes the following form:

$$\mathrm{sys}\pi_1(\mathcal{G})^2 \leq \tfrac{\pi}{2}\,\mathrm{area}(\mathcal{G}), \tag{2.2}$$

for every metric $\mathcal{G}$ on $\mathbb{RP}^2$.

For a proof, see [Ka07, Section 6.5].

Pu's inequality can be generalized as follows.

Theorem 2.1. *Every surface $(\Sigma, \mathcal{G})$ different from S^2 satisfies the optimal bound (2.2), attained precisely when, on the one hand, the surface Σ is a real projective plane, and on the other, the metric $\mathcal{G}$ is of constant Gaussian curvature.*

The extension to surfaces of nonpositive Euler characteristic follows from Gromov's inequality (2.3) below (by comparing the numerical values of the two constants). Namely, every aspherical compact surface $(\Sigma, \mathcal{G})$ admits a metric ball

$$B = B_p\left(\tfrac{1}{2}\,\mathrm{sys}\pi_1(\mathcal{G})\right) \subset \Sigma$$

of radius $\frac{1}{2}\,\mathrm{sys}\pi_1(\mathcal{G})$ which satisfies [Gro83, Corollary 5.2.B]

$$\mathrm{sys}\pi_1(\mathcal{G})^2 \leq \frac{4}{3}\,\mathrm{area}(B). \tag{2.3}$$

3. Loewner's torus inequality

Historically, the first lower bound for the volume of a Riemannian manifold in terms of a systole is due to Charles Loewner. In 1949, Loewner proved the first systolic inequality, in a course on Riemannian geometry at Syracuse University, *cf.* [Pu52]. Namely, he showed the following result.

Theorem 3.1 (C. Loewner)**.** *Every Riemannian metric $\mathcal{G}$ on the torus $\mathbb{T}^2$ satisfies the inequality*

$$\mathrm{sys}\pi_1(\mathcal{G})^2 \leq \gamma_2 \ \mathrm{area}(\mathcal{G}), \tag{3.1}$$

where $\gamma_2 = \frac{2}{\sqrt{3}}$ is the Hermite constant. A metric attaining the optimal bound (3.1) *is necessarily flat, and is homothetic to the quotient of $\mathbb{C}$ by the lattice spanned by the cube roots of unity.*

For a proof, see [Ka07, Section 6.2].

4. Stable norm and stable systoles

We recall the definition of the stable norm in the real homology of a polyhedron X with a piecewise Riemannian metric, following [BaK03, BaK04].

Definition 4.1. The stable norm $\|h\|$ of $h \in H_k(X, \mathbb{R})$ is the infimum of the volumes

$$\mathrm{vol}_k(c) = \Sigma_i |r_i| \, \mathrm{vol}_k(\sigma_i) \tag{4.1}$$

over all real Lipschitz cycles $c = \Sigma_i r_i \sigma_i$ representing h.

Note that $\| \ \|$ is indeed a norm, *cf.* [Fed74] and [Gro99, 4.C].

We denote by $H_k(X, \mathbb{Z})_\mathbb{R}$ the image of $H_k(X, \mathbb{Z})$ in $H_k(X, \mathbb{R})$ and by $h_\mathbb{R}$ the image of $h \in H_k(X, \mathbb{Z})$ in $H_k(X, \mathbb{R})$. Recall that $H_k(X, \mathbb{Z})_\mathbb{R}$ is a lattice in $H_k(X, \mathbb{R})$. Obviously

$$\|h_\mathbb{R}\| \leq \mathrm{vol}_k(h) \tag{4.2}$$

for all $h \in H_k(X, \mathbb{Z})$, where $\mathrm{vol}_k(h)$ is the infimum of volumes of all integral k-cycles representing h. Moreover, one has $\|h_\mathbb{R}\| = \mathrm{vol}_n(h)$ if $h \in H_n(X, \mathbb{Z})$. H. Federer [Fed74, 4.10, 5.8, 5.10] (see also [Gro99, 4.18 and 4.35]) investigated the relations between $\|h_\mathbb{R}\|$ and $\mathrm{vol}_k(h)$ and proved the following.

Proposition 4.2. *If $h \in H_k(X, \mathbb{Z})$, $1 \leq k < n$, then*

$$\|h_\mathbb{R}\| = \lim_{i \to \infty} \frac{1}{i} \mathrm{vol}_k(ih). \tag{4.3}$$

Equation (4.3) is the origin of the term *stable norm* for $\| \ \|$. The stable k-systole of a metric $(X, \mathcal{G})$ is defined by setting

$$\mathrm{stsys}_k(\mathcal{G}) = \lambda_1 \left(H_k(X, \mathbb{Z})_{\mathbb{R}}, \| \ \| \right), \tag{4.4}$$

where λ_1 denotes the first successive minimum of the lattice $\left(H_k(X, \mathbb{Z})_{\mathbb{R}}, \| \ \| \right)$, *i.e.* the least norm of a nonzero lattice element.

5. Gromov's inequality for complex projective space

We now discuss systolic inequalities on projective spaces.

Theorem 5.1 (M. Gromov)**.** *Let $\mathcal{G}$ be a Riemannian metric on complex projective space $\mathbb{CP}^n$. Then*

$$\mathrm{stsys}_2(\mathcal{G})^n \le n! \, \mathrm{vol}_{2n}(\mathcal{G});$$

equality holds for the Fubini-Study metric on $\mathbb{CP}^n$.

Proof. Following Gromov's notation in [Gro99, Theorem 4.36], we let

$$\alpha \in H_2(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z} \tag{5.1}$$

be the positive generator in homology, and let

$$\omega \in H^2(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}$$

be the dual generator in cohomology. Then the cup power ω^n is a generator of $H^{2n}(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}$. Let $\eta \in \omega$ be a closed differential 2-form. Since wedge product $\wedge$ in $\Omega^*(X)$ descends to cup product in $H^*(X)$, we have

$$1 = \int_{\mathbb{CP}^n} \eta^{\wedge n}. \tag{5.2}$$

Now let $\mathcal{G}$ be a metric on $\mathbb{CP}^n$. Recall that the pointwise comass norm for a simple k-form coincides with the natural Euclidean norm on k-forms associated with $\mathcal{G}$. In general, the comass is defined as follows.

Definition 5.2. The comass of an exterior k-form is its maximal value on a k-tuple of unit vectors.

The comass norm of a differential k-form is, by definition, the supremum of the pointwise comass norms. Then by the Wirtinger inequality we obtain

$$\begin{aligned} 1 &\le \int_{\mathbb{CP}^n} \|\eta^{\wedge n}\| \, d\mathrm{vol} \\ &\le n! \left(\|\eta\|_\infty \right)^n \mathrm{vol}_{2n}(\mathbb{CP}^n, \mathcal{G}) \end{aligned} \tag{5.3}$$

where $\| \, \|_\infty$ is the comass norm on forms. See [Gro99, Remark 4.37] for a discussion of the constant in the context of the Wirtinger inequality. A more detailed discussion appears in [Ka07, Section 13.1].

The infimum of (5.3) over all $\eta \in \omega$ gives

$$1 \leq n! \left(\|\omega\|^*\right)^n \mathrm{vol}_{2n}\left(\mathbb{CP}^n, \mathcal{G}\right), \tag{5.4}$$

where $\| \, \|^*$ is the comass norm in cohomology. Denote by $\| \, \|$ the stable norm in homology. Recall that the normed lattices $(H_2(M;\mathbb{Z}), \| \, \|)$ and $(H^2(M;\mathbb{Z}), \| \, \|^*)$ are dual to each other [Fed69]. Therefore the class α of (5.1) satisfies

$$\|\alpha\| = \frac{1}{\|\omega\|^*},$$

and hence

$$\mathrm{stsys}_2(\mathcal{G})^n = \|\alpha\|^n \leq n! \, \mathrm{vol}_{2n}(\mathcal{G}). \tag{5.5}$$

Equality is attained by the two-point homogeneous Fubini-Study metric, since the standard $\mathbb{CP}^1 \subset \mathbb{CP}^n$ is calibrated by the Fubini-Study Kahler 2-form, which satisfies equality in the Wirtinger inequality at every point. □

Example 5.3. Every metric $\mathcal{G}$ on the complex projective plane satisfies the optimal inequality

$$\mathrm{stsys}_2(\mathbb{CP}^2, \mathcal{G})^2 \leq 2 \, \mathrm{vol}_4(\mathbb{CP}^2, \mathcal{G}).$$

6. Other inequalities due to Gromov

There is a number of inequalities in the systolic literature that could be described as Gromov's inequality.

The deepest result in systolic geometry is Gromov's inequality for the homotopy 1-systole of essential manifolds. Gromov's original definition of an essential manifold M depends on the choice of the coefficient ring A, taken to be $\mathbb{Z}$ if M is orientable, or $\mathbb{Z}_2$ otherwise. We then have a nonzero fundamental homology class $[M] \in H_n(M, A)$.

Definition 6.1. A closed n-dimensional manifold M is called *essential* if there exists a map from M to a suitable Eilenberg-MacLane space $K(\pi, 1)$ such that the induced homomorphism

$$h : H_n(M, A) \to H_n(K(\pi, 1), A)$$

maps the fundamental class $[M]$ to a nonzero class in the homology group $H_n(K(\pi, 1), A)$, *i.e.* $h([M]) \neq 0$.

A more general definition of an n-essential space X, in the context of an arbitrary polyhedron X, can be defined in terms of arbitrary local coefficients.

The following theorem was proved in [Gro83, Section 0] and [Gro83, Appendix 2, p. 139, item B_1'].

Theorem 6.2 (M. Gromov)**.** *Every n-essential, compact, n-dimensional polyhedron X satisfies the inequality*

$$\mathrm{sys}\pi_1(X)^n \le C_n \,\mathrm{vol}_n(X). \tag{6.1}$$

where the constant C_n depends only on n. If X is a manifold, the constant C_n can be chosen to be on the order of n^{2n^2}.

In other words, the quotient

$$\frac{\mathrm{vol}_n}{(\mathrm{sys}\pi_1)^n} > 0 \tag{6.2}$$

is bounded away from zero for such polyhedra.

A summary of a proof appears in [Ka07, Section 12.2].

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