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DETAILED SOLUTIONS FOR EXAM IN 88-201 MOED A JULY '17

Solution to Problem 1: Prove the theorema egregium of Gauss.

Consider the third partial derivative $x_{ijk} = \frac{\partial^3 x}{\partial u^i \partial u^j \partial u^k}$. Let us calculate its tangential component relative to the basis (x_1, x_2, n) for $\mathbb{R}^3$. Recall that $n_k = L^p{}_k x_p$ and $x_{jk} = \Gamma^{\ell}_{jk} x_\ell + L_{jk} n$. Thus, we have

$$\begin{aligned}
(x_{ij})_k &= \left(\Gamma^m_{ij} x_m + L_{ij} n\right)_k \\
&= \Gamma^m_{ij;k} x_m + \Gamma^m_{ij} x_{mk} + L_{ij} n_k + L_{ij;k} n \\
&= \Gamma^m_{ij;k} x_m + \Gamma^m_{ij}\left(\Gamma^p_{mk} x_p + L_{mk} n\right) + L_{ij}\left(L^p{}_k x_p\right) + L_{ij;k} n.
\end{aligned}$$

Grouping together the tangential terms, we obtain

$$\begin{aligned}
(x_{ij})_k &= \Gamma^m_{ij;k} x_m + \Gamma^m_{ij}\left(\Gamma^p_{mk} x_p\right) + L_{ij}\left(L^p{}_k x_p\right) + (\ldots)n \\
&= \left(\Gamma^q_{ij;k} + \Gamma^m_{ij}\Gamma^q_{mk} + L_{ij} L^q{}_k\right) x_q + (\ldots)n \\
&= \left(\Gamma^q_{ij;k} + \Gamma^m_{ij}\Gamma^q_{km} + L_{ij} L^q{}_k\right) x_q + (\ldots)n,
\end{aligned}$$

since the symbols Γ^q_{km} are symmetric in the two subscripts. The symmetry in j, k (equality of mixed partials) implies the identity $x_{i[jk]} = 0$. Hence $0 = x_{i[jk]} = (x_{i[j})_{k]} = \left(\Gamma^q_{i[j;k]} + \Gamma^m_{i[j}\Gamma^q_{k]m} + L_{i[j} L^q{}_{k]}\right) x_q + (\ldots)n$ and therefore $\Gamma^q_{i[j;k]} + \Gamma^m_{i[j}\Gamma^q_{k]m} + L_{i[j} L^q{}_{k]} = 0$ for each $q = 1, 2$. We now choose the valus $i = j = 1$ and $k = q = 2$ for the indices, to obtain $\Gamma^2_{1[1;2]} + \Gamma^m_{1[1}\Gamma^2_{2]m} = -L_{1[1} L^2{}_{2]} = g_{1i} L^i{}_{[1} L^2{}_{2]} = g_{11} L^1{}_{[1} L^2{}_{2]}$ since the term $L^2{}_{[1} L^2{}_{2]} = 0$ vanishes. This yields the desired formula for $K = \det(W_p)$ and completes the proof of the *theorema egregium*.

Solution to Problem 2: Consider the quadratic form $Q(x, y) = -3x^2 + 4xy - 6y^2$.

(a) To characterize the plane curve $Q(x, y) = -1$, we write down the explicit equation $-3x^2 + 4xy - 6y^2 = -1$. Changing the sign, we obtain $3x^2 - 2 \cdot 2xy + 6y^2 = 1$. Thus we have the coefficients $a = 3, b = -2, c = 6$. Note two points:

1. (1) $ac - b^2 = 18 - 4 = 14 > 0$ (the expression $ac - b^2$ is positive);
2. (2) This conic section is not degenerate since it contains at least two points ($x = \frac{\pm 1}{\sqrt{3}}, y = 0$).

Therefore by the classification theorem proved in class, this conic section is an ellipse.

(b) To characterize the surface $z = Q(x, y)$ we write down the explicit equation of the quadric surface: $z = -3x^2 + 4xy - 6y^2$, or $-3x^2 + 4xy - 6y^2 - z = 0$ or equivalently $3x^2 - 4xy + 6y^2 + z = 0$. The corresponding matrix S is the matrix $S = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note the following:

(1) The matrix S has rank 2 and can be diagonalized in coordinates (x', y', z);
(2) the coefficent of the linear term z is nonzero.

By the classification theorem proved in class, the surface is a paraboloid. Recall that paraboloids are of two types: elliptic paraboloid and hyperbolic paraboloid. Note that $\det \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} > 0$ and therefore the eigenvalues have the same sign. By the classification theorem, the quadric surface is an *elliptic paraboloid*.

Finally, the origin is a critical point of the function $Q(x, y)$. By the theorem proved in class, at such a point the Gaussian curvature K of the surface given by the graph of the function equals the determinant of the Hessian H_Q of Q. Differentiating, we obtain $H_Q = \begin{pmatrix} -6 & 4 \\ 4 & -12 \end{pmatrix}$, and therefore $K = \det(H_Q) = 56$.

Solution to Problem 3: We look for the point of maximal curvature of curves. We use the formula $k = \frac{|D_B(F)|}{|\nabla F|^3}$ to compute the curvature. Here $D_B(F) = F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2$.

(a) Curve $x + y^2 = 1$. We define $F(x, y) = x + y^2 - 1$. We have $F_x = 1$, $F_y = 2y$, $F_{xx} = 0$, $F_{xy} = 0$, $F_{yy} = 2$. Therefore $D_B(F) = F_{yy}F_x^2 = 2 \cdot 1 = 2$. Meanwhile $|\nabla F| = \sqrt{1 + 4y^2}$. Thus $k = \frac{2}{(1+4y^2)^{3/2}}$. The curvature is maximal when the expression $1 + 4y^2$ is minimal, i.e., when $y = 0$. At this point $x = 1$ from the equation of the curve. Thus the point $(x, y) = (1, 0)$ (the apex of the parabola) is the unique point where the curvature is maximal.

(b) Curve $xy + 1 = 0, x > 0$. We set $F(x, y) = xy + 1$. Then $F_x = y$, $F_y = x$, $F_{xx} = 0$, $F_{xy} = 1$, $F_{yy} = 0$. Therefore $D_B(F) = -2F_{xy}F_xF_y = -2xy$. From the equation of the curve we have $xy = -1$ and therefore $D_B(F) = 2$. Thus $k = \frac{2}{(x^2+y^2)^{3/2}}$. It remains to mimimize the expression $x^2 + y^2$ along the curve. We have $x > 0$ by definition of the curve. We exploit the defining relation $xy = -1$ of the curve to obtain $x^2 + y^2 = (x - y)^2 + 2xy = (x - y)^2 - 2 = \left(x + \frac{1}{x}\right)^2 - 1$. To minimize this expression, it suffices to minimize the quantity $x + \frac{1}{x}$. We will show that the sum $x + \frac{1}{x}$ is minimal when $x = 1$. Namely, to show that $x + \frac{1}{x} \geq 2$, rewrite the inequality as $x^2 + 1 \geq 2x$ or $x^2 + 1 - 2x \geq 0$ or equivalently $(x - 1)^2 \geq 0$ which is a true inequality. Hence the maximum of the curvature is when $x = 1$ and so $y = -\frac{1}{x} = -1$. Thus the maximum of curvature is attained at the point $(x, y) = (1, -1)$.

(c) Curve $x + \ln y = 0$. We set $F(x, y) = x + \ln y$. We have $F_x = 1$, $F_y = \frac{1}{y}$, $F_{xx} = 0$, $F_{xy} = 0$, $F_{yy} = \frac{1}{-y^2}$. Hence $D_B(F) = \frac{1}{-y^2} = -y^{-2}$. The curvature is $k = \frac{y^{-2}}{(\sqrt{1+y^{-2}})^3} = \frac{1}{y^2(1+y^{-2})^{3/2}} = \frac{y^3}{y^2(y^2+1)^{3/2}} = \frac{y}{(y^2+1)^{3/2}}$. To maximize k_C it is sufficient to maximize $k^2 = \frac{y^2}{(y^2+1)^3}$. We will use an auxiliary variable $z = y^2$ to simplify calculations. We are therefore interested in maximizing the expression $g(z) = \frac{z}{(z+1)^3}$. Differentiating we obtain $g'(z) = \frac{(z+1)^3 \cdot 1 - z \cdot 3(z+1)^2}{(z+1)^6} = \frac{z+1-3z}{(z+1)^4} = \frac{1-2z}{(z+1)^4} = 0$. We obtain an extremum when $1 - 2z = 0$, i.e., $z = \frac{1}{2}$. Checking that the second derivative is negative at the point, we conclude that this is a point of maximum. Thus the maximum of the curvature of the curve is attained when $y^2 = \frac{1}{2}$, i.e., $y = 2^{-1/2}$. Hence $x = \ln(2^{-1/2}) = -\frac{1}{2} \ln 2$.

Solution to Problem 4: Let M be a surface with a parametrisation $\underline{x}(u, v)$.

(a) To prove the formula $\Delta \underline{x} = -2f^2 H \underline{n}$ in isothermal coordinates (u, v), we calculate as follows. We use the formula $\underline{x}_{ij} = \Gamma_{ij}^k x_k + L_{ij} n$ to write $\Delta\, \underline{x} = x_{11} + x_{22} = \Gamma_{11}^1 x_1 + \Gamma_{11}^2 x_2 + L_{11} n + \Gamma_{22}^1 x_1 + \Gamma_{22}^2 x_2 + L_{22} n = (\Gamma_{11}^1 + \Gamma_{22}^1)\, x_1 + (\Gamma_{11}^2 + \Gamma_{22}^2)\, x_2 + (L_{11} + L_{22})\, n$. By a theorem proved in the lectures, with respect to isothermal coordinates we necessarily have the identities $\Gamma_{11}^1 + \Gamma_{22}^1 = 0$ and $\Gamma_{11}^2 + \Gamma_{22}^2 = 0$. Recall that with respect to isothermal coordinates, we have $L_{ii} = -f^2 L_{\underline{i}}^{\underline{i}}$. Therefore $\Delta\, \underline{x} = (L_{11} + L_{22})n = -(L_1^1 + L_2^2) f^2 n = -2H f^2 n$ as required.

(b) To prove that the catenoid is a minimal surface, recall that the catenoid is the surface obtained by rotating the graph of $x = \cosh z$ around the z-axis. Thus the *catenoid* parametrized by means of the formula $\underline{x}(\theta, \phi) = (\cosh\phi \cos\theta, \cosh\phi \sin\theta, \phi)$. The generating curve is the curve $r(\phi) = \cosh\phi$ and $z(\phi) = \phi$ (the catenary). Then according to the general formula, $g_{11} = r^2 = \cosh^2\phi$. Also, $g_{22} = (\frac{dr}{d\phi})^2 + (\frac{dz}{d\phi})^2 = (\sinh\phi)^2 + = \cosh^2\phi = g_{11}$, and $g_{12} = 0$. We conclude that the coordinates (θ, ϕ) are isothermal. Finally, $x_{11} + x_{22} =$

$(-a\cosh\phi\cos\theta, -a\cosh\phi\sin\theta, 0)^t + (a\cosh\phi\cos\theta, a\cosh\phi\sin\theta, 0)^t = (0,0,0)$. By the result of (a), we have $H = 0$ and therefore the catenoid is a minimal surface.

(c) Let us prove that the Scherk surface is a minimal surface. First we note the following Fact. If $f(x) = \ln\cos x$ and $h(x) = f'(x)$, then we have $\frac{1+h^2(x)}{f''(x)} = -1$ identically in x. Indeed, if $f(x) = \ln\cos x$ then $h(x) = f'(x) = -\tan x$ and $f''(x) = \frac{-1}{\cos^2 x}$. Therefore $\frac{1+h^2(x)}{f''(x)} = -(1 + \tan^2 x)\cos^2 x = -1$ as required.

The Scherk surface by definition is parametrized by the map $\underline{x}(x,y) = (x, y, f(y) - f(x))$. Clearly, we have $x_{12} = 0$. Therefore $L_{12} = \langle x_{12}, n\rangle = 0$. Thus the matrix (L_{ij}) is diagonal. The mean curvature H satisfies $2H = \operatorname{trace} W_p = L_{11}g^{11} + L_{22}g^{22}$ and therefore the condition $H = 0$ is equivalent to $L_{11}g^{11} + L_{22}g^{22} = 0$. Let $g = \det(g_{ij})$ (note that $g_{12} \neq 0$). Then $g^{11} = \frac{g_{22}}{g}$ and $g^{22} = \frac{g_{11}}{g}$. Thus the condition becomes becomes $\frac{L_{11}g_{22} + L_{22}g_{11}}{g} = 0$ where $g \neq 0$. Therefore the minimality condition is $\frac{L_{11}}{g_{11}} + \frac{L_{22}}{g_{22}} = 0$. Now let $h(x) = f'(x)$. We have $x_1 = (1, 0, -h(x))^t$ and $x_2 = (0, 1, h(y))^t$. Hence $g_{11} = 1 + h^2(x)$ and $g_{22} = 1 + h^2(y)$. The normal vector is the normalisation of the cross product $(-h(x), h(y), 1)^t$. Let $C = \sqrt{1 + h^2(x) + h^2(y)}$, so that $n = \frac{1}{C}(h(x), -h(y), 1)^t$. Since $x_{11} = (0, 0, -f''(x))^t$, we have $L_{11} = \langle n, \underline{x}_{11}\rangle = -\frac{f''(x)}{C}$ and similarly $L_{22} = \frac{f''(y)}{C}$. Thus we have $H = 0 \iff \frac{f''(x)}{1+h^2(x)} = \frac{f''(y)}{1+h^2(y)}$, and the Fact above proves minimality.

Solution to Problem 5: Calculations with index notation.

(a) Consider the expression $\langle x_j, x_{pq}\rangle g^{jp}$. Here j and p are summation indices, and q is a free index. We have $\langle x_j, x_{pq}\rangle = \langle x_j, \Gamma^i_{pq}x_i + L_{pq}n\rangle = \langle x_j, \Gamma^i_{pq}x_i\rangle + \langle x_j, +L_{pq}n\rangle = \Gamma^i_{pq}\langle x_j, x_i\rangle + L_{pq}\langle x_j, n\rangle$. Since tangent vectors are orthogonal to n, the second summand vanishes and we are left with $\Gamma^i_{pq}\langle x_j, x_i\rangle = \Gamma^i_{pq} = \Gamma^i_{pq}g_{ij}$ by symmetry of the metric coefficients. Therefore $\langle x_j, x_{pq}\rangle g^{jp} = \Gamma^i_{pq}g_{ij}g^{jp} = \Gamma^i_{pq}\delta^p_i$ since (g^{ab}) is the inverse matrix of (g_{ab}). Finally $\Gamma^i_{pq}\delta^p_i = \Gamma^p_{pq}$ by definition of Kronecker delta. The final expression can also be written as $\Gamma^1_{1q} + \Gamma^2_{2q}$ if the indices run from 1 to 2.

The remaining formulas are treated similarly.