88-826 Differential Geometry, moed A
Bar Ilan University, Prof. Katz
Date: 25 july '21
Duration of the exam: 3 hours
Each of 4 problems is worth 25 points; the bonus problem is 10 points
All answers must be justified by providing complete explanations and proofs

1. Let M be a closed connected 10-dimensional manifold. Assume that $b_2(M) = 1$ and that a class $\omega \in H^2_{dR}(M)$ satisfies $\omega^{\cup 5} \neq 0$.
   (a) Consider a metric g on M. Give detailed definitions of the norm $\| \ \|$ in $\Lambda^2(T_pM)$; the norm $\| \ \|_\infty$ in $\Omega^2 M$; and the norm $\| \ \|^*$ in de Rham cohomology.
   (b) Give detailed definitions of the stable norm and of the duality between the stable norm and the comass norm.
   (c) Give a detailed definition of what it means for a de Rham class $\omega \in H^2_{dR}(M)$ to be an integer class.
   (d) Let $\eta \in \omega$ be a representative differential 2-form. Estimate the integral $\int_M \eta^{\wedge 5}$ in terms of the comass of η as well as the total volume $\mathrm{vol}(M)$ of M.
   (e) Use part (d) to provide (with proof) the best upper bound for the ratio $\mathrm{stsys}_2(g)^5/\mathrm{vol}(g)$.

2. This problem deals with de Rham cohomology.
   (a) Compute (with proof) the de Rham cohomology group $H^0_{dR}(\mathbb{R}/\mathbb{Z})$.
   (b) Compute (with proof) the group $H^1_{dR}(\mathbb{R}/\mathbb{Z})$.
   (c) Compute (with proof) the group $H^2_{dR}(\mathbb{R}/\mathbb{Z})$.
   (d) Let $L \subseteq \mathbb{C}$ be the Gaussian integers. Compute (with proof) the de Rham cohomology group $H^2_{dR}(\mathbb{C}/L)$.

3. For each of the lattices $L_n \subseteq \mathbb{C}$, compute the conformal parameter $\tau(\mathbb{C}/L_n)$:
   (a) L_1 spanned by the roots of the polynomial $z^3 + 27$;
   (b) L_2 spanned by $1 + i$ and $2 - 2i$;
   (c) L_3 spanned by i and $1 + 4i$.

4. This question deals with orientations on manifolds.
   (a) Let M be a 3-manifold with boundary S. Suppose ori_M is a differential form representing an orientation on M. Give a detailed definition of the notion of the induced orientation represented by ori_S on the boundary S.
   (b) Let $\rho > 0$. Consider the bounded region $D = \{z \in \mathbb{C} \colon z\bar{z} \leq \rho^2\}$ endowed with the standard orientation $\frac{i}{2} dz \wedge \overline{dz}$. Calculate the induced orientation on ∂D and compare it to $d\theta$.

B. (**bonus**) Let $M_n = S^1 \times S^n$. Determine for which n is there a uniform upper bound (valid for all metrics) for the ratio $\frac{\mathrm{sys}_1(M)^{n+1}}{\mathrm{vol}(M)}$.

GOOD LUCK!