88-826 Differential Geometry, moed B
Bar Ilan University, Prof. Katz
Date: 27 aug '18
Duration of the exam: 3 hours
Each of 5 problems is worth 20 points; bonus problem is worth 10 points
All answers must be justified by providing complete proofs

1. Does there exist a constant $C \in \mathbb{R}$ such that the following relation holds for all infinitesimal ε, and if so for which C exactly does the relation hold?
   (a) $\tan\varepsilon \sqcap C\varepsilon^2$;
   (b) $1 - e^{\varepsilon} \sqcap C\varepsilon$;
   (c) $\ln(\cos\varepsilon) \sqcap C\varepsilon$.

2. Consider the real projective space $\mathbb{RP}^n$.
   (a) Specify a collection of coordinate charts covering $\mathbb{RP}^n$.
   (b) Prove that $\mathbb{RP}^n$ is a manifold.
   (c) Verify the metrizability condition for $\mathbb{RP}^n$.

3. This problem deals with flows on manifolds.
   (a) Define the notion of a flow $\theta(t,p)$ on a manifold M and the notion of an infinitesimal generator X of the flow θ.
   (b) Prove that a differentiable vector field on a manifold is invariant under its flow θ_t.

4. Let Φ and G be D^1 prevector fields on $\mathbb{R}^n$ generated respectively by displacements δ_Φ and δ_G.
   (a) Prove that $a \mapsto a + \delta_\Phi + \delta_G$ is also a D^1 prevector field.
   (b) Prove that $\Phi \circ G$ and $G \circ \Phi$ are equivalent prevector fields.

5. This problem deals with oscillations of a pendulum.
   (a) Give the equation of motion of a pendulum of length ℓ subject to acceleration g under gravity.
   (b) Present the problem as a first order system.
   (c) Define an associated hyperreal walk.
   (d) Prove that for an appropriate choice of infinitesimal λ and prevector field, the hyperreal walk associated with oscillations with amplitude $\prec \lambda$ is periodic with period independent of the amplitude, and specify the period.

Bonus question. Let f be a (possibly discontinuous) function on $I = [a,b] \subseteq \mathbb{R}$ satisfying $|f(x) - f(y)| \le (x-y)^2$ for all $x, y \in I$, and let $H \in {}^*\mathbb{N} \setminus \mathbb{N}$. Use the partition of ${}^*I$ defined by H to prove that f is constant.

GOOD LUCK!