March 26, 2020

Differential geometry 88-826 homework set 2

1. Let $M = T^2$ be the 2-torus. Prove that the tangent bundle of M can be naturally identified with the product $T^2 \times \mathbb{R}^2$.

2. Consider the unit sphere S^2 in spherical coordinates (θ, φ). Consider the vector fields $\frac{\partial}{\partial \theta}$ and $\sin\varphi \, \frac{\partial}{\partial \varphi}$.

(a) Explain why both vector fields can be viewed as continuous vector fields defined everywhere on S^2.
(b) Find the zeros of the vector field $\frac{\partial}{\partial \theta}$ if any.
(c) compute the length of the vector field $\frac{\partial}{\partial \theta}$ at an arbitrary point with coordinates (θ, φ);
(d) Find the zeros of the vector field $\sin\varphi \, \frac{\partial}{\partial \varphi}$ if any.
(e) compute the length of the vector field $\sin\varphi \, \frac{\partial}{\partial \varphi}$ at an arbitrary point with coordinates (θ, φ).

3. Consider the real projective plane $\mathbb{RP}^2$ defined in the lecture as the collection of equivalence classes $[x]$ where $x \in \mathbb{R}^3 \setminus \{0\}$ (see choveret of the course, section 1.6, pages 14–16). Prove that the following two definitions are naturally equivalent to the one given in the lecture:

(1) Let S^2 be the unit 2-sphere. Then $\mathbb{RP}^2$ is the set of unordered pairs $\{p, -p\}$ where $p \in S^2$.
(2) Let $U \subseteq S^2$ be the upper hemisphere, namely the set $U = \big\{(x, y, z) \in S^2 \colon z \geq 0\big\}$. Then $\mathbb{RP}^2$ is obtained from U by identifying antipodal points on the equator by an equivalence relation $\sim$ where by definition $(x, y, 0) \sim (-x, -y, 0)$ whenever $x^2 + y^2 = 1$.