88-826 DIFFERENTIAL GEOMETRY HOMEWORK SET 1

1. The lattice L_E of Eisenstein integers is the lattice in $\mathbb{C} = \mathbb{R}^2$ spanned by the cube roots of unity. Find the dual lattice L_E^* to the lattice L_E and compute $\lambda_1(L_E^*)$.

2. The lattice of Gaussian integers is the lattice $L_G \subset \mathbb{C} = \mathbb{R}^2$ consisting of elements with integer coordinates, i.e. spanned by $(1, i)$. Find its dual lattice L_G^* in $\mathbb{C} = \mathbb{R}^2$ and compute $\lambda_1(L_G^*)$.

3. Let $a, b > 0$. Consider the lattice $L_{a,b} \subset \mathbb{R}^2$ spanned by ae_1 and be_2. Find the lattice $L_{a,b}^*$ dual to $L_{a,b}$ and compute $\lambda_1(L_{a,b}^*)$.

4. The 1-forms dr and $rd\theta$ form an orthonormal basis for the cotangent plane T_p^* at a point p of the plane other than the origin, in polar coordinates (r, θ). Find an orthonormal basis for the tangent plane T_p, by modifying the basis $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$.

5. Consider the cotangent line $T_p^* S^1$ at a point p of the circle of radius $r_0 > 0$. Consider the lattice $L_0 \subset T_p^*$ spanned by the 1-form $d\theta$. Calculate $\lambda_1(L_0)$.

6. Consider the tangent line T_p at a point p of the circle of radius $r_0 > 0$. Consider the lattice $L \subset T_p$ spanned by $\frac{\partial}{\partial \theta}$. Calculate $\lambda_1(L_1)$.

7. Prove that every skew-symmetric 3 by 3 matrix is orthogonally conjugate to a matrix of the form $\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and conjecture a 4-dimensional generalisation.

8. Consider the unit circle $S^1 \subset \mathbb{R}^2$, and its tangent bundle (eged hameshik) TS^1. Prove that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

9. Let V be a finite dimensional real vector space equipped with an real inner product $\langle\ ,\ \rangle$. Construct a natural isomorphism between V and its dual V^*.

10. A natural basis for the tangent plane of the (x, y)-plane is given by the vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. Consider the curve $\alpha(t)$ parametrizing the standard parabola: $\alpha(t) = (t, t^2)$. Identify the tangent vector $\alpha'(0)$ in terms of the natural basis.

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