

88-826 Differential Geometry, moed B

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Each of 4 problems is worth 25 points

All answers must be justified by providing complete explanations and proofs

1. Let M be an orientable 8-dimensional manifold with $b_2(M) = 1$, with an integer de Rham class $\omega \in L^2_{\text{dR}}(M)$ such that $\omega^{\cup 4}$ is the fundamental cohomology class of M . Find an upper bound for the ratio $\frac{(\text{stsys}_2(g))^4}{\text{vol}(g)}$ valid for all Riemannian metrics g on M , with proof.

2. Determine which of the following 12-dimensional manifolds satisfy a stable systolic inequality for stsys_2 with a constant independent of the metric:

- (1) $S^2 \times S^{10}$;
- (2) $S^6 \times \mathbb{CP}^3$;
- (3) $S^2 \times S^4 \times S^6$.

3. Let $n \geq 1$. Determine which of the following manifolds satisfy a stable systolic inequality relating stsys_2 to the volume, with a constant independent of the metric:

- (1) $S^2 \times S^2 \times \mathbb{CP}^n$;
- (2) $\mathbb{CP}^2 \times S^n$;
- (3) $\mathbb{CP}^n \times T^2$.

The answer may depend on n .

4. Let T^2 be a torus with a Riemannian metric g , and suppose T^2 contains an annulus $A = \mathbb{R}/\mathbb{Z} \times I$ such that the class of $\mathbb{R}/\mathbb{Z}$ has nonzero image in $H_1(T^2; \mathbb{Z})$.

- (a) Define the capacity of the annulus A .
- (b) Suppose the capacity of the annulus $A \subseteq T^2$ is C . Prove an optimal inequality relating $\text{sys}_1(g)$, C , and $\text{area}(g)$.
- (c) State Loewner's inequality for the torus and determine for which values of C the inequality obtained in part (b) is stronger than Loewner's inequality for g .

Good Luck!