A Two-Track Tour of Cauchy's *Cours*

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1 Rigour, then and now

Building upon pioneering work by Kepler, Fermat, Cavalieri, Gregory, Wallis, Barrow and others, Isaac Newton and Gottfried Wilhelm von Leibniz invented calculus in the 17th century. While immediately acquiring an enthusiastic following, the new methods proved to be controversial in the eyes of some of their contemporaries, who employed the more traditional methods of their predecessors. One of the controversial aspects of the new technique was Leibniz's distinction between assignable and unassignable quantities (including infinitesimals and infinite quantities [1]).

At the French Academy, the opposition to the new calculus was led by Michel Rolle, and across the Channel, by George Berkeley. The scientific success of the new methods ultimately silenced the opposition, but lingering doubts persisted (fed in part by doctrinal theological issues [2]). A new era was ushered in by Augustin-Louis Cauchy's textbook *Cours d'Analyse*, addressed to the students of the École Polytechnique in Paris.

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

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Cauchy and the title page of Cours d'Analyse

Cauchy published his *Cours d'Analyse* (CDA) 200 years ago. The book was of fundamental importance for the development of both real and complex analysis. Hans Freudenthal mentioned in his essay on Cauchy (for the *Dictionary of Scientific Biography* [3]) that Niels Henrik Abel described the CDA as 'an excellent work which should be read by every analyst who loves mathematical rigor.' But what did *rigour* mean to Abel and Cauchy?

In the early 19th century context, the term *rigour* referred to the standard of mathematical precision set by the geometry of Euclid. This context enables us to understand Cauchy's intention when, in the introduction to CDA [4, p. 1], he referred to 'all the rigor which one demands from geometry,' committing himself further to 'never rely on arguments drawn from the generality of algebra.' We see that Cauchy's notion of rigour in CDA is distinct from 'what has been called the nineteenth-century "rigorization" of real analysis' [5, p. 221].

Like the term *rigour*, Cauchy's term *generality of algebra* requires explanation to be comprehensible to modern readers. It refers to certain techniques used by his predecessors, particularly Euler and Lagrange, that today would be considered cavalier, specifically (a) proofs based on the algebraic manipulation of divergent series and (b) the idea that algebraic rules and formulas valid in the real domain remain valid in the complex domain. Some of the proofs that fall under item (a) have since been justified using summation techniques developed later, whereas some of the techniques under item (b) have been verified in terms of analytic continuation.

By the standards of the current century, some of Freudenthal's comments could be considered controversial. Thus, Freudenthal writes [3, p. 137]:

> Terms like 'infinitesimally small' prevail in Cauchy's limit arguments and epsilontics still looks far away, but there is one exception. His proof … of the well-known theorem
> If $\lim_{x\to\infty}(f(x+1)-f(x))=\alpha$,
> then $\lim_{x\to\infty} x^{-1}f(x)=\alpha$
> is a paragon, and the first example, of epsilontics – the character ε even occurs there.

The claim that infinitesimals 'prevail' in Cauchyan foundations of analysis, whereas (ε,δ) arguments 'look far away' and are limited to a small number of exceptions, may surprise a reader whose perceptions of Cauchyan rigour are influenced by Judith Grabiner's views [6] and publications that followed, especially if they tend to identify *rigour* with the jettisoning of infinitesimals in favour of (ε,δ) arguments based on an 'algebra of inequalities'. Some historians today would view both Freudenthal's and Grabiner's perspectives as outdated. But the dual view of Cauchyan analysis persists in the current literature.

2 Dual tracks

The track-A view holds that Cauchy, ahead of his time, worked primarily with an Archimedean continuum and pioneered many of the techniques that would become known as (ε,δ) in the next century.

The track-B view holds that Cauchy, like most of his contemporaries and colleagues at the École Polytechnique, based his analysis primarily on variable quantities and infinitesimals; see Laugwitz [7, 8].

What were then the Cauchyan foundations of analysis? While we will not purport to provide a definitive answer in this short note, we will let Cauchy speak for himself (using the translation by Bradley and Sandifer [4]). Cauchy writes in the introduction to CDA [4, p. 1]:

> In speaking of the continuity of functions, I could not dispense with a treatment of the principal properties of infinitely small quantities, properties which serve as the foundation of the infinitesimal calculus.

Track-A advocates read this passage as a concession to the management of the École and argue that Cauchy 'could not dispense with a treatment of … infinitely small quantities' because of explicit mandates from the École, against his better judgement.

Track-B advocates read this passage as a recognition by Cauchy (in a departure from his pre-1820 approaches) that a convincing and accessible treatment of continuity necessitates infinitesimals, and they note that Cauchy's favourable judgement of infinitesimals is corroborated by their use in his research long after the end of his teaching stint at the École [9].

Readers searching for an (ε, δ) definition of limit in CDA may be surprised to find instead the following definition [4, p. 6]:

> We call a quantity *variable* if it can be considered as able to take on successively many different values. ... When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the *limit* of all the other values.

Here the notion of a variable quantity is taken as primary, and limits are defined in terms of variable quantities. Variable quantities similarly provide the basis for the definition of infinitesimals [4, p. 7]:

> When the successive numerical values of such a variable decrease indefinitely, in such a way as to fall below any given number, this variable becomes what we call *infinitesimal*, or an *infinitely small quantity*. A variable of this kind has zero as its limit.

Track-A advocates read this as asserting that an infinitesimal is merely a null sequence (i.e. a sequence tending to zero), and take the last sentence to refer to infinitesimals. Track-B advocates point out that Cauchy did not write that a variable quantity *is* an infinitesimal but rather that a variable quantity *becomes* an infinitesimal, implying a change in nature (from being a variable quantity to being an infinitesimal). They take the last sentence to refer to the variable quantity mentioned at the beginning of the passage.

Returning to limits, Cauchy writes [4, p. 12]:

> When a variable quantity converges towards a fixed limit, it is often useful to indicate this limit with particular notation. We do this by placing the abbreviation lim in front of the variable quantity in question.

Bradley and Sandifer note that the 1821 edition of the CDA used the notation 'lim.' (with a full stop). Cauchy's description of 'lim.' as an abbreviation suggests that he viewed it as secondary to, or an aspect of, the concept of a variable quantity.

In Chapter 2, Cauchy returns to the definition of infinitesimals [4, p. 21]:

> We say that a variable quantity becomes *infinitely small* when its numerical value decreases indefinitely in such a way as to converge towards the limit zero.

The relation between the concepts of variable quantity and infinitesimal was already discussed above, as well as the possible ambiguity of the verb *becomes*; limits again play a secondary role. Cauchy proceeds next to the properties of infinitesimals [4, p. 22]:

> Infinitely small and infinitely large quantities enjoy several properties that lead to the solution of important questions, which I will explain in a few words. Let α be an infinitely small quantity, that is a variable whose numerical value decreases indefinitely.

Track-A advocates point out that here Cauchy states that an infinitesimal is a variable quantity. Track-B advocates note that here Cauchy is no longer dealing with detailed definitions, and this particular formulation is merely shorthand for the more careful definition in terms of *becoming* elaborated earlier.

3 Continuity in 1817, 1821 and beyond

Cauchy's first documented characterisation of continuity is found in a record of a course summary dating from March 1817 (a month before the earliest written mention of Bolzano's *Rein analytischer Beweis* in an Olms catalogue). The definition can be described as reasonably precise in the sense of enabling a straightforward transcription as an impeccable modern definition [10]. In modern mathematics, a real function φ is continuous at $c \in \mathbb{R}$ if and only if for each sequence (x_n) converging to c, one has

$$\varphi(c) = \varphi\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} \varphi(x_n), \tag{1}$$

or briefly $\varphi \circ \lim = \lim \circ \varphi$ at c, expressing the commutation of φ and lim. In 1817, Cauchy wrote [10, p. 209]:

> The limit of a continuous function of several variables is [equal to] the same function of their limit. Consequences of this Theorem with regard to the continuity of composite functions dependent on a single variable.

(Being part of a summary, the second phrase is not a complete sentence.) Cauchy's 1817 characterisation of continuity in terms of the commutation of φ and lim as in (1) does not use infinitesimals and thus contrasts with his definitions involving infinitesimals given four years later in CDA. Surprisingly, it is the 1817 characterisation that is actually used in CDA; see Section 5.

Here is Cauchy's first definition of continuity in CDA [4, p. 26]:

> Let $f(x)$ be a function of the variable x, and suppose that for each value of x between two given limits, the function always takes a unique finite value. If, beginning with a value of x contained between these limits, we add to the variable x an *infinitely small increment* α, the function itself is incremented by the difference $f(x+\alpha) - f(x)$, which depends both on the new variable α and on the value of x. Given this, the function $f(x)$ is a *continuous* function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference $f(x+\alpha) - f(x)$ decreases indefinitely with the numerical value of α.

Note that, while the increment α is described as infinitesimal, the resulting change $f(x+\alpha) - f(x)$ is not. This 1821 definition can be seen as intermediary between the March 1817 characterisation in terms of variables (not mentioning infinitesimals) and his second 1821 definition stated purely in terms of infinitesimals [4, p. 26]:

> In other words, *the function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.*

Significantly, Cauchy's abbreviation 'lim.' appeared in none of the definitions of continuity given in CDA. It is the second definition purely in terms of infinitesimals that reappears in the following works by Cauchy:

- *Résumé des Leçons* (1823), English translation [11, p. 9]
- *Leçons sur le Calcul Différentiel* (1829) [12, p. 9]
- *Mémoire sur l'Analyse Infinitésimale* (1844) [13, p. 17]
- the 1853 article [14] on the sum theorem (see Section 6)

Nonetheless, the perception that Cauchy allegedly 'sought to establish foundations for real analysis that gave no role to infinitesimals'(e.g. [15]) has firmly entered the canonical creation narrative of modern mathematical analysis.

4 Infinitesimals without choice

Cauchy's definition of continuity in CDA, in its B-track interpretation, is harder to follow for modern readers more familiar with the (ε, δ) definition of continuity à la Weierstrass and Dini than with the definition in a modern infinitesimal theory. We therefore provide a formalisation of Cauchy's procedures involving continuity in terms of the theory SPOT (acronym of its axioms) developed in [16]. SPOT has the advantage of being conservative over the traditional Zermelo–Fraenkel set theory (ZF) and therefore depends on neither the axiom of choice nor the existence of ultra-filters.

The language of ZF is limited to the two-place membership relation $\in$. The language of SPOT includes also a predicate ST, where ST(x) reads 'x is standard.' Such a distinction between standard and non-standard entities can be thought of as formalising the Leibnizian distinction between assignable and unassignable quantities [1]. The standard ordered field $\mathbb{R}$ has both standard and non-standard elements. An element α is *infinitesimal* if $|\alpha| < r$ for each standard $r > 0$. Let x be a standard point in the domain of a real standard function f. Then f is continuous at x (in the traditional sense of the (ε, δ) definition) if and only if

$$\begin{array}{l}\text{infinitesimal } \alpha \\ \text{produce infinitesimal changes } f(x+\alpha) - f(x),\end{array} \tag{2}$$

whenever $x + \alpha$ is in the domain of f. Continuity in an interval, say $(0, 1)$, is equivalent to the satisfaction of condition (2) at every standard point $x \in (0, 1)$.

It should be mentioned that, according to most scholars, Cauchy did not formulate the notion of continuity at a point, but only 'continuity between limits' (i.e. in an interval) or in a neighbourhood of a point. Freudenthal remarks [3, p. 137]:

> It is the weakest point in Cauchy's reform of calculus that he never grasped the importance of uniform continuity.

From an A-track viewpoint, the weakness is that there seems to be no trace in Cauchy of the idea that a significant issue is whether the allowable error is independent of the point x or not.

From a B-track viewpoint, the weakness is that Cauchy did not make it clear whether condition (2) is expected to be satisfied only at assignable points x or at all points of the interval. Note that uniform continuity of f on, say, $(0, 1)$ is equivalent to (2) being satisfied at all points of $(0, 1)$. For example, $1/x$ fails to be uniformly continuous because of the failure of (2) at an infinitesimal input $x = \beta > 0$: indeed, the change

$$\frac{1}{\beta + \alpha} - \frac{1}{\beta}$$

is not infinitesimal if, say, $\alpha = \beta$. See further in Section 6.

By Section 2.3, Cauchy reaches the theorem described by Freudenthal as a *paragon* of (ε, δ) arguments. What Cauchy actually shows is that if $f(x+1) - f(x)$ is between $k - \varepsilon$ and $k + \varepsilon$ then (assuming monotonicity) $f(x)/x$ is similarly between $k - \varepsilon$ and $k + \varepsilon$. If anything this is a paragon of $(\varepsilon, \varepsilon)$ arguments, since here the δ equals ε! The trademark feature of modern (ε, δ) arguments, namely an explicit (non-trivial) dependence of δ on ε, does not appear anywhere in Cauchy's alleged 'algebra of inequalities', lending support to Freudenthal's sentiment that 'epsilontics looks far away ' [3, p. 137].

Here Cauchy mentions that an infinite limit is 'larger than any *assignable* number' (e.g. [4, p. 37]) indicating familiarity with this Leibnizian term (which occurs nine times in CDA).

5 Functional equations and continuity

In Chapter 5, Cauchy studies functional relations for continuous functions, and treats the following problem [4, p. 71]:

> Problem I. – To determine the function $\varphi(x)$ in such a manner that it remains continuous between any two real limits of the variable x and so that for all real values of the variables x and y, we have $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Cauchy arrives at

$$\varphi\Big(\frac{m}{n}\alpha\Big) = \frac{m}{n}\varphi(\alpha)$$

and argues as follows [4, p. 72]:

> Then, by supposing that the fraction m/n varies in such a way as to converge towards any number μ, and passing to the limit, we find that $\varphi(\mu\alpha) = \mu\varphi(\alpha)$.

Here Cauchy exploits the 1817 characterisation of the continuity of φ in terms of the commutation of φ and lim as summarised in formula (1), rather than the definitions presented in Chapter 2 of CDA (the 1817 characterisation is also used in the proof of the intermediate value theorem). Curiously, Cauchy provides no explanatory comment. Possibly, Cauchy wrote the material in Chapter 5 with definition (1) in mind, and introduced the definitions in Chapter 2 at a later stage in the writing of the book. Cauchy applies a similar technique to study the functional relation $\varphi(x+y) = \varphi(x)\varphi(y)$ and other variations.

6 Sum theorem and convergence *always*

Chapter 6 includes Cauchy's controversial sum theorem. We summarise the historical facts. The 1821 formulation of the theorem appears to be incorrect to the modern reader, as it seems to assert that pointwise convergence of a series of continuous functions implies the continuity of the sum. Already in 1826 Abel pointed out that the theorem 'suffers exceptions'. Cauchy was curiously silent on the subject of the sum theorem for several decades. Then in 1853 he presented a modified statement of the sum theorem, mentioned an example similar to Abel's (without mentioning Abel by name) and explained why the example does not contradict the (modified) theorem. Numerous scholars have attempted to explain what the modification was (if any) and to interpret Cauchy's sum theorem in terms intelligible to modern audiences.

Cauchy considers the series obtained as the sum of the terms of the sequence $u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$, denoted (1). In its 1821 formulation, the theorem asserts the following [4, p. 90]:

> When the various terms of series (1) are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value.

Here is the 1853 formulation [14, pp. 456–457, our translation]:

> If the various terms of the series
>
> $$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots \tag{1}$$
>
> are functions of a real variable x and are continuous with respect to this variable between the given bounds, and if, furthermore, the sum
>
> $$u_n + u_{n+1} + \ldots + u_{n'-1} \tag{3}$$
>
> *always* becomes infinitely small for infinitely large values of the whole numbers n and $n' > n$, then the series (1) will converge and the sum s of the series will be, between the given bounds, a continuous function of the variable x.

Note that Cauchy adds the word *toujours* (always). But what exactly is supposed to happen always in 1853, and how does this modify the 1821 hypothesis? Cauchy himself provides a hint in his 1853 analysis of the example

$$\sum_n \frac{\sin nx}{n},$$

representing a (discontinuous) sawtooth waveform. But the hint he provides is itself puzzling. Cauchy evaluates $u_n + u_{n+1} + \ldots + u_{n'-1}$ at $x = 1/n$ and shows that the sum does not become arbitrarily small, and in fact can be made 'sensibly equal' to the integral

$$\int_1^\infty \frac{\sin x}{x}\, \mathrm{d}x = 0.6244\ldots$$

Here Cauchy explicitly describes n (and n') as infinite; then $x = 1/n$ is infinitesimal.

Many scholars of both A-track and B-track persuasions have argued that Cauchy meant to add what is known today as the condition of uniform convergence.

A-track advocates may interpret the condition as requiring the sums $u_n + u_{n+1} + \ldots + u_{n'-1}$ to be small independently of n, n' and also of the input x, a condition that can be stated formally in terms of an alternating quantifier string of the type

$$\forall \varepsilon > 0, \exists \delta > 0, \forall n < n' \in \mathbb{N}, \forall x,$$
$$\left(n > \frac{1}{\delta} \to |u_n(x) + u_{n+1}(x) + \ldots + u_{n'-1}(x)| < \varepsilon \right).$$

Something along these (long) lines would have to be attributed to Cauchy, in inchoate form, in order to interpret the addition of uniform convergence in an A-track fashion. What is unclear is how the word 'always' manages to allude to such independence, particularly since Cauchy seems to have overlooked its significance in the context of continuity (see Freudenthal's remark quoted in Section 4 and the ensuing discussion). Moreover, the evaluation at what seems to be a new type of point, namely $x = 1/n$, appearing in Cauchy's discussion of

$$\sum_n \frac{\sin nx}{n},$$

suggests that the insistence on the qualifier *always* indicates an extension of the inputs x to include additional ones (that were not 'always' included before).

A B-track reading of Cauchy's 1853 hypothesis interprets the qualifier *always* as referring to additional unassignable inputs x (including infinitesimal values such as $1/n$ for infinite n). If one requires the sum $u_n + u_{n+1} + \ldots + u_{n'-1}$ to be infinitesimal for all infinite n, n' and all inputs x (standard and non-standard) then one indeed obtains a condition equivalent to uniform convergence [17, Theorem 4.6.1, p. 116], guaranteeing the continuity of the limit function.

7 Conclusion

Though we have sketched widely divergent readings of Cauchy's definitions and his theorems, we hope to have conveyed to the reader a sense that not only the concept of *rigour* but also *continuity* and *limit* may have had a different meaning to Cauchy than they do to us today, underscoring the contingency of the historical evolution of mathematics. While Cauchy incontestably made extensive use of bona fide infinitesimals in fields as varied as differential geometry, elasticity theory and geometric probability [9], the nature of his foundational stance remains controversial. But perhaps this is as it should be: with a genius of Cauchy's calibre, tidy construals of his work may necessarily amount to a flattening of his multi-dimensional vision.

References

1 Bair, J. et al. (2021) Procedures of Leibnizian infinitesimal calculus: An account in three modern frameworks, *Br. J. Hist. Math.*, vol. 36, no. 2, pp. 1–40.

2 Bair, J., Katz, M. and Sherry, D. (2018) Fermat's dilemma: Why did he keep mum on infinitesimals? and the European theological context, *Found. Sci.*, vol. 23, no. 3, pp. 559–595.

3 Freudenthal, H. (1981) Cauchy, Augustin-Louis, in *Dictionary of Scientific Biography, Vol. 3*, ed. Gillispie, C., Charles Scribner's Sons, New York.

4 Bradley, R.E. and Sandifer, C.E. (2009) *Cauchy's Cour d'Analyse. An Annotated Translation*, in *Sources and Studies in the History of Mathematics and Physical Sciences*, Springer, New York.

5 Kanamori, A. (2021) Cantor and continuity, in *The History of Continua: Philosophical and Mathematical Perspectives*, eds Shapiro, S. and Hellman, G., Oxford University Press, Oxford, pp. 219–254.

6 Grabiner, J. (1981) *The Origins of Cauchy's Rigorous Calculus*, MIT Press, Cambridge MA, London.

7 Laugwitz, D. (1987) Infinitely small quantities in Cauchy's textbooks, *Hist. Math.*, vol. 14, no. 3, pp. 258–274.

8 Laugwitz, D. (1989) Definite values of infinite sums: Aspects of the foundations of infinitesimal analysis around 1820, *Arch. Hist. Exact Sci.*, vol. 39, no. 3, pp. 195–245.

9 Bair, J. et al. (2020) Cauchy's work on integral geometry, centers of curvature, and other applications of infinitesimals, *Real Anal. Exch.*, vol. 45, no. 1, pp. 127–150.

10 Bair, J. et al. (2020) Continuity between Cauchy and Bolzano: Issues of antecedents and priority, *Br. J. Hist. Math.*, vol. 35, no. 3, pp. 207–224.

11 Cates, D. (2019) *Cauchy's Calcul Infinitésimal. An Annotated English Translation*, Springer, Cham.

12 Cauchy, A.L. (1829) *Leçons sur le Calcul Différentiel*, Bure Frères, Paris.

13 Cauchy, A.L. (1844) *Exercices d'Analyse et de Physique Mathématique. Mémoire sur l'Analyse Infinitésimale*, Bachelier, Paris.

14 Cauchy, A.L. (1853) Note sur les séries convergentes dont les divers termes sont des fonctions continues d'une variable réelle ou imaginaire, entre des limites données, *Compt. Rend. Acad. Roy. Sci.*, vol. 36, pp. 454–459.

15 Oppy, G. et al. (2021) Infinity, in *Stanford Encyclopedia of Philosophy*, plato.stanford.edu/entries/infinity.

16 Hrbacek, K. and Katz, M. (2021) Infinitesimal analysis without the Axiom of Choice, *Ann. Pure Appl. Logic*, vol. 172, no. 6, p. 102959.

17 Robinson, A. (1966) *Non-Standard Analysis*, North-Holland, Amsterdam.

letters

Musical Necklaces Reprise

Further to my recent letter [1], one of the difficulties mentioned in David Thomson's article on counting chord types in a p-note equi-tempered scale is the calculation of the article's functions J_t and $J_{t,m}$ [2]. These functions count certain chord types but are defined implicitly. Möbius inversion of the formulas given in the article produces the following explicit results by which the function values can be calculated directly:

$$J_t = \sum_{g|t} \mu(g) 2^{t/g} \qquad \text{for } t \geq 1 \text{ and } t|p,$$

$$J_{t,m} = \sum_{g|\gcd(t,m)} \mu(g) \binom{t/g}{m/g} \qquad \text{for } t \geq 1, m \geq 0,$$

where $\mu(g)$ is the Möbius function [3, A008683]. For example, in a 12-note equi-tempered scale, the number of chords with 6 distinct transposed versions is

$$\begin{aligned} J_6 &= \mu(1)2^6 + \mu(2)2^3 + \mu(3)2^2 + \mu(6)2^1 \\ &= 64 - 8 - 4 + 2 = 54, \end{aligned}$$

and the number of chord types comprising chords with 12 transposed versions, each with 4 notes, is

$$\begin{aligned} I^{12}_{12,4} &= \frac{J_{12,4}}{12} \\ &= \frac{\mu(1)\binom{12}{4} + \mu(2)\binom{6}{2} + \mu(4)\binom{3}{1}}{12} \\ &= \frac{495 - 15 + 0}{12} = 40, \end{aligned}$$

in agreement with the values given in the article.

Returning to the general musical polygon/necklace isomorphism noted in [1], the functions J_t and $J_{t,m}$ derived above for certain chord types are each related to certain necklace types. In the article's notation:

$$\begin{aligned} I_t &= J_t/t, \\ I^p_{t,n} &= J_{t,nt/p}/t, \end{aligned}$$

are discusssed in [3, A001037 and A185158], and see [4] for further connections.

It is also a useful student exercise to use the formulas for I_t and J_t above to derive the alternative formula for the total number of chord types given in [1] directly; i.e., show

$$\begin{aligned} \sum_{t|p} I_t &= \sum_{t|p} \sum_{g|t} \mu(g) 2^{t/g}/t \\ &= \frac{1}{p} \sum_{g|p} \varphi(g) 2^{p/g}, \end{aligned}$$

where $\varphi(g)$ is the Euler totient function [3, A000010], as observed along the bottom row of the article's table [2, p. 51]. Similarly, the formulas for $I^p_{t,n}$ and $J_{t,m}$ may be used to produce general results corresponding to the table's marginal totals.

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References

1 Ollerton, R.L. (2021) Musical necklaces, *Math. Today*, vol. 57, no. 3, p. 109.

2 Thomson, D. (2021) Musical polygons, *Math. Today*, vol. 57, no. 2, pp. 50–51, ima.org.uk/16507/musical-polygons.

3 The On-Line Encyclopedia of Integer Sequences (2021), oeis.org.

4 Meštrović, R. (2018) Different classes of binary necklaces and a combinatorial method for their enumerations, arXiv:1804.00992.

The d'Alembertian

James Moffat's excellent article on Quaternions and the Discovery of Antimatter (*Mathematics Today*, April 2021) used the Laplacian operator extended to four dimensional spacetime without noting that this extension is commonly referred to as the d'Alembertian. The article also used the del symbol (typically associated with the Laplacian) rather than the box symbol (typically associated with the d'Alembertian). The Editor apologies for any confusion caused by these slight deviations from the common approach.